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Run order considerations for Plackett and Burman designs

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ABSTRACT

Run order considerations for two-level full and fractional factorial designs have been studied in depth, but are lacking for Plackett and Burman designs. We look at the level change problems in Plackett and Burman designs. When a systematic run order is appropriate (as opposed to the conventional random run order), minimizing level changes implies the minimization of experiment costs. We thus aim to find optimal run orders with respect to minimizing level changes. It is shown that level changes are a constant for saturated Plackett and Burman designs. Methods for obtaining the minimum/maximum level changes are given. Tables with example run orders for the cases where $N = 12$ and $N = 20$ are tabulated for practical uses. By finding minimum level change designs, we also produce maximum level change designs and such results can be directly extended to Trend Robust designs.

1. Introduction

Plackett and Burman (1946) proposed a class of two-level factorial designs which only require $N = 4t$ runs as opposed to $N = 2^{k-p}$ for a fractional factorial design. Thus, Plackett and Burman designs are often used for screening purposes when there are a large number of factors. For example, a Plackett and Burman design with 36 runs can be applied to a study with 33 factors as opposed to the 64 runs required for a regular 2^{k-p} fractional factorial design. When using these designs, we assume that only a small number of factors are active, and that all interaction terms are negligible. This is known as effect sparsity (Box and Meyer, 1986).

Our focus here is the run order. The principle behind systematically choosing a run order is that we should use information we have about the experiment to optimize the run order of our design. For example, if one knows that the quality of our materials will degrade over time, one would ideally have a design which does not confound main effects with this time trend. Although in theory, randomization will balance out these unwanted effects as the number of runs increases, this is not always the case in practice. As will be shown in this paper, under some optimality criteria, randomization alone will rarely achieve the optimum run order. Therefore, we should not leave it up to chance that our run order will avoid a known issue.

If we wish to systematically produce a run order, we should select one which is optimal with respect to a given criteria. This has been studied for factorial designs, with guarding against a time trend or minimizing cost. Some examples of the former include Trend Robust Designs (Cheng and Steinberg, 1991), and Trend-Free Run Orders (Cheng and Jacroux, 1988); while examples of the later include One-at-a-time level changes (Lin and Lam, 1997) and Hard-to-Change Easy-to-change (Ju and Lucas, 2002). With so much written about this topic for factorial designs, it is interesting that these criteria have not been studied for Plackett and Burman designs. In this paper we will describe optimal run orders for Plackett and Burman designs under various criteria.

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Table 1
A 12 run Plackett and Burman design.

Run	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

This paper is organized as follows. In Section 2, we present a theorem about saturated Plackett and Burman designs with respect to level changes and then provide some optimal run orders for unsaturated designs with respect to cost in Section 3. In Section 4, we present trend robust designs and a conclusion is given in Section 5.

2. Saturated designs

Most Plackett and Burman designs have a cyclic structure. This means that each row, prior to the last one is a rotation of the first. Therefore, to construct a cyclic Plackett and Burman design we only need to be given the first row. First one would rotate each value one position to the right (or left) and move the furthest value to the other side. Then repeat this process until there are $N-1$ rows. Finally, add a row of all '−' (low level) to the design. For a design rotated to the right when $N=12$, the first row will be $++-+++---+-$. The second row is $-++-+++---+$, and continuing in this way we obtain our saturated design as presented in Table 1.

As previously discussed, one may wish to optimize their run order given a certain criteria. We will first consider minimizing cost. A level change occurs when a factor goes from its high (low) level on one run to its low (high) level on the next. This corresponds to the experimenter being forced to reset this factor to its other setting. If it is expensive to change levels, it may be in the experimenters' best interest to minimize the level changes. This cost could be monetary or it may be the amount of time it takes to run the experiment. Thus, the total cost can be minimized by minimizing the total number of level changes.

Let x_{ij} represent the level of the ith factor on the jth run. Then a level change occurs between row j and $j+1$ if $x_{ij} \neq x_{i(j+1)}$. Further, define $l_{mn} = \sum_{i=1}^{r} I_{\{x_{im} \neq x_{in}\}}$ to be the total number of level changes between row m and row n when there are r total factors. Since any Plackett and Burman design is in fact a Hadamard matrix, the following result for a saturated Plackett and Burman designs can be established.

Theorem 1. *The number of total level changes for a Placket and Burman design is a constant. For an N run Placket and Burman design, $l_{mn} = N/2$ for all $m \neq n$.*

Proof. By definition, a Hadamard Matrix is any square matrix which satisfies $H_N H_N' = N I_N$. Consider $O_N = \left(1/\sqrt{N}\right) H_N$. The columns of O_N are an orthonormal basis, and thus, the rows of O_N are orthogonal with each entry either taking $1/\sqrt{N}$ or $-1/\sqrt{N}$. Hence, it must be the case that there are $N/2$ level changes, as otherwise the rows would not be orthogonal. □

The important implication of this theorem is that with respect to level changes the run order of a saturated Plackett and Burman design does not matter.

3. Unsaturated designs

In this section we look at the properties of Plackett and Burman designs with only $N-q-1$ factors, where there are q design columns removed from the $N-1$ columns in the saturated design. In this case, the choice of columns removed may result in different designs. Therefore, one needs to consider all intrinsically different unsaturated designs for different values of q. Draper and Lin (1988, 1990) characterized all possible projections of Plackett and Burman designs for $N = 12, 20$, and 24. Thus, for the given Plackett and Burman designs, all cases can be covered by only looking at the given columns. By Theorem 1, we can minimize the number of level changes in our design by maximizing the number of level changes for the factors removed. One can use the following methods to obtain an optimum run order. Any run order which satisfies the following conditions will be optimal. In the cases of $q=1$ or 2, the optimum is obtained no matter which column is selected, but with $q=3$ the optimal run order is dependent on the selection of the columns removed. Note that these methods still hold for all non-equivalent Hadamard matrices.

For $q = 1$, alternate the levels of the column to be removed. This results in $N/2 - 1$ level changes between any two rows in the design matrix. For example, if the first column from Table 1 is removed, a run order of 1, 2, 3, 4, 7, 5, 8, 6, 9, 10, 11, 12 will have minimum level changes for the Plackett and Burman design with $N = 12$.

For $q = 2$, we use the fact that the columns are orthogonal. First alternate the $N/4$ rows that are at their high level for both factors with the $N/4$ that are both at their low level. Next, do the same with the $N/4$ that are "$-+$" and "$+-$". This gives us $N/2 - 2$ level changes between any two rows except for the change from the $N/2$th and $N/2 + 1$th row where there are $N/2 - 1$ level changes. Any choice of the four possible combinations as the first run will give the same result as long as the same procedure is followed. If factors x_1 and x_2 are removed from Table 1, then a run order of 1, 5, 8, 6, 9, 12, 2, 3, 4, 7, 10, 11 will satisfy this condition for $N = 12$. We could also select 2, 3, 4, 7, 10, 11, 1, 5, 8, 6, 9, 12 if we started with "$-+$" instead of "$++$".

For $q = 3$, when N is between 12 and 36 there is a nice structure if one removes the "proper" columns. Lin and Draper (1995) characterized the projections of saturated Plackett and Burman designs into three factors. For each projection, there exist three columns which will provide either repeats of a 2^3 factorial design, or a combination of repeated 2^3 designs plus a 2^{3-1} design. In either case, we can use this to systematically generate an optimal run order. First, the reverse foldover algorithm presented by Cheng and Steinberg (1991) can be used to produce a 2^3 design with maximum level changes. This design will consist of blocks of "mirror" pairs (for example, $+++$ and $---$). If the projection into 3 factors contains $r2^3$ designs, then each block of mirror pairs is repeated r times so there are three level changes between runs within each block. Then, the blocks are arranged so there are only two level changes between them. If there is an additional 2^{3-1}, each run from it will belong to a different mirror pair (as there are two level changes between each pair from the 2^{3-1}). We simply repeat the method for repeated 2^3, and add the runs from the remaining 2^{3-1} to their respective blocks. In either situation, if there are N runs in the design, then there are a total of $3N - 6$ level changes for the columns removed.

Following the classical notation of Hicks (1964), for example, the maximum level changes for a 2^3 design could be generated by, 1, abc, c, ab, ac, b, a, bc; where a, b, and c correspond to x_1, x_2, and x_3 in Table 1 respectively. If a letter is in the string, that means it is at its high level, and if not it is at its low level. To match this run order, one would use runs (6, 9), (5, 1), (3, 4), (7, 2) with parenthesis to indicate mirror pairs. There are no additional full factorials, only a fractional factorial is remaining. Therefore, our final design would have the run order 6, 9, 12, 8, 5, 1, 3, 4, 11, 10, 7, 2.

However, in some situations the design may not permit the selection of columns with this structure. Let $m_-(m_+)$ be the total number of the number of runs which are all at low (high) level. Take m to be $\min(m_-, m_+)$. The following theorem allows us to create a maximum level change design for any structure for $q = 3$.

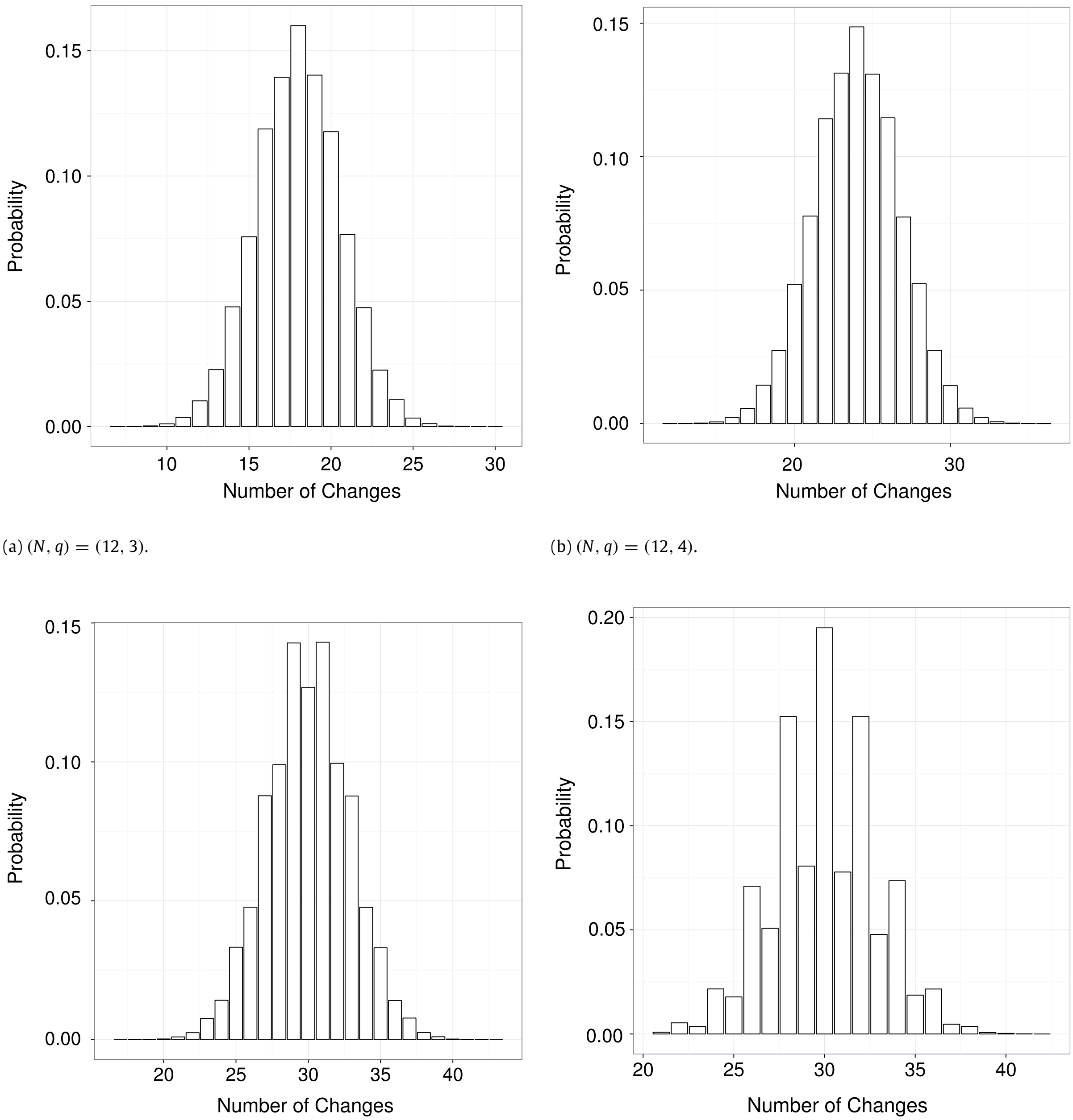
Theorem 2. *In an N run Plackett and Burman design projected into three columns there will either be m runs or $N/4 - m$ runs for each combination of factor levels.*

Proof. All factors must be orthogonal to one another and balanced. If we have m runs that are at all low level, then there must be $N/4 - m$ runs for each factor combination with one factor at its high level. This result holds by orthogonality, as each pair of factors must have $N/4$ factors at its low level. Similarly, there must be m runs which have two factors at their high level, and $N/4 - m$ which have all factors at their high level. □

Using Theorem 2, the structure can be determined by checking the number of runs for any combination of factor levels. The maximum number of level changes is $2N + 8m - 2$ for all cases except for $m = N/8$ in which case the maximum is still $3N - 6$. This maximum can be obtained by first proceeding as in the previous case when $q = 3$ by using the reverse foldover algorithm and adding to the blocks of mirror pairs. Once no mirror pairs remain, each block will start and end with the same factor levels. We can repeat the order of the blocks to obtain the maximum number of level changes. For example, if columns 1, 3, and 6 are removed when $N = 20$, m will be 1 as there are 4 runs with all low level and 1 with all high level. Start with the mirror pairs to get an order of (7, 16, 10), (3, 8, 11), (1, 17, 9), (2, 4, 5). Then repeat the pattern from the blocks to obtain a run order of 7, 16, 10, 3, 8, 11, 1, 17, 9, 2, 4, 5, 12, 13, 14, 6, 20, 15, 19, 18. Since the maximum level changes is dependent on the choice of m, we can only provide upper and lower bounds for the maximum level changes. Using the smallest and largest possible values of m, the maximum level changes ranges from $2N - 2$ to $3N - 6$. Since the maximum level changes is determined by m, the practitioner can search for the three columns which produce the largest m to obtain the optimal design.

For $q > 3$, the structure of the projected design is not as well behaved. Certainly, it is only possible to use a method similar to that of $q = 3$ if $2^q < N$. Otherwise, the number of level changes can be maximized by removing the q columns which contain the most mirror pairs. This allows us to match the mirror pairs, and achieve the maximum of q level changes for each of those pairs. Then, if one can obtain $q - 1$ level changes between any two rows that are not mirror pairs, we are guaranteed to have the optimal design. However, for the purpose of minimizing total level changes it is not necessary to use a design with N runs, and instead a design with a smaller number of runs (e.g. $N - 4$ or $N - 8$) is more appropriate. This will reduce q and allow us to apply the methods listed above.

Fig. 1 gives the distribution of run orders under randomization and was obtained by evaluating the total number of level changes for all possible 12! permutations of the three removed columns. When $q = 3$, only 384 run orders out of the 12! possible achieve the optimal number of level changes. This means the probability of reaching the optimal number of level changes of 30, by randomizing it is 8.01×10^{-7}, and on average, one would expect 18 level changes. Therefore, if cost is of primary concern, there is almost no chance of obtaining the optimal number of level changes by randomizing. While



(a) $(N, q) = (12, 3)$.

(b) $(N, q) = (12, 4)$.

(c) $(N, q) = (12, 5\text{–}1)$.

(d) $(N, q) = (12, 5\text{–}2)$.

Fig. 1. Probabilities under randomization for the number of level changes when $N = 12$ and q columns are removed when: (a) $q = 3$ or 8 with a range of $\{7, 30\}$. (b) $q = 4$ or 7, with a range of $\{12, 36\}$ (c) $q = 5$–1 or 6-1 (as given in Table 2) with a range of $\{17, 43\}$ (d) $q = 5$–2 or 6–2 with a range of $\{21, 42\}$.

Fig. 1(a)–(c) all appear to be relatively bell shaped, the distribution of Fig. 1(d) appears to be a mixture of two bell shaped curves. We conjecture that this is due to the fact that between any two runs, there are at least two level changes for the second unique projection of 5 columns. Hence, for $q = 5$, the projection of the second type, given in Table 2 would be more likely to have an even number of total level changes than the other choices of q. To quickly reference multiple projections we introduce a new notation. When we write $q = m - 1$ it represents the first type of projection for m columns of the design, and $q = m-2$ represents the second type. To avoid possible confusion with the notation used for fractional factorials we will always pair this with a q to make it clear what we are referencing. As an example, the second unique projection mentioned above when $q = 5$ is written as $q = 5$–2.

Table 2
Minimum level change designs when $N = 12$, to maximize level changes, switch the roles of the columns removed and the design. $E(lc)$ represents the expectation of the level change distribution. Note the exact expectation differs by $o(10^{-4})$ from what is listed in the table (e.x. $E(lc) = 18.00031$ for $q = 3$). n_{opt} represents the total number of run orders whose level changes are optimal among all $12! = 479,001,600$ possibilities. The notation $q = 5$–1 represents that there are five columns removed from the design, and that this is the first type. An example choice of columns is given in the table, but any projection with the same structure would be classified as the same type.

q	Columns removed	Example run order	Level changes	$E(lc)$	n_{opt}
1	1	1, 2, 3, 4, 7, 5, 8, 6, 9, 10, 11, 12	$11 \times 1 = 11$	6	1036800
2	1, 2	1, 5, 8, 6, 9, 12, 2, 3, 4, 7, 10, 11	$10 \times 2 + 1 \times 1 = 21$	12	10368
3	1, 2, 3	12, 9, 6, 2, 7, 10, 1, 5, 8, 3, 4, 11	$8 \times 3 + 3 \times 2 = 30$	18	384
4	1, 2, 3, 4	3, 12, 11, 8, 5, 1, 2, 6, 9, 4, 7, 6	$3 \times 4 + 8 \times 3 = 36$	24	1438
5–1	1, 2, 3, 4, 5	5, 1, 12, 11, 8, 10, 7, 2, 3, 4, 9, 6	$1 \times 5 + 8 \times 4 + 2 \times 3 = 43$	30	519
5–2	1, 2, 3, 5, 8	6, 9, 4, 11, 10, 7, 2, 3, 1, 5, 8, 12	$9 \times 4 + 2 \times 3 = 42$	30	480
6–1	6, 7, 8, 9, 10, 11	7, 11, 3, 10, 4, 1, 8, 9, 2, 5, 12, 6	$5 \times 5 + 6 \times 4 = 49$	36	520
6–2	4, 6, 7, 9, 10, 11	3, 11, 9, 10, 8, 7, 5, 4, 2, 1, 12, 6	$1 \times 6 + 9 \times 4 + 1 \times 3 = 45$	36	381771
7	5, . . . , 11	9, 3, 11, 1, 8, 7, 12, 5, 2, 10, 4, 6	$1 \times 6 + 8 \times 5 + 2 \times 4 = 54$	42	1918
8	4, . . . , 11	12, 6, 7, 1, 8, 4, 2, 10, 9, 3, 11, 5	$4 \times 6 + 7 \times 5 = 59$	48	2304
9	3, . . . , 11	1, 8, 9, 2, 4, 10, 5, 6, 12, 3, 7, 11	$8 \times 6 + 3 \times 5 = 63$	54	10368
10	2, . . . , 11	1, 3, 7, 8, 9, 11, 2, 4, 5, 6, 10, 12	$10 \times 6 + 1 \times 5 = 65$	60	1036800

Table 3
Minimum level change designs when $N = 20$, to maximize level changes, switch the roles of the columns removed and the design. As in Table 2 the notation $q = 3$–1, for example, represents the first type of projection when $q = 3$.

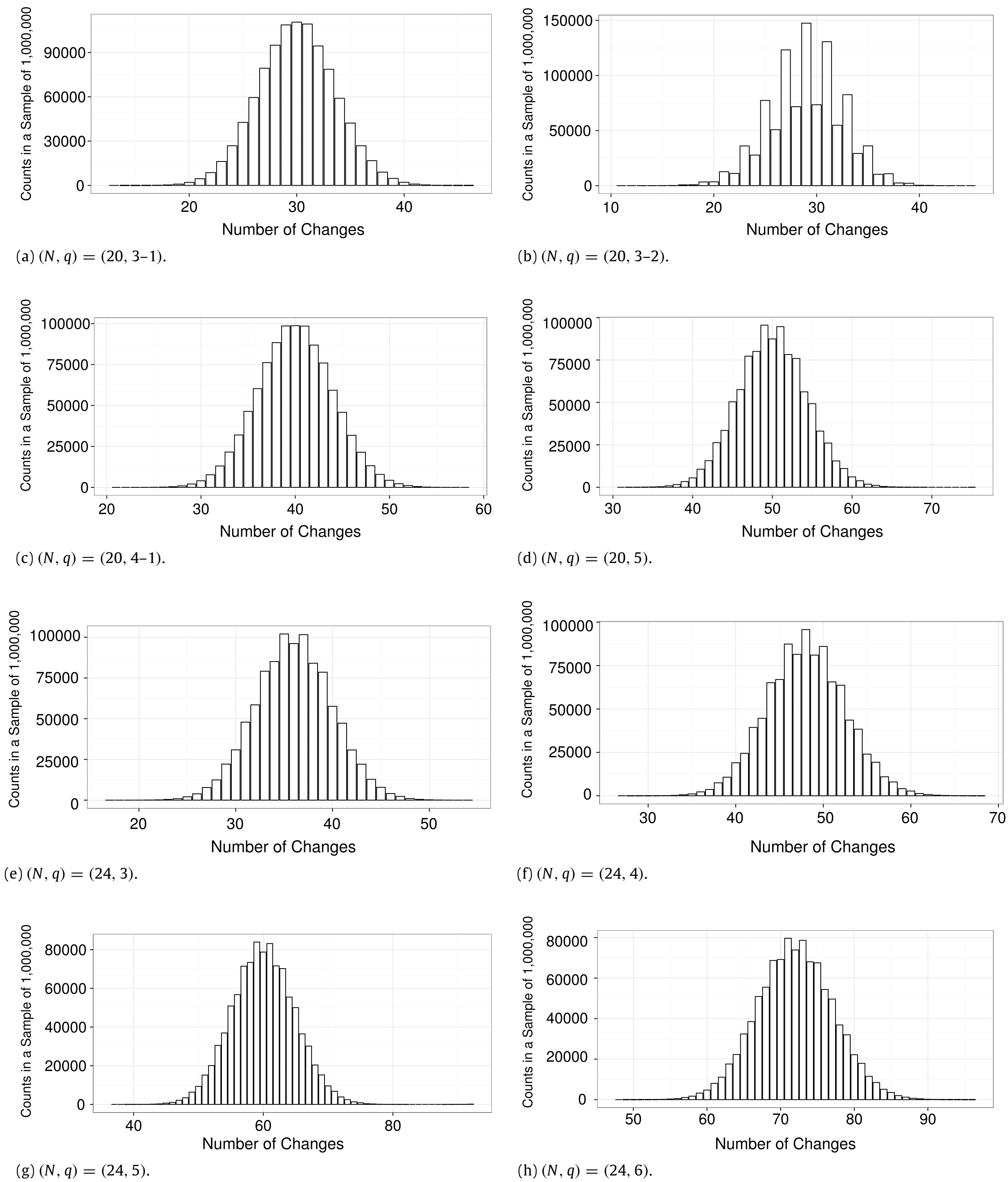
q	Columns removed	Example run order	Level changes
1	1	1, 2, 3, 5, 4, 6, 9, 7, 11, 8, 13, 10, 14, 12, 15, 17, 16, 18, 19, 20	$19 \times 1 = 19$
2	1, 2	1, 6, 4, 7, 14, 8, 15, 18, 16, 20, 2, 3, 5, 9, 10, 11, 12, 13, 17, 19	$18 \times 2 + 1 \times 1 = 37$
3–1	1, 2, 3	1, 6, 4, 18, 14, 7, 15, 8, 16, 20, 3, 10, 11, 12, 13, 2, 9, 5, 19, 17	$16 \times 3 + 3 \times 2 = 54$
3–2	1, 3, 6	7, 16, 10, 3, 8, 11, 1, 17, 9, 2, 4, 5, 12, 13, 14, 6, 20, 15, 19, 18	$8 \times 3 + 11 \times 2 = 46$
4–1	1, 2, 3, 4	9, 17, 11, 12, 13, 4, 8, 16, 20, 14, 6, 1, 18, 10, 3, 2, 19, 5, 7, 15	$9 \times 4 + 10 \times 3 = 66$
4–2	1, 2, 3, 6	8, 15, 6, 4, 18, 12, 9, 17, 19, 10, 13, 1, 11, 14, 3, 2, 7, 16, 20, 5	$7 \times 4 + 12 \times 3 = 64$
4–3	1, 2, 3, 16	11, 4, 18, 14, 7, 17, 19, 2, 9, 6, 15, 8, 16, 20, 10, 13, 12, 3, 5, 1	$11 \times 4 + 8 \times 3 = 68$
15–1	5, . . . , 19	8, 20, 10, 2, 5, 15, 16, 17, 6, 18, 7, 12, 4, 14, 1, 9, 11, 13, 3, 19	$5 \times 10 + 14 \times 9 = 176$
15–2	4, 5, 7, . . . , 19	17, 15, 4, 10, 12, 7, 20, 8, 6, 18, 2, 5, 16, 1, 14, 9, 19, 3, 11, 13	$8 \times 10 + 10 \times 9 + 1 \times 8 = 178$
15–3	4, . . . , 15, 17, 18, 19	15, 16, 5, 18, 7, 8, 20, 6, 3, 13, 11, 9, 19, 10, 12, 2, 17, 14, 4, 1	$8 \times 10 + 6 \times 9 + 3 \times 8 = 172$
16–1	4, . . . , 19	7, 8, 20, 6, 18, 2, 5, 17, 15, 16, 3, 11, 13, 9, 19, 1, 4, 14, 10, 12	$12 \times 10 + 7 \times 9 = 183$
16–2	2, 4, 5, 7, . . . , 19	7, 20, 10, 12, 8, 6, 18, 2, 5, 17, 15, 3, 11, 13, 4, 9, 19, 1, 14, 16	$12 \times 10 + 7 \times 9 = 183$
17	3, . . . , 19	1, 4, 14, 15, 16, 2, 5, 10, 12, 17, 6, 7, 8, 18, 20, 3, 9, 11, 13, 19	$16 \times 10 + 3 \times 9 = 187$
18	2, . . . , 19	1, 3, 4, 9, 11, 13, 14, 15, 16, 19, 2, 5, 6, 7, 8, 10, 12, 17, 18, 20	$18 \times 10 + 1 \times 9 = 189$

In Table 2, example run orders are given for the optimal unsaturated designs when $N = 12$. q greater than 4 is included in the table, but for practical purposes it is more efficient to use a minimum level change design with a smaller N with fewer columns removed. To obtain the design from the table, first produce the cyclic Plackett and Burman design by rotating to the right (This is given in Table 1.) Then, remove the columns under the "Columns Removed" section. Finally, use the example run order given to produce a design with minimum level changes. The table also gives the number of level changes in the columns to be removed. These are listed as (number of occurrences) $\times$ (level changes) and listed in decreasing order of number of changes. For example, when $q = 4$, removing any four columns will be identical to removing the first four. An example run order is given, and for this run order, the columns removed have 3 transitions with 4 level changes, and 8 with 3 level changes. Which implies our design will have 3 transitions with only 2 level changes, and every other transition will have 3 level changes. The expected level changes for each q is given, and denoted by $E(lc)$. It is interesting to note that $E(lc) = q \cdot N/2$ for all cases in the table. The number of run orders out of 12! which obtain the optimal number of level changes is also provided. As an example checking $q = 4$ tells us that only 1438 run orders out of 12! reach this optimum run order. In some situations (such as $q = 5$) there are two different sets of columns removed. These columns were not chosen arbitrarily, but rather to cover all non-equivalent projections as stated by Draper and Lin (1995).

Table 3 gives designs when $N = 20$ for values of q up to 4 and from $q = 15$ to $q = 18$. This was done in part because of the large number of unique projections into q when it is greater than 4. If cost is the primary concern, instead of removing more than 3 columns it may be more useful to use the Plackett and Burman design with $N - 4$ runs instead. Table 3 is used the same way as Table 2, but the total number of run orders out of 20! which achieve the optimal design are not included. When $N \geq 24$, one can use the guidelines presented above to construct optimum run orders for $q = 1, 2,$ and 3. Finally, note that the given run orders are not unique, but are all optimal with respect to our criteria. It is possible to further optimize the run order within some other constraints.

4. Trend robust designs

Thus far, we have only considered designs which are optimal with respect to minimizing the level changes (and thus the cost). In some situations, the cost associated with level changes may be negligible, and thus we can consider other optimality



(a) $(N, q) = (20, 3–1)$.

(b) $(N, q) = (20, 3–2)$.

(c) $(N, q) = (20, 4–1)$.

(d) $(N, q) = (20, 5)$.

(e) $(N, q) = (24, 3)$.

(f) $(N, q) = (24, 4)$.

(g) $(N, q) = (24, 5)$.

(h) $(N, q) = (24, 6)$.

Fig. 2. Random samples of 1 million run orders for $N = 20$ and $N = 24$ for q columns removed. For the choice of columns when $N = 20$ see Table 3. In the case of $q = 5$ for $N = 20$ and all cases for $N = 24$, the first q columns of the design were selected. These plots show a continuation of the pattern of a bell shape with increasing range as N is increasing.

criteria. A common concern is that there will be a time trend that will confound with the factor effects. Note that one method to deal with this is to select a design which is orthogonal to a prespecified time trend. Alternatively, Cheng and Steinberg (1991) found that maximizing the number of level changes in an experiment with k two-level factors will give optimal or near optimal run order if we assume auto regressive errors in a main effects only model. These designs are considered trend robust, as they still perform well with respect to different time trends, while a design which is orthogonal to a prespecified time trend (trend free) may perform poorly under different circumstances. These results can be applied directly to Plackett and Burman designs.

First, note that run orders with maximum level changes can be obtained from Tables 2 and 3 by switching the role of the removed columns and the design matrix. One can also produce optimal run orders systematically. In a similar fashion to Section 2, we consider cases with q columns removed.

$q = 1$: first repeat all of the low level runs, and then repeat all of the high level runs. This will give us one level change for the column removed. For example, in Table 1 one can remove factor x_1 and the resulting run order will be 1, 3, 7, 8, 9, 11, 2, 4, 5, 6, 10, 12 for a 12 run Plackett and Burman design.

$q = 2$: In this case, the two columns removed are orthogonal, so the projection into $q = 2$ will always be a repeat of a 2^2 full factorial. Therefore, the one at a time level change method (Lin and Draper, 1995) can be applied to obtain the minimum level changes for a 2^2 full factorial. Then, the repeat runs can be blocked together to obtain a design with 3 total level changes. In Table 1 if factors x_1 and x_2 are removed, one can start with both factors at their high level to obtain a run order of 1, 8, 9, 2, 4, 10, 5, 6, 12, 3, 7, 11 to obtain a maximum level change design for $N = 12$.

$q = 3$: There are two cases, either there are repeated fractional factorial designs, or at least one full factorial design. This is a direct result of Theorem 2. If there is at least one full factorial, the minimum number of level changes is 7. This result can be obtained by first finding the 2^3 design with minimum level changes using the same method as in $q = 2$. If factors x_1, x_2, and x_3 in Table 1 are removed, a run order of 12, 6, 7, 1, 8, 4, 2, 10, 9, 3, 11, 5 is obtained. If instead, a repeated fractional factorial designs is given, the minimum number of level changes is six. In this case, after blocking all runs with the same factor levels, any run within a block will have two level changes with any run outside of the block. When $N = 16$, if we remove columns 1, 2, and 13, then the run order which maximizes level changes is 3, 4, 7, 16, 1, 9, 14, 15, 2, 6, 10, 12, 5, 8, 11, 13.

$q > 3$: In these cases one can follow the method from $q = 2$, however there is no guarantee that we will obtain all of the design points for a full factorial design when we project into larger q, so some additional work may be required to obtain the optimal design.

An interesting feature is that the minimum number of level changes for the removed columns does not directly depend on N, while the maximum number of level changes increases with N. We remark that if the distribution of the level changes follows a similar shape as in Fig. 1, then the expected number of level changes would move further from the optimal value when we randomize (minimum or maximum level changes). This point is illustrated by taking random samples of 1 million run orders in Fig. 2. We see a similar bell curve when N is increased to 20 or 24. Given the relatively low probability of obtaining the optimal run order by randomization as shown in Table 2, it is likely that the range of values for the plots in Fig. 2 is smaller than the actual range given by the full level change distribution.

5. Conclusions

In this paper, it is shown that the total number of level changes for a saturated Plackett and Burman design is a constant. Using this result, minimum and maximum level changes for unsaturated designs were obtained. The former was used as a way to minimize the cost, while the latter was used to obtain a trend robust design. Run orders were optimized with respect to some optimality criteria in the popularly used Plackett and Burman designs. Note that we have assumed all factors have an equal cost associated with their level changes. If this is not the case, our approach can be applied to meet these needs. Level changes over the entire design have been minimized/maximized for $q = 1, 2, 3, N-3, N-2, N-1$ columns, solving some cases when one set of factors is more expensive than the rest. The more general case when each factor may have a different cost associated with its level changes will require more modification of the methods presented in this paper.

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