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UNIFORM FRACTIONAL FACTORIAL DESIGNS

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The minimum aberration criterion has been frequently used in the selection of fractional factorial designs with nominal factors. For designs with quantitative factors, however, level permutation of factors could alter their geometrical structures and statistical properties. In this paper uniformity is used to further distinguish fractional factorial designs, besides the minimum aberration criterion. We show that minimum aberration designs have low discrepancies on average. An efficient method for constructing uniform minimum aberration designs is proposed and optimal designs with 27 and 81 runs are obtained for practical use. These designs have good uniformity and are effective for studying quantitative factors.

1. Introduction. The minimum aberration criterion [Fries and Hunter (1980)] has been frequently used in the selection of regular fractional factorial (FF) designs with nominal factors, as it provides nice design properties. This is especially important when the experimenter has little knowledge about the potential significance of factorial effects. The readers are referred to Mukerjee and Wu (2006) and Wu and Hamada (2009) for existing theory and results on minimum aberration designs. Deng and Tang (1999), Tang and Deng (1999) and Xu and Wu (2001) further proposed generalized minimum aberration criteria for comparing nonregular fractional factorial designs.

Cheng and Wu (2001) and Fang and Ma (2001) found that designs may have different geometrical structures and statistical properties, even though they share the identical word-length pattern. In view of this, Cheng and Ye (2004) pointed out that the distinction in the analysis objective and strategy for experiments with nominal or quantitative factors requires different selection criteria and classification methods. For designs with quantitative factors, they proposed to describe design properties using the geometrical structures. Two designs are said to be *geometrically isomorphic* if one can be obtained from the other by a permutation of factors and/or reversing the level order of one or more factors. For example, consider the two designs in Table 1. Design A is a regular 3^{3-1} FF design with

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TABLE 1
Two combinatorially isomorphic designs with different geometrical structures

Design A			Design B		
F_1	F_2	F_3	F_1	F_2	F_3'
0	0	0	0	0	2
0	1	1	0	1	0
0	2	2	0	2	1
1	0	1	1	0	0
1	1	2	1	1	1
1	2	0	1	2	2
2	0	2	2	0	1
2	1	0	2	1	2
2	2	1	2	2	0

$F_3 = F_1 + F_2 \pmod 3$, and Design B is formed by $F_3' = F_1 + F_2 + 2 \pmod 3$. It is obvious that these two designs are *combinatorially isomorphic* to each other, because one can be obtained from the other by permuting the levels in the third column [i.e., map (0, 1, 2) to (2, 0, 1)]. However, they have different geometrical structures and thus are geometrically nonisomorphic. Design B contains the center run with all ones, while Design A does not. If we reverse the level order [i.e., map (0, 1, 2) to (2, 1, 0)] for all three columns, Design B is invariant while Design A is not. These two designs have different statistical properties due to their different geometrical structures.

To further classify geometrically nonisomorphic designs, Cheng and Ye (2004) generalized the concept of minimum aberration and used indicator function to define the β-word-length pattern based on a polynomial model. Despite its theoretical beauty, the complexity of indicator function prohibits its use for design construction. On the other hand, Fang and Ma (2001) suggested using uniformity to compare the performance of geometrically nonisomorphic designs. Various discrepancies have been used as measures of uniformity; see Fang et al. (2000) and Fang, Li and Sudjianto (2006). These discrepancies all have their geometrical meanings and can be interpreted as the difference between the empirical distribution and the uniform distribution. Among them, the centered L_2-discrepancy (CD), proposed by Hickernell (1998), is the most frequently used.

Both the β-word-length pattern and the centered L_2-discrepancy reflect the geometrical structure of the design. Here we use the discrepancy to choose FF designs mainly for two reasons. First, the centered L_2-discrepancy has a simple analytic formula; it is much faster to calculate the discrepancy than the β-word-length pattern. The difference between the computational times is substantial. The second and more important reason is that the β-word-length pattern is model-dependent while the centered L_2-discrepancy is model free. Cheng and Ye (2004) defined the

β-word-length pattern based on a polynomial model but they further pointed out that the β-word-length pattern needs to be modified in other situations. Optimal designs constructed based on the β-word-length pattern would rely on the specific model used. In contrast, designs with low discrepancy tend to have good space filling properties and are model robust in the sense that they can guard against inaccurate estimates caused by model misspecification [Hickernell and Liu (2002)].

Here we propose to construct uniform FF designs from existing minimum aberration designs via level permutations. Obviously, for two-level designs, there is no difference when levels are permuted, but for high-level designs, there are many unknowns to be studied. For convenience, we will focus on three-level designs in this paper, but the basic ideas can be extended to higher-level designs.

The paper is organized as follows. In Section 2, we obtain a key theorem and show that minimum aberration designs tend to have low discrepancy on average. Then we introduce the concept of uniform minimum aberration design. In Section 3, we present an efficient way for constructing three-level regular uniform FF designs and construct uniform minimum aberration designs with 27 runs and 81 runs for practical use. These newly-constructed designs often outperform existing uniform designs, especially when the number of factors is large. In Section 4, we examine the relationship between the discrepancy and the β-word-length pattern. Uniform minimum aberration designs appear to perform well with respect to the β-word-length pattern. The last section gives a brief conclusion. For clarity, we defer all proofs to the Appendix.

2. Uniform minimum aberration designs. A design with N runs, n factors and s levels, denoted by (N, s^n), is an $N \times n$ matrix. Throughout the paper, the s levels are denoted as $0, 1, \dots, s-1$. For an (N, s^n)-design D, consider an ANOVA model

$$Y = X_0\alpha_0 + X_1\alpha_1 + \cdots + X_n\alpha_n + \varepsilon,$$

where Y is the vector of N observations, α_0 is the intercept and X_0 is an $N \times 1$ vector of 1's, α_j is the vector of all j-factor interactions and X_j is the matrix of orthonormal contrast coefficients for α_j, and ε is the random error. Denote $n_j = (s-1)^j\binom{n}{j}$ and $X_j = (x_{ik}^{(j)})_{N\times n_j}$, then the (generalized) word-length pattern of design D can be defined by

$$A_j(D) = N^{-2}\sum_{k=1}^{n_j}\left|\sum_{i=1}^{N} x_{ik}^{(j)}\right|^2 \qquad \text{for } j = 0, \dots, n. \tag{2.1}$$

For two designs $D^{(1)}$ and $D^{(2)}$, $D^{(1)}$ is said to have less aberration than $D^{(2)}$ if there exists an $r \in \{1, 2, \dots, n\}$, such that $A_r(D^{(1)}) < A_r(D^{(2)})$ and $A_i(D^{(1)}) = A_i(D^{(2)})$ for $i = 1, \dots, r-1$. $D^{(1)}$ is said to have (generalized) minimum aberration if there is no other design with less aberration than $D^{(1)}$.

For a regular design, the traditional definition of $A_j(D)$ is the number of words of length j. Following Xu and Wu (2001), $A_j(D)$ defined in (2.1) is the number of degrees of freedom associated with all words of length j. Therefore, two definitions are equivalent and generalized minimum aberration reduces to minimum aberration for regular designs. For simplicity, in the following we use the notion of word-length pattern and minimum aberration for both regular and nonregular designs.

For an (N, s^n)-design $D = (x_{ik})_{N\times n}$, its centered L_2-discrepancy (CD) is defined as

$$\begin{aligned}\phi(D) = {} & \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\prod_{k=1}^{n}\left(1+\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|+\frac{1}{2}\left|u_{jk}-\frac{1}{2}\right|-\frac{1}{2}|u_{ik}-u_{jk}|\right)\\ & -\frac{2}{N}\sum_{i=1}^{N}\prod_{k=1}^{n}\left(1+\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|-\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|^2\right)+\left(\frac{13}{12}\right)^n,\end{aligned} \tag{2.2}$$

where $u_{ik} = (2x_{ik}+1)/(2s)$. Note that $0 < u_{ik} < 1$.

It is well known that word-length pattern remains the same for combinatorially isomorphic designs. However, the centered L_2-discrepancy will not be the same when levels of factors are permuted.

EXAMPLE 2.1. Consider the two designs given in Table 1. Both designs have one word of length three and share the same word-length pattern $(A_1, A_2, A_3) = (0, 0, 2)$. Their CD values are 0.033186 and 0.033034, respectively. So Design B is better than Design A in terms of CD.

There is a close relationship between minimum aberration and uniformity for two-level designs. Fang and Mukerjee (2000) showed that for a two-level regular design D, the centered L_2-discrepancy of D can be linearly expressed by its word-length pattern $(A_1(D), A_2(D), \ldots, A_n(D))$. Later on, Ma and Fang (2001) generalized it to the nonregular case. Obviously, their results cannot be generalized to high-level designs.

For an s-level factor, there are $s!$ possible level permutations. Given an (N, s^n)-design D, we apply all $s!$ level permutations to each column and obtain $(s!)^n$ combinatorially isomorphic designs. Denote the set of these designs as $\mathcal{P}(D)$. Some of them may be geometrically nonisomorphic and have different CD values. We compute the CD value for each design and define $\bar{\phi}(D)$ as the average CD value of all designs in $\mathcal{P}(D)$, that is,

$$\bar{\phi}(D) = \frac{1}{(s!)^n}\sum_{D'\in\mathcal{P}(D)}\phi(D').$$

Note that all designs in $\mathcal{P}(D)$ share the same word-length pattern. The following result shows that the average CD value, $\bar{\phi}(D)$, is closely related to the word-length pattern of D.

THEOREM 2.2. *For an* $(N, 3^n)$*-design* D,

$$\bar{\phi}(D) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + \left(\frac{29}{27}\right)^n \sum_{i=1}^{n} \left(\frac{2}{29}\right)^i A_i(D).$$

Theorem 2.2 implies that the average centered L_2-discrepancy and the minimum aberration criterion are approximately equivalent, as $(2/29)^i$ decreases geometrically when i increases. Thus designs permuted from a minimum aberration design tend to be more likely to have low discrepancies. As will be seen, Theorem 2.2 is very useful in finding uniform FF designs.

EXAMPLE 2.3. Consider designs from the commonly used orthogonal array $\mathrm{OA}(18, 3^7)$; see, for example, Table 2(a) of Xu, Cheng and Wu (2004). There are 3, 4, 4 and 3 combinatorially nonisomorphic designs when projected onto 3, 4, 5 and 6 factors, respectively. We rank these designs based upon the minimum aberration criterion, and denote them as 18-3.1, 18-3.2, 18-3.3 and etc. For each design, we conduct all possible level permutations and compute their CD values. Table 2 shows the average, minimum, maximum and standard deviation of the CD values of all permuted designs, as well as one representative of the columns, and A_3 and A_4 of the word-length pattern. Note that $A_1 = A_2 = 0$ for all designs here. It can be seen from Table 2 that the rankings of average, minimum and maximum CD values are all consistent with the minimum aberration ranking; that is, less aberration leads to lower CD values. It is interesting to note that designs 18-4.3 and 18-4.4 have the same word-length pattern but different standard deviations; so do designs 18-6.2 and 18-6.3. This implies that the word-length pattern does not uniquely determine the variance of the CD values of permuted designs.

We further compare the minimum aberration designs with the uniform designs listed on the Uniform Design (UD) homepage (http://www.math.hkbu.edu.hk/UniformDesign/). These uniform designs, labeled as UD18-3, UD18-4, etc., appear to be orthogonal arrays of strength 2 so that $A_1 = A_2 = 0$. The minimum aberration design 18-3.1 has the same minimum CD value as the uniform design UD18-3; however, the former has less aberration ($A_3 = 0.5$ vs. $A_3 = 0.67$) than the latter. Design 18-4.1 and UD18-4 have the same properties, and they are indeed combinatorially isomorphic. Design 18-5.1 has a slightly larger minimum CD value (0.065265 vs. 0.065248) and less aberration ($A_3 = 5$ vs. $A_3 = 6.17$) than UD18-5. The same phenomenon also appears for design 18-6.1 and UD18-6. UD18-7 has a smaller CD value than design 18-7 although they have the same word-length pattern. The existing uniform designs have minimum discrepancy for all cases because the run size is small here; nevertheless, the level-permuted minimum aberration designs are competitive. In summary, by permuting minimum aberration designs from the commonly used $\mathrm{OA}(18, 3^7)$, we can obtain good uniform FF designs.

TABLE 2
Comparison of 18-*run designs*

Design	**Columns**	**Ave ϕ**	**Min ϕ**	**Max ϕ**	**Sd ϕ**	A_3	A_4
18-3.1	1 2 3	0.032526	0.032500	0.032538	0.000018	0.5	
18-3.2	1 2 5	0.032729	0.032500	0.032958	0.000163	1	
18-3.3	1 3 4	0.033135	0.033034	0.033186	0.000072	2	
UD18-3			0.032500			0.67	
18-4.1	2 3 4 5	0.047407	0.047357	0.047446	0.000023	2	1.5
18-4.2	1 2 3 5	0.047611	0.047391	0.047866	0.000166	2.5	1
18-4.3	1 2 3 4	0.048017	0.047849	0.048077	0.000087	3.5	0
18-4.4	1 2 5 6	0.048017	0.047849	0.048306	0.000139	3.5	0
UD18-4			0.047357			2	1.5
18-5.1	2–6	0.065273	0.065265	0.065337	0.000019	5	7.5
18-5.2	1–3 5 6	0.065883	0.065706	0.066193	0.000150	6.5	4.5
18-5.3	1–5	0.066086	0.065722	0.066423	0.000197	7	3.5
18-5.4	1 2 5–7	0.066492	0.066197	0.067107	0.000211	8	1.5
UD18-5			0.065248			6.17	5.17
18-6.1	2–7	0.086964	0.086914	0.087145	0.000057	10	22.5
18-6.2	1–6	0.088184	0.087769	0.088591	0.000215	13	13.5
18-6.3	1–3 5–7	0.088184	0.087769	0.088974	0.000240	13	13.5
UD18-6			0.086896			12.33	15.5
18-7	1–7	0.115386	0.114505	0.116556	0.000347	22	34.5
UD18-7			0.113591			22	34.5

As suggested by Example 2.3, an efficient way for constructing uniform FF designs is to start with a minimum aberration design, permute its levels and choose the level permutation with the minimum CD value. These designs have minimum aberration, and good uniformity, and are suitable for investigation of both nominal and quantitative factors.

DEFINITION 2.4. Let D be a minimum aberration design. If $D_* \in \mathcal{P}(D)$ has the minimum centered L_2-discrepancy over $\mathcal{P}(D)$, then D_* is said to be a uniform minimum aberration design.

EXAMPLE 2.5. Consider 27-run designs. For $n = 4$ to 13 columns, we evaluate average CD values for the existing regular minimum aberration designs [see Xu (2005)] and compare with the CD values of the best designs listed on the UD homepage. For $n = 8$ to 10, the average CD values of the minimum aberration designs are even smaller than the CD values of the best existing designs; see Table 3 below.

To find uniform minimum aberration designs, we further conduct all possible level permutations for these minimum aberration designs and calculate the mini-

TABLE 3
Comparison of 27*-run designs*

	Minimum aberration designs					Designs on UD homepage		
n	Ave ϕ	Min ϕ	Max ϕ	A_2	A_3	ϕ	A_2	A_3
4	0.046549	0.046547$^{\diamond}$	0.046553	0	0	0.046547	0	0
5	0.063818	0.063689	0.063878	0	2	0.063525	0	2.67
6	0.083786	0.083475$^{\diamond}$	0.083923	0	4	0.083475	0	5.33
7	0.108701	0.108061*	0.109118	0	10	0.108698	0.10	12.17
8	0.137749*	0.136644*	0.138483*	0	16	0.138657	0.35	18.44
9	0.172783*	0.170996*	0.174090*	0	24	0.175343	0.69	30.05
10	0.218927*	0.213994*	0.221241	0	42	0.219131	1.36	40.99
11	0.273255	0.264549*	0.276195	0	60	0.272383	2	56
12	0.338698	0.325027*	0.343084	0	80	0.336401	2.32	75.46
13	0.418900	0.397890*	0.425576	0	104	0.414783	3.53	96.20

$^{\diamond}$The same CD value as the best existing design;
*Smaller CD value than the best existing design.

mum and maximum CD values. Table 3 shows the comparison between permuted minimum aberration (PMA) designs and the best designs listed on UD homepage in terms of discrepancy and aberration. For all designs, $A_1 = 0$ is not listed in the table. For $n = 4$, the PMA design is geometrically isomorphic to the one listed on UD homepage. For $n = 5$, the PMA design has a larger CD value than the one listed on UD homepage, but the PMA design has less aberration. For $n = 6$, the PMA design has the same CD value as the one listed on UD homepage and has less aberration. For $n > 6$, PMA designs always outperform the best ones listed on UD homepage. Note that those designs listed on UD homepage have resolution 2 ($A_2 > 0$) whereas our designs have resolution 3 ($A_2 = 0$), when $n > 6$. This shows the advantage of our approach and the disadvantage of the purely algorithmic approach. Further notice for $n = 8$ and 9, even the maximum CD values of all permuted designs, are less than those of the best existing ones.

3. Construction of regular three-level uniform minimum aberration designs. This section is devoted to providing an efficient method for constructing uniform minimum aberration designs. For an $(N, 3^n)$-design D, the total number of designs in $\mathcal{P}(D)$ is 6^n. However, when D is a regular FF design, many designs in $\mathcal{P}(D)$ are geometrically isomorphic and have the same CD values. So it will be much easier to find the uniform FF design when a regular minimum aberration design is permuted.

For a three-level factor, exchange of levels 0 and 2 does not change the geometrical structure and such a "mirror image" operation keeps its centered L_2-discrepancy unchanged according to formula (2.2). Denote $\pi_{i_0 i_1 i_2}$ as a permutation of (0, 1, 2), that is, $\pi_{i_0 i_1 i_2}$ maps (0, 1, 2) to (i_0, i_1, i_2). In view of the "mirror image" operation, we only need to consider three permutations π_{012}, π_{120} and π_{201} for a three-level design. Notice that π_{012} is the identity map, π_{120} maps x to $x+1 \pmod 3$ and π_{201} maps x to $x+2 \pmod 3$. So each permutation is equivalent to a linear permutation, which transforms x to $x+b \pmod 3$, where $b = 0$, 1, or 2.

A regular 3^{n-k} FF design D has $n-k$ independent columns, denoted as $\mathbf{x}_1, \dots, \mathbf{x}_{n-k}$, and k dependent columns, denoted as $\mathbf{y}_1, \dots, \mathbf{y}_k$. It is specified by k linear equations:

$$\begin{cases} \mathbf{y}_1 = c_{11}\mathbf{x}_1 + c_{12}\mathbf{x}_2 + \cdots + c_{1,n-k}\mathbf{x}_{n-k} + b_1, \\ \mathbf{y}_2 = c_{21}\mathbf{x}_1 + c_{22}\mathbf{x}_2 + \cdots + c_{2,n-k}\mathbf{x}_{n-k} + b_2, \\ \cdots \\ \mathbf{y}_k = c_{k1}\mathbf{x}_1 + c_{k2}\mathbf{x}_2 + \cdots + c_{k,n-k}\mathbf{x}_{n-k} + b_k, \end{cases}$$

where c_{ij} and b_i are constants in GF(3), the finite field of size 3. Note that here and after, all algebra operations are performed in GF(3). The standard design corresponds to $b_1 = \cdots = b_k = 0$ and is an $(n-k)$-dim linear space over GF(3). Now any linear permutation of factor levels only alters the coefficient vector $(b_1, \dots, b_k)^T$. Obviously, designs corresponding to the same vector $(b_1, \dots, b_k)^T$ are actually the same. Thus among all the 3^n linearly permuted designs, there are at most 3^k intrinsic differences. Moreover, each design corresponding to a specific $(b_1, \dots, b_k)^T$ can be obtained by only conducting linear permutations to the k dependent columns (mapping $\mathbf{y}_j$ to $\mathbf{y}_j + b_j$ for $j = 1, \dots, k$), while keeping the $n-k$ independent columns unchanged. So we have the following lemma.

LEMMA 3.1. *For a regular 3^{n-k} FF design, when all possible linear level permutations are considered, the set of all 3^n permuted designs consists of 3^{n-k} copies of the 3^k designs obtained by permuting the k dependent columns.*

For a design corresponding to vector $(b_1, \dots, b_k)^T$, consider the "mirror image" permutation for all factors, that is, substituting $\mathbf{x}_i$ by $(2 - \mathbf{x}_i)$ for $i = 1, \dots, n-k$ and $\mathbf{y}_j$ by $(2 - \mathbf{y}_j)$ for $j = 1, \dots, k$. The resulting "mirror image" design actually corresponds to the coefficient vector $(2 - 2\sum_{i=1}^{n-k} c_{1i} - b_1, \dots, 2 - 2\sum_{i=1}^{n-k} c_{ki} - b_k)^T$. Because the "mirror image" permutation does not change the geometrical structures, these two designs are geometrically isomorphic and have the same CD value. When two coefficient vectors are the same, these two designs are identical. Thus we have the following lemma.

LEMMA 3.2. *For a regular 3^{n-k} FF design, there are at most $(3^k + 1)/2$ geometrically nonisomorphic designs when all possible level permutations are considered.*

Applying the above results, we conduct level permutations of three-level minimum aberration designs with 27 runs and 81 runs given by Xu (2005) to find designs with minimum discrepancy. The results are concisely presented as follows. For 27-run designs, when $n = 4$ to 6, the first n columns of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + 2$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 1$ and $\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_3 + 1$ form a regular uniform minimum aberration design; when $n = 7$ to 13, the first n columns of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + 1$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 1$, $\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_3$, $\mathbf{x}_1 + \mathbf{x}_3 + 2$, $\mathbf{x}_2 + 2\mathbf{x}_3 + 1$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 2\mathbf{x}_3 + 2$, $\mathbf{x}_1 + \mathbf{x}_2 + 2$, $\mathbf{x}_2 + \mathbf{x}_3 + 2$, $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3$ and $\mathbf{x}_1 + 2\mathbf{x}_3 + 1$ form a regular uniform minimum aberration design, where $\mathbf{x}_1, \mathbf{x}_2$, $\mathbf{x}_3$ are independent columns. Their CD values are listed as "Min ϕ" in Table 3.

For 81-run designs, according to Xu (2005), when $n = 5$ to 11, the first n columns of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3$, $\mathbf{x}_1 + 2\mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 2\mathbf{x}_4$, $\mathbf{x}_2 + \mathbf{x}_3 + 2\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_3 + 2\mathbf{x}_4$ and $\mathbf{x}_1 + \mathbf{x}_2$ form the minimum aberration design; when $n = 12$ to 20, the first n columns of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3$, $\mathbf{x}_1 + 2\mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 2\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_2 + 2\mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + 2\mathbf{x}_3$, $\mathbf{x}_1 + 2\mathbf{x}_3 + 2\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_3$, $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_4$, $\mathbf{x}_2 + \mathbf{x}_3$, $\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + 2\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_3$, $\mathbf{x}_2 + 2\mathbf{x}_3 + 2\mathbf{x}_4$, $\mathbf{x}_1 + \mathbf{x}_4$ and $\mathbf{x}_2 + \mathbf{x}_4$ form the minimum aberration design, where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $\mathbf{x}_4$ are independent columns. Table 4 summarizes the results when the minimum aberration designs are permuted. For example, when $n = 7$, the best linear permutation conducted to three dependent columns is $(0, 2, 1)$, which means that the best design with minimum CD value 0.102515 is formed by seven columns $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4$, $\mathbf{x}_1 + 2\mathbf{x}_2 + \mathbf{x}_3 + 2$ and $\mathbf{x}_1 + 2\mathbf{x}_3 + \mathbf{x}_4 + 1$. As three-level designs with 81 runs are not listed on UD homepage, the best designs found in Table 4 are apparently new.

TABLE 4
Results of 81-run minimum aberration designs

n	Ave ϕ	Min ϕ	Best level permutations															
5	0.062691	0.062690	0															
6	0.081294	0.081290	0	1														
7	0.102528	0.102515	0	2	1													
8	0.126795	0.126764	0	2	1	0												
9	0.154565	0.154497	0	2	1	0	1											
10	0.186393	0.186255	0	2	1	0	1	0										
11	0.226648	0.225969	1	1	0	0	0	0	2									
12	0.270884	0.269750	1	1	0	0	2	0	0	2								
13	0.324370	0.322305	1	0	2	0	2	0	2	1	2							
14	0.385994	0.382976	0	0	1	1	2	2	2	0	2	2						
15	0.457704	0.453338	0	0	1	1	2	2	2	0	2	2	2					
16	0.540883	0.534813	0	0	1	1	2	2	2	0	2	2	2	2				
17	0.640085	0.631437	0	0	1	1	2	2	2	0	2	2	2	2	0			
18	0.755854	0.743782	0	0	1	1	2	2	2	0	2	2	2	2	0	1		
19	0.898270	0.883749	0	0	1	1	2	2	2	0	2	2	2	2	0	1	2	
20	1.066298	1.048120	0	0	1	1	2	2	2	0	2	2	2	2	0	1	2	1

As stated in Lemma 3.2, for a regular 3^{n-k} FF design, there are at most $(3^k + 1)/2$ geometrically nonisomorphic designs when all possible level permutations are considered. Now consider the simplest case with $k = 1$. A regular 3^{n-1} minimum aberration design can be specified by $\mathbf{y} = 2\mathbf{x}_1 + \cdots + 2\mathbf{x}_{n-1} + b$, where $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$ are the independent columns, $\mathbf{y}$ is the dependent column and $b \in \mathrm{GF}(3)$. This is equivalent to $\mathbf{x}_1 + \cdots + \mathbf{x}_{n-1} + \mathbf{y} = b \pmod 3$. When levels of $\mathbf{y}$ are permuted, they will generate two geometrically nonisomorphic designs. To be specific, denote D_i as the design corresponds to $b = n + i \pmod 3$ for $i = 0, 1, 2$. Then D_0 contains a row of ones, and D_1 and D_2 are geometrically isomorphic. The next theorem provides explicit formulas for the CD values of D_0 and D_1.

THEOREM 3.3. *Let D_0 and D_1 be the two geometrically nonisomorphic regular 3^{n-1} minimum aberration designs, where D_0 represents the design with the all-one row. Then the centered L_2-discrepancies of D_0 and D_1 are*

$$\phi(D_0) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n + \frac{2(-1)^n}{3^{3n}}$$

and

$$\phi(D_1) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n + \frac{(-1)^{n+1}}{3^{3n}}.$$

As an immediate implication of Theorem 3.3, when n is odd, $\phi(D_1) > \phi(D_0)$, so D_0 is the uniform minimum aberration design; when n is even, $\phi(D_0) > \phi(D_1)$, so D_1 is the uniform minimum aberration design.

4. Connection to the β-word-length pattern. Under the hierarchical principle, Cheng and Ye (2004) defined the β-word-length pattern. Specifically, for an (N, s^n)-design, let θ_j be the vector of all j-degree interactions and Z_j be the matrix of orthogonal polynomial contrast coefficients for θ_j. Then the response Y can be fitted by a polynomial model $Y = Z_0\theta_0 + Z_1\theta_1 + \cdots + Z_K\theta_K + \varepsilon$. Denote $Z_j = (z_{ik}^{(j)})_{N \times n'_j}$, where n'_j is the number of effects with degree j. The β-word-length pattern is defined by

$$\beta_j(D) = N^{-2} \sum_{k=1}^{n'_j} \left| \sum_{i=1}^{N} z_{ik}^{(j)} \right|^2 \qquad \text{for } j = 0, \ldots, K,$$

where $K = n(s-1)$ represents the highest polynomial degree. Cheng and Ye (2004) argued that a good design should minimize $\beta_1, \beta_2, \ldots, \beta_K$ in a sequential order.

It is interesting to see how uniform minimum aberration designs perform under the β-word-length pattern. In principle, given a design, one can always find the best design related to the β-word-length pattern by permuting levels for all factors. However, the computational burden makes it infeasible to evaluate all $\beta_j(D)$

values even for three-level designs with moderate number of factors. Here we only consider permuting levels of regular minimum aberration designs with 27 runs and $n = 4$ to 10 columns and compute their β-word-length patterns. To our surprise, for all cases, the permuted designs with best β-word-length patterns always have the least centered L_2-discrepancies and vice verse; that is, the uniform minimum aberration designs are the best designs under the β-word-length pattern. Of course, there are cases where different designs have the same CD value but different β-word-length pattern, and vice verse. Moreover, for $n = 4$ to 8, the β-word-length pattern and centered L_2-discrepancy give the exactly same ordering of the permuted designs. For $n = 9$ or 10, the orderings under the two criteria are quite consistent, though not identical.

We end this section with a theoretical result. Notice that a regular 3^{n-1} minimum aberration design has resolution n so that $A_1 = \cdots = A_{n-1} = 0$, which implies $\beta_1 = \cdots = \beta_{n-1} = 0$. The following theorem gives an interesting relationship between the CD value and β_n.

THEOREM 4.1. *For a regular 3^{n-1} minimum aberration design D,*

$$\phi(D) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n - 2\left(\frac{1}{27}\right)^n + \left(\frac{2}{27}\right)^n \beta_n(D).$$

Theorem 4.1 shows that the two criteria, centered L_2-discrepancy and β-word-length pattern, are exactly equivalent for regular 3^{n-1} minimum aberration designs.

5. Conclusion. Uniform FF designs are useful for studying quantitative factors with multiple levels; however, the construction of such designs is challenging. We establish a connection between uniformity and aberration by showing that the average centered L_2-discrepancy is a function of the word-length pattern. We propose to construct uniform FF designs by permuting levels of existing minimum aberration designs. Using this strategy, we construct regular uniform minimum aberration designs with 27 runs and 81 runs for practical use. We further evaluate the performance of the uniform minimum aberration designs for polynomial models. They perform well under the β-word-length pattern.

APPENDIX: PROOFS OF ALL THEOREMS

For an (N, s^n)-design $D = (x_{ik})$, let $d_H(i, j)$ be the Hamming distance of rows i and j of D, that is, $d_H(i, j) = \sharp\{k : x_{ik} \neq x_{jk}, k = 1, \ldots, n\}$, where $\sharp(S)$ is the cardinality of S. The distance distribution of D is $(B_0(D), B_1(D), \ldots, B_n(D))$, where

$$B_j(D) = N^{-1}\sharp\{(a, b) : d_H(a, b) = j \text{ and } a, b = 1, \ldots, N\} \qquad \text{for } j = 0, \ldots, n.$$

Xu and Wu (2001) showed that the (generalized) word-length pattern can be calculated by the MacWilliams transform of the distance distribution, that is,

$$A_j(D) = N^{-1} \sum_{i=0}^{n} B_i(D) P_j(i; n, s) \qquad \text{for } j = 0, \dots, n,$$

where $P_j(x; n, s) = \sum_{i=0}^{j} (-1)^i (s-1)^{j-i} \binom{x}{i} \binom{n-x}{j-i}$ are the Krawtchouk polynomials.

By the orthogonality of the Krawtchouk polynomials, we also have

$$B_j(D) = N \cdot s^{-n} \sum_{i=0}^{n} P_j(i; n, s) A_i(D).$$

The following existing property related to Krawtchouk polynomials was stated in MacWilliams and Sloane (1977).

LEMMA A.1. *For nonnegative integers n, x and s with $n \geq x$, $s \geq 2$ and $0 < y < 1$,*

$$\sum_{j=0}^{n} P_j(x; n, s) y^j = [1 + (s-1)y]^{n-x} (1-y)^x.$$

To prove Theorem 2.2, we need the following lemma.

LEMMA A.2. *For an (N, s^n)-design D, denote δ_{ij} as the number of positions where rows i and j take the same value, that is, $\delta_{ij} = n - d_H(i, j)$. Then for any positive number z greater than* 1,

$$\sum_{i,j=1}^{N} z^{\delta_{ij}} = N^2 \left(\frac{z+s-1}{s} \right)^n \sum_{i=0}^{n} \left(\frac{z-1}{z+s-1} \right)^i A_i(D).$$

PROOF. According to the definition of distance distribution, Lemma A.1 and the relationship between distance distribution and word-length pattern, we have

$$\begin{aligned}
\sum_{i,j=1}^{N} z^{\delta_{ij}} &= N \sum_{j=0}^{n} B_j(D) z^{n-j} = z^n s^{-n} N^2 \sum_{j=0}^{n} \sum_{i=0}^{n} P_j(i; n, s) A_i(D) z^{-j} \\
&= z^n s^{-n} N^2 \sum_{i=0}^{n} \left(1 + \frac{s-1}{z} \right)^{n-i} \left(1 - \frac{1}{z} \right)^i A_i(D) \\
&= N^2 \left(\frac{z+s-1}{s} \right)^n \sum_{i=0}^{n} \left(\frac{z-1}{z+s-1} \right)^i A_i(D).
\end{aligned}$$

□

PROOF OF THEOREM 2.2. Notice that for an $(N, 3^n)$-design $D = (x_{ik})$ with $x_{ik} = 0, 1$, or 2, u_{ik} and u_{jk} in formula (2.2) can only take values 1/6, 1/2 and 5/6. If $u_{ik} = 1/2$, $1 + \frac{1}{2}|u_{ik} - \frac{1}{2}| - \frac{1}{2}|u_{ik} - \frac{1}{2}|^2$ takes value 1; otherwise, it takes value 10/9. Furthermore, if $u_{ik} = u_{jk} = 1/6$ or $u_{ik} = u_{jk} = 5/6$, $1 + \frac{1}{2}|u_{ik} - \frac{1}{2}| + \frac{1}{2}|u_{jk} - \frac{1}{2}| - \frac{1}{2}|u_{ik} - u_{jk}|$ takes value 4/3; otherwise, it takes value 1. Thus for any two rows of an $(N, 3^n)$-design D, denoted by $(x_{i1}, x_{i2}, \ldots, x_{in})$ and $(x_{j1}, x_{j2}, \ldots, x_{jn})$, if we define $\gamma_i(D) = \sharp\{k : x_{ik} \neq 1, k = 1, \ldots, n\}$ and $\gamma_{ij}(D) = \sharp\{k : x_{ik} = x_{jk} \neq 1, k = 1, \ldots, n\}$, the CD value of D can be determined by the distributions of its γ_i and γ_{ij} values. That is, formula (2.2) can be simplified to

$$\begin{aligned}\phi(D) = {} & \left(\frac{13}{12}\right)^n - \frac{2}{N}\sum_{i=1}^{N}\left(\frac{10}{9}\right)^{\gamma_i(D)} \\ & + \frac{1}{N^2}\sum_{i=1}^{N}\left(\frac{4}{3}\right)^{\gamma_i(D)} + \frac{1}{N^2}\sum_{i\neq j}\left(\frac{4}{3}\right)^{\gamma_{ij}(D)}.\end{aligned} \tag{A.1}$$

Moreover, for any fixed row i of D, when all level permutations of D are considered, each n-tuple in $\mathrm{GF}(3)^n$ occurs 2^n times. For any element x_{ik} in the kth column, there are three possible choices, that is, 0, 1, 2, corresponding to $\frac{10}{9}$, 1, $\frac{10}{9}$ for $1 + \frac{1}{2}|u_{ik} - \frac{1}{2}| - \frac{1}{2}|u_{ik} - \frac{1}{2}|^2$. So

$$\sum_{D'\in\mathcal{P}(D)}\left(\frac{10}{9}\right)^{\gamma_i(D')} = 2^n \cdot \left(\frac{10}{9} + \frac{10}{9} + 1\right)^n = 2^n \cdot \left(\frac{29}{9}\right)^n.$$

Similarly,

$$\sum_{D'\in\mathcal{P}(D)}\left(\frac{4}{3}\right)^{\gamma_i(D')} = 2^n \cdot \left(\frac{4}{3} + \frac{4}{3} + 1\right)^n = 2^n \cdot \left(\frac{11}{3}\right)^n = 6^n \cdot \left(\frac{11}{9}\right)^n.$$

For any two rows i and j of D with Hamming distance $d_H(i, j) = n - \delta_{ij}$, when level permutations of corresponding columns are considered, each identical pair, that is, $(l, l), l \in \mathrm{GF}(3)$, occurs twice in δ_{ij} positions where rows i and j coincide, and each distinct pair occurs once in corresponding $d_H(i, j)$ positions where rows i and j differ. So

$$\sum_{D'\in\mathcal{P}(D)}\left(\frac{4}{3}\right)^{\gamma_{ij}(D')} = 2^{\delta_{ij}} \cdot \left(\frac{4}{3} + \frac{4}{3} + 1\right)^{\delta_{ij}} \cdot (6 \times 1)^{n-\delta_{ij}} = 6^n\left(\frac{11}{9}\right)^{\delta_{ij}}.$$

Then

$$\begin{aligned}\sum_{D'\in\mathcal{P}(D)}\phi(D') &= 6^n\left(\frac{13}{12}\right)^n - \frac{2}{N} \cdot N \cdot 2^n\left(\frac{29}{9}\right)^n + \frac{1}{N^2}6^n\sum_{i,j=1}^{N}\left(\frac{11}{9}\right)^{\delta_{ij}} \\ &= 6^n\left(\frac{13}{12}\right)^n - 2 \cdot 6^n\left(\frac{29}{27}\right)^n + \frac{6^n}{N^2}\sum_{i,j=1}^{N}\left(\frac{11}{9}\right)^{\delta_{ij}}\end{aligned}$$

and

$$\bar{\phi}(D) = \left(\frac{13}{12}\right)^n - 2\left(\frac{29}{27}\right)^n + \frac{1}{N^2}\sum_{i,j=1}^{N}\left(\frac{11}{9}\right)^{\delta_{ij}}.$$

Finally, the result follows from Lemma A.2 and the fact $A_0 = 1$. $\square$

PROOF OF THEOREM 3.3. We will use formula (A.1) to calculate the CD values. First, we prove that the distributions of γ_{ij} values for the two designs D_0 and D_1 are the same. Because D_1 is obtained by adding 1 (mod 3) to the last column of D_0, $\gamma_{ij}(D_0) \neq \gamma_{ij}(D_1)$ if and only if both last positions of rows i and j of D_0 have the same value 0 or 1. For any two distinct rows of D_0 with the last positions both taking value 0, denoted by $(x_{i1}, x_{i2}, \dots, x_{i,n-1}, 0)$ and $(x_{j1}, x_{j2}, \dots, x_{j,n-1}, 0)$, respectively, there exists a unique pair of rows of D_1, $(x_{i1}, \dots, x_{i,l-1}, x_{il} - 1, x_{i,l+1}, \dots, x_{i,n-1}, 2)$ and $(x_{j1}, \dots, x_{j,l-1}, x_{jl} - 1, x_{j,l+1}, \dots, x_{j,n-1}, 2)$, where l is the first position such that $x_{il} \neq x_{jl}$. These two pairs have the same γ_{ij} value. Similarly, for any two distinct rows of D_0 with the last positions both taking value 1, $(x_{i1}, x_{i2}, \dots, x_{i,n-1}, 1)$ and $(x_{j1}, x_{j2}, \dots, x_{j,n-1}, 1)$, there exists a unique pair of rows of D_1, $(x_{i1}, \dots, x_{i,l-1}, x_{il} + 1, x_{i,l+1}, \dots, x_{i,n-1}, 1)$ and $(x_{j1}, \dots, x_{j,l-1}, x_{jl} + 1, x_{j,l+1}, \dots, x_{j,n-1}, 1)$, with the same γ_{ij} value, where l is the first position such that $x_{il} \neq x_{jl}$. It is easy to see that the above correspondence between two pairs of rows in D_0 and D_1 is actually one-to-one (bijective). This completes our claim on the distributions of γ_{ij} values.

Now we consider the distributions of γ_i values of D_0 and D_1. For a design D, denote $\eta_j(D) = \sharp\{i : \gamma_i(D) = j, i = 1, \dots, N\}$, $j = 0, 1, \dots, n$, as the distribution of γ_i values of D. Obviously, $\eta_0(D_0) = 1$ as D_0 contains the all-one row, and $\eta_0(D_1) = 0$. Moreover, for any $j = 0, 1, \dots, n$, the total number of rows in $\mathrm{GF}(3)^n$ with $\gamma_i = j$ is $\binom{n}{j}2^j$; therefore, $\eta_j(D_0) + 2\eta_j(D_1) = \binom{n}{j}2^j$. For convenience, if a vector with length j only contains 0 or 2, it will be called a $(0, 2)^j$-vector. Then $\eta_j(D_0)/\binom{n}{j}$ is the number of possible $(0, 2)^j$-vectors with sum j (mod 3) and $\eta_j(D_1)/\binom{n}{j}$ is the number of possible $(0, 2)^j$-vectors with sum $j + 1$ (mod 3). Notice that a $(0, 2)^j$-vector with sum $j - 1$ (mod 3) can be obtained by conducting a "mirror image" operation to a $(0, 2)^j$-vector with sum $j + 1$ (mod 3). Thus $\eta_j(D_1)/\binom{n}{j}$ also represents the number of possible $(0, 2)^j$-vectors with sum $j - 1$ (mod 3). Each $(0, 2)^j$-vector with sum j (mod 3) can be formed by adding 2 or 0 to a $(0, 2)^{j-1}$-vector with sum $(j - 1) - 1$ or $(j - 1) + 1$ (mod 3). So we have $\eta_j(D_0)/\binom{n}{j} = 2\eta_{j-1}(D_1)/\binom{n}{j-1}$. Combining this with $\eta_j(D_0) + 2\eta_j(D_1) = \binom{n}{j}2^j$ and $\eta_0(D_1) = 0$, we obtain $\eta_j(D_1) = \binom{n}{j}\frac{2^j - (-1)^j}{3}$ and $\eta_j(D_0) = \binom{n}{j}\frac{2^j + 2(-1)^j}{3}$. Thus $\eta_j(D_0) - \eta_j(D_1) = \binom{n}{j}(-1)^j$. Using formula (A.1) and $N = 3^{n-1}$, we have

$$\begin{aligned}\phi(D_0)-\phi(D_1) &= \frac{1}{N^2}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(\frac{4}{3}\right)^j - \frac{2}{N}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(\frac{10}{9}\right)^j \\ &= \frac{1}{N^2}\left[\left(-\frac{1}{3}\right)^n - 2N\left(-\frac{1}{9}\right)^n\right] \\ &= \frac{3^n-2N}{N^2}\left(-\frac{1}{9}\right)^n = \frac{(-1)^n}{3^{3n-1}}.\end{aligned}$$

From Theorem 2.2, we also have

$$\phi(D_0)+2\phi(D_1) = 3\left[\left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n\right].$$

Then we have

$$\phi(D_0) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n + \frac{2(-1)^n}{3^{3n}}$$

and

$$\phi(D_1) = \left(\frac{13}{12}\right)^n - \left(\frac{29}{27}\right)^n + 2\left(\frac{2}{27}\right)^n + \frac{(-1)^{n+1}}{3^{3n}}. \qquad \square$$

PROOF OF THEOREM 4.1. For a regular 3^{n-1} minimum aberration design $D=(x_{ik})$ with resolution n, its $\beta_n(D)$ is determined by the product of linear polynomials as follows:

$$\beta_n(D) = N^{-2}\left|\sum_{i=1}^{N} p_1(x_{i1})\times\cdots\times p_1(x_{in})\right|^2, \tag{A.2}$$

where $N=3^{n-1}$ and $p_1(x)=\sqrt{3/2}(x-1)$. Because $p_1(x)=0$ when $x=1$, we only need to consider rows with 0 or 2 only, that is, $(0,2)^n$-vectors. Notice that D_0 is an $(n-1)$-dim linear space or a coset over GF(3) containing the all-one vector, thus run $(2-z_1,\dots,2-z_n)$ occurs in D_0 if and only if $(z_1,\dots,z_n)$ occurs in D_0. So for any odd n, $\beta_n(D_0)=0$ according to (A.2). To calculate $\beta_n(D_1)$ for odd n, we will establish a recursive formula. For this purpose, we use D_i^n to denote a design D_i with n columns; that is, the sum of each row of the design is congruent to $n+i$ modulo 3 for $i=0,1,2$. Then, up to row permutations, we can express D_1^n as follows:

$$D_1^n = \begin{bmatrix} D_2^{n-1} & \mathbf{0} \\ D_1^{n-1} & \mathbf{1} \\ D_0^{n-1} & \mathbf{2}\end{bmatrix}. \tag{A.3}$$

Let $\delta(D_i^{n-1})$ be the difference between the number of $(0,2)^{n-1}$-vectors in D_i^{n-1} with even number of zeros and the number of those with odd number of zeros for $i = 0, 1, 2$. Then, according to (A.2), we have

$$\beta_{n-1}(D_i^{n-1}) = 3^{-2(n-2)}(3/2)^{n-1}|\delta(D_i^{n-1})|^2 \qquad \text{for } i = 0, 1, 2. \tag{A.4}$$

Furthermore, when $n-1$ is even, $\delta(D_0^{n-1}) + \delta(D_1^{n-1}) + \delta(D_2^{n-1}) = 0$ and $\delta(D_0^{n-1}) = -2\delta(D_1^{n-1}) = -2\delta(D_2^{n-1})$. Then, according to (A.2) and (A.3), for odd n, we have

$$\begin{aligned} \beta_n(D_1^n) &= 3^{-2(n-1)}(3/2)^n|-\delta(D_2^{n-1}) + \delta(D_0^{n-1})|^2 \\ &= 3^{-2(n-1)}(3/2)^n|-3\delta(D_1^{n-1})|^2. \end{aligned} \tag{A.5}$$

Combining (A.4) and (A.5), we obtain $\beta_n(D_1^n) = (3/2)\beta_{n-1}(D_1^{n-1})$. In the same vein, for even n, we can establish $\beta_n(D_1^n) = (1/6)\beta_{n-1}(D_1^{n-1})$ and $\beta_n(D_0^n) = (2/3)\beta_{n-1}(D_1^{n-1})$. So for odd n, $\beta_n(D_1^n) = (3/2)\beta_{n-1}(D_1^{n-1}) = (1/4) \times \beta_{n-2}(D_1^{n-2})$. It is easy to verify that $\beta_3(D_1^3) = 3/8$; thus for odd n, $\beta_n(D_1^n) = (3/8)(1/4)^{(n-3)/2} = 3/2^n$. For even n, we obtain $\beta_n(D_1^n) = (1/6)(3/2^{n-1}) = 1/2^n$ and $\beta_n(D_0^n) = (2/3)(3/2^{n-1}) = 1/2^{n-2}$. Then the result follows from Theorem 3.3. □

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