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# On *D*-optimal robust designs for lifetime improvement experiments

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ABSTRACT

Locating the optimal operating conditions of the process parameters is critical in a lifetime improvement experiment. For log-normal lifetime distribution with compound error structure (i.e., symmetry, inter-class and intra-class correlation error structures), we have developed methods for construction of *D*-optimal robust first order designs. It is shown that *D*-optimal robust first order designs are always robust first order rotatable but the converse is not always true.



## 1. Introduction

It is known to be difficult to derive optimal designs for linear models with correlated observations. For some cases, the *D*-optimal design may not exist. Bischoff (1996) pointed that *D*-optimal design does not exist for linear model with some correlated error structures, and for the tri-diagonal structure he derived *maximin* designs. Kiefer and Wynn (1981) suggested seeking optimal designs only in the class of uncorrelated and homoscedastic errors. For special factorial linear models with correlated observations, optimal designs are known (see, for example, Kiefer and Wynn, 1981, 1984). Gennings et al. (1989) studied optimality under a non-linear model with correlated error. Little is known on optimal designs of linear regression models for correlated observations (Bischoff, 1992, 1995, 1996). Recently, Das and Park (2008a) derived *efficient robust rotatable* designs for autocorrelated error structure instead of *D*-optimal designs, and they (2008b) developed *D*-optimal robust first order designs under tri-diagonal correlation structure with lag *n*. Improvement of performance of a system plays an important role in reliability theory. Statistical design of experiments has been popularly used in reliability theory in order to improve the quality of a system (see, for example, Taguchi, 1986, 1987; Condra, 1993).

In response surface methodology (RSM), when we are remote from the optimum operating conditions for the system, we usually assume that a first order model is an adequate approximation to the true surface in a small region of the explanatory variables *x*'s. If there is a curvature in the system, then a polynomial of higher degree, such as the second order model must be used. The method of steepest ascent helps us for moving sequentially along the path of steepest ascent, that is, the direction of maximum increase in response through sequentially selecting appropriate response function.

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Practitioner is experimenting with a system in which the goal is not to find a *point of optimum response*, but to search for a new *region* in which the process or product is improved (see, for example, Draper and Lin, 1996).

In reliability theory, the random variable under consideration is the life of a system. A question of fundamental importance is how to improve the mean life or reliability of the system for a given mission time. Mean life is conveniently written as a function of the exploratory variables. For exponential lifetime distribution a similar study was done by Mukhopadhyay et al. (2002). Aitkin and Clayton (1980) fitted complex censored survival data to exponential, Weibull and extreme value distribution. Design of experiments and concepts of optimality have been abundantly used in reliability theory, in order to improve the mean life.

The conventional response surface designs are *not* appropriate in lifetime distributions, because they do not satisfy the assumptions of lifetime distributions (i.e., non-linearity, dependence of errors, non-normality, etc.). In this paper we have derived the necessary and sufficient conditions of first order *D*-optimality for log-normal lifetime distribution with a general correlated error structure; specifically, compound symmetry, inter-class and intra-class correlation structures. Construction methods of *D*-optimal robust first order designs under each of the correlation structure are proposed.

The rest of the paper is organized as follows. In Section 2, correlated log-normal lifetime distribution model is developed. In Section 3, *D*-optimal robust first order designs for log-normal lifetime distribution is discussed. *D*-optimal robust first order designs are constructed for some well-known correlation structures in Section 4. The concluding remark is given in Section 5.

## 2. Log-normal lifetime distribution with correlated experimental errors

Let $T$ be the lifetime of a component or a system, measured in some unit, which follows log-normal distribution. Let $x_1, x_2, \ldots, x_k$ be $k$ controllable explanatory variables in the system which are highly related with lifetime $T$. The probability density function (p.d.f.) of lifetime $T$ given the vector $\mathbf{x}=(x_1,x_2,\ldots,x_k)'$, is of the form

$$f(t)=\frac{1}{t\delta(2\pi)^{1/2}}\exp\left[-\frac{(\ln t-\ln h(\mathbf{x}))^2}{2\delta^2}\right],\quad t\geq 0,\ \delta\geq 0. \tag{2.1}$$

$\ln T$ is assumed to follow Normal distribution with mean $\ln h(\mathbf{x})$ and variance $\delta^2$ for a given $\mathbf{x}$. In fact that $\delta$ is independent of $\mathbf{x}$ implies proportional hazards for lifetimes and constant variance for log lifetimes of individuals. For more details of such a popular model and its applications, see, Pike (1966); Peto and Lee (1973); Nelson (1972); and Meeker and Escobar (1998).

The probability density function of the log lifetime $Y^*(=\ln T)$, given $\mathbf{x}$, is a Normal distribution with mean $\ln h(\mathbf{x})$ and variance $\delta^2$. Take $\ln h(\mathbf{x})=g(\mathbf{x},\boldsymbol{\beta})$, thus, $E(T)=h(\mathbf{x})\exp(\delta^2/2)=e^{g(\mathbf{x},\boldsymbol{\beta})}\exp(\delta^2/2)>0$, where $\boldsymbol{\beta}$ is a vector of unknown coefficients. Once the data are obtained, the point is to estimate the parameters $\boldsymbol{\beta}$ in an appropriate manner. It is assumed that given $\mathbf{x}$, $T$ follows a log-normal distribution in an *ideal situation* with mean $e^{g(\mathbf{x},\boldsymbol{\beta})}\exp(\delta^2/2)$. This is the same as assuming $\ln T=g(\mathbf{x},\boldsymbol{\beta})+\delta\tau$, where $\tau$ follows the Standard Normal distribution.

For estimating the parameters $\boldsymbol{\beta}$ we have to conduct experiment to collect the data on the basis of which the estimation to be done. Once an experiment is conducted, the experiment introduces some noise factors which may be numerous, some may be unidentifiable. The total impact on ($\ln T$) of all these noise factors represented by the experimental condition is denoted by '$e$'. Consider the response surface $g(\mathbf{x},\boldsymbol{\beta})$ is of the first order. We write $y=\ln T+e=\beta_0+\beta_1x_1+\cdots+\beta_kx_k+\delta\tau+e$, or

$$y_u=\beta_0+\beta_1x_{u1}+\beta_2x_{u2}+\cdots+\beta_kx_{uk}+\delta\tau_u+e_u,\quad 1\leq u\leq N, \tag{2.2}$$

where $e_u$'s represent the experimental errors. The most convenient distributional choice of $\mathbf{e}$, the vector of errors $e_u$'s is a multivariate Normal distribution with $E(\mathbf{e})=\mathbf{0}$, $D(\mathbf{e})=\sigma_1^2W_1$, and rank $(W_1)=N$, where $W_1$ is any unknown general variance covariance structure of errors. The prescription of the proper design matrix is a problem of regression design of experiments (see Box and Hunter, 1957; Pukelsheim, 1993; Khuri and Cornell, 1996; Box and Draper, 2007; Panda and Das, 1994; Das, 1997, 2003, 2004; Das and Park, 2008a, 2008b). Little is known in the literature for our specific situation of Log Normal lifetime distribution. In Section 4 we have developed some *D*-optimal robust first order designs for some special correlation structures of errors $W_1$ for the situation as mentioned herein.

## 3. *D*-optimal robust first order designs for log-normal lifetime distribution

The model as explained in (2.2) can be written as

$$\mathbf{Y}=X\boldsymbol{\beta}+\boldsymbol{\varepsilon}, \tag{3.1}$$

where $\varepsilon_u=\delta\tau_u+e_u$, $\mathbf{Y}=(y_1,y_2,\ldots,y_N)'$ is the vector of recorded observations, $X=(1:(x_{ui});1\leq u\leq N,1\leq i\leq k)$ is the model matrix; $\boldsymbol{\beta}=(\beta_0,\beta_1,\ldots,\beta_k)'$ is the vector of regression coefficients and $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_N)'$. Then $E(\varepsilon_u)=0$, $\mathrm{Var}(\varepsilon_u)=(\delta^2+\sigma_1^2)=\sigma^2$ say, $\mathrm{Cov}(\varepsilon_u,\varepsilon_{u'})=\mathrm{Cov}(e_u,e_{u'})$; $E(\boldsymbol{\varepsilon})=\mathbf{0}$, $D(\boldsymbol{\varepsilon})=\delta^2I_N+\sigma_1^2W_1=\sigma^2W$ say, and $\boldsymbol{\varepsilon}\sim \mathrm{N}(\mathbf{0},\sigma^2W)$, where $W$ is an $N\times N$ matrix whose $(i,j)$th element $(i\neq j)$ is $\mathrm{Cov}(e_i,e_j)/\sigma^2$ and all diagonal elements are unity. Note that if $\mathrm{Cov}(e_i,e_j)=0$ for all $i\neq j$, then $D(\boldsymbol{\varepsilon})=\sigma^2 I_N$. The best linear unbiased estimator (BLUE) of $\boldsymbol{\beta}$, for known $W$ is $\hat{\boldsymbol{\beta}}=(X'W^{-1}X)^{-1}(X'W^{-1}\mathbf{Y})$, with

$$D(\hat{\boldsymbol{\beta}})=\sigma^{-2}(X'W^{-1}X)^{-1}=\sigma^{-2}(\{v_{ij}\})^{-1} \quad \text{say } 0\le i,j\le k, \tag{3.2}$$

where $v_{00}=\mathbf{1}'W^{-1}\mathbf{1}, v_{0j}=\mathbf{1}'W^{-1}\mathbf{x}_j, v_{ij}=\mathbf{x}_i'W^{-1}\mathbf{x}_j$ and $\mathbf{x}_i=(x_{1i},x_{2i},\ldots,x_{Ni})'$.

The objective of the $D$-optimality criteria is to *minimize* $|(X'W^{-1}X)^{-1}|$, or equivalently *maximize* $|X'W^{-1}X|$ where $(X'W^{-1}X)$ is known as the moment matrix.

**Definition 3.1** (*Rotatable design*). A design is said to be *rotatable* if the variance of the estimated response at a point is a function of only the distance from the design center (i.e., center of the coordinate axes, or at $(0,0,\ldots,0)$) to that point.

**Definition 3.2** (*Robust first order rotatable design*). A design $D$ on $k$ factors under the correlated model (3.1) which remains first order rotatable for *all* the variance–covariance matrices belonging to a well-defined class $W_0=\{W$ positive definite: $W_{N\times N}$ defined by a particular correlation structure possessing a definite pattern$\}$ is called a *robust first order rotatable design* (RFORD).

**Definition 3.3** (*D-optimal robust first order design*). A first order regression design $\xi$ of $k$ factors in the correlated model (3.1) is said to be *D-Optimal Robust First Order Design* ($D$-ORFOD) if the determinant $|X'W^{-1}X|$ is uniformly maximum (over the design space $\mathcal{X}=\{|x_{ui}|\le 1; 1\le u\le N,\ 1\le i\le k\}$) for *all* the variance–covariance matrices belonging to a well-defined class $W_0$.

**Theorem 3.1.** *The necessary and sufficient conditions for a D-optimal robust first order design in the model* (3.1) *are* (*for all* $1\le i,j\le k$)

(i) $v_{0j}=\mathbf{1}'W^{-1}\mathbf{x}_j=0$,
(ii) $v_{ij}=\mathbf{x}_i'W^{-1}\mathbf{x}_j=0,\ i\ne j$,
(iii) $v_{ii}=\mathbf{x}_i'W^{-1}\mathbf{x}_i=\mu$,

*where* $\mu$ *is the maximum possible value* (*a positive constant*) *over the design space* $\mathcal{X}$ *as in Definition*3.3.

**Proof.** For a first order design $\xi$, the moment matrix is $M(\xi)=X'W^{-1}X=(\{v_{ij}\})$ as given in (3.2). Let $M_{i,i}$ be the $i\times i$ leading principal submatrix of $M(\xi)$, $1\le i\le k+1$. Since $M(\xi)$ is a positive definite matrix, with $v_{ij}=v_{ji}$, for $0\le i,j\le k$; $v_{ii}>0$, for $0\le i\le k$; and $M_{i,i}$ is positive definite, for $1\le i\le k+1$; we have

$$M(\xi)=M=M_{k+1,k+1}=\begin{pmatrix} M_{k,k} & \mathbf{v} \\ \mathbf{v}' & v_{kk}\end{pmatrix}=\begin{pmatrix} M_{k,k} & \mathbf{0} \\ \mathbf{v}' & 1\end{pmatrix}\times\begin{pmatrix} I_k & M_{k,k}^{-1}\mathbf{v} \\ \mathbf{0}' & v_{kk}-\mathbf{v}'M_{k,k}^{-1}\mathbf{v}\end{pmatrix},$$

where $\mathbf{v}=(v_{0k},\ldots,v_{(k-1)k})'$. Therefore

$$\det(M)=\det(M_{k,k})\det(v_{kk}-\mathbf{v}'M_{k,k}^{-1}\mathbf{v}).$$

Since both $\det(M_{k,k})$ and $v_{kk}$ are positive, both $M_{k,k}$ and $M_{k,k}^{-1}$ are positive definite and $\mathbf{v}'M_{k,k}^{-1}\mathbf{v}\ge 0$, this implies

$$\det(M)\le\det(M_{k,k})v_{kk}$$

with the equality holds *if and only if* $\mathbf{v}=\mathbf{0}$ (equivalently, $\mathbf{v}'M_{k,k}^{-1}\mathbf{v}=\mathbf{0}$), that is $v_{ik}=0$, for all $0\le i\le k-1$.

Similarly, expanding $M_{k,k}$, we can show that $\det(M)\le\det(M_{k-1,k-1})v_{(k-1)(k-1)}v_{kk}$, with equality holds if and only if $v_{i(k-1)}=0, 0\le i\le k-2$. By applying induction on $M_{k-1,k-1},M_{k-2,k-2},\ldots,M_{2,2}$, we have

$$\det(M)\le\prod_{i=0}^{k}v_{ii},$$

with equality holds if and only if $v_{ij}=0, 0\le i<j, 1\le j\le k$. This inequality is also known as the Hadamard's Inequality (Anderson, 1984, p. 54).

Since $M$ is symmetric matrix, this implies that $v_{ij}=0, 0\le i\ne j\le k$. Thus, the maximum value of $\det(M)$ is $v_{00}v_{11}\ldots v_{kk}$ if and only if $v_{ij}=0, 0\le i\ne j\le k$. Based on design matrix $X$ over design space $\mathcal{X}$ given in Definition 3.3, the maximum value possible for each $v_{ii}$ is the same (equal to $\mu$, say). Therefore, from the above result $\det(M)$ will attain maximum possible value ($=v_{00}\mu^k$) *if and only if* $v_{ij}=0, 0\le i\ne j\le k$; and $v_{ii}=\mu, 1\le i\le k$. This completes the proof. □

The variance of the estimated response $\hat{y}_{\mathbf{x}}$ at $\mathbf{x}$ of a first order $D$-optimal design with correlated error is given by $V(\hat{y}_{\mathbf{x}})=1/v_{00}+(1/\mu)\sum_{i=1}^{k}x_i^2=f(r^2)$, where $r^2=\sum_{i=1}^{k}x_i^2$, $v_{00}=\mathbf{1}'W^{-1}\mathbf{1}$ and $\mu$ as in Theorem 3.1. If $v_{ii}$ is constant $=\lambda(<\mu)$, $1\le i\le k$, then the design is a robust first order rotatable as the variance of the estimated response $\hat{y}_{\mathbf{x}}$ is a function of only the distance from the design center, but not a $D$-optimal (as $\lambda<\mu$) robust first order design. This can be stated as the following theorem.

**Theorem 3.2.** *A D-optimal robust first order regression design is always a robust first order rotatable design but the converse is not always true.*

## 4. *D*-optimal designs for well-known correlation structures

In this section, we study the necessary and sufficient conditions for *D*-optimality of a first- order regression design, the variance function and the corresponding *D*-optimal robust first order regression designs under compound symmetry, inter-class and intra-class structure of errors commonly encountered in practice.

### 4.1. Compound symmetry correlation structure

Compound symmetry variance covariance matrix will have such a pattern when the observations are divided into '*m*' sets of size '*n*' each such that every observation has the same variance, and within each set of observations, the covariances are equal and the covariance between two observations from two distinct sets is a constant which may be different from the constant intra-class covariance in a set. Its variance covariance matrix of errors is

$$D(\mathbf{e}) = \sigma_1^2[I_m \otimes (A-B) + E_{m\times m} \otimes B] = \sigma_1^2 W_1(\rho^*,\rho_1^*), \tag{4.1}$$

where $A=(1-\rho^*)I_n+\rho^* E_{n\times n}, B=\rho_1^* E_{n\times n}$, $I_n$ indicates an identity matrix of order $n$, $E_{n\times n}$ is an $n\times n$ matrix with all elements 1, $N = mn$ and $\otimes$ denotes Kronecker product.

Under the compound symmetry variance–covariance structure $\sigma_1^2 W_1(\rho^*,\rho_1^*)$, $\sigma^2 W$ will be reduced to $\sigma^2 W(\rho,\rho_1)$, as in (4.1), where $\rho = q\rho^*$, $\rho_1 = q\rho_1^*$, and $q=\sigma_1^2/(\sigma_1^2+\delta^2)$. Also note that, $W^{-1}(\rho,\rho_1) = [I_m \otimes (A_1 - B_1) + E_{m\times m} \otimes B_1]$, where $A_1 = (\delta_2-\gamma_2)I_n + \gamma_2 E_{n\times n}, B_1 = \delta_3 E_{n\times n}, \delta_2 = \gamma_2 + (1/(1-\rho)), \gamma_2 = [(m-1)n\rho_1^2 - (m-2)n\rho\rho_1 - \{1+(n-1)\rho\}\rho]/\ R, \delta_3 = (\{1-\delta_2\ -(n-1)\rho\gamma_2\}/(m-1)n\rho_1)$ and $R=(1-\rho)[(m-2)n\rho_1\{1+(n-1)\rho\} + \{1+(n-1)\rho\}^2 - (m-1)n^2\rho_1^2]$.

Following Theorem 3.1, the necessary and sufficient conditions for *D*-optimal robust first order regression design (*D*-ORFOD) under the variance–covariance structure $\sigma^2 W(\rho,\rho_1)$ can be summarized in Theorem 4.1 below.

**Theorem 4.1.** *A set of necessary and sufficient conditions for a D-optimal robust first order regression design under the variance–covariance structure $\sigma^2 W(\rho,\rho_1)$ as over the design space $\mathcal{X}$ are*

(i) $v_{0j} = 0$, *i.e.*, $\sum_{u=1}^{N} x_{uj} = 0$;
(ii) $v_{ij} = 0$, $i \neq j$;
(iii) *if* $(\gamma_2 - \delta_3) < 0$,
$$\sum_{u=1}^{N} x_{ui}^2 = N; \sum_{u=1}^{n} x_{ui} = \sum_{u=(n+1)}^{2n} x_{ui} = \cdots = \sum_{u=n(m-1)+1}^{mn} x_{ui} = 0;$$
(iv) *if* $(\gamma_2 - \delta_3) > 0$,
$$\sum_{u=1}^{N} x_{ui}^2 = N; \sum_{u=1}^{n} x_{ui} = \sum_{u=(n+1)}^{2n} x_{ui} = \cdots = \sum_{u=n(m-1)+1}^{mn} x_{ui} = \pm n. \tag{4.2}$$

Note that $(\gamma_2-\delta_3) > 0$ holds in the following two cases: Case I: $\rho_1 > \rho > 0, 0 < \rho_1 \leq 0.5; m,n < 6$; and Case II: $0 < \rho_1 < \rho; m,n < 12$. The basic parameters characterizing the correlation structure (4.1) being $\rho$ and $\rho_1$, the interpretation of $(\gamma_2-\delta_3) > 0$ can be obtained from the above two cases only, and $(\gamma_2-\delta_3) < 0$ everywhere else. Given below only the conditions either $(\gamma_2-\delta_3) < 0$ or $(\gamma_2-\delta_3) > 0$ will be considered.

The variance function of a *D*-ORFOD under the structure $\sigma^2 W(\rho,\rho_1)$ as in (4.1) is

$$V(\hat{y}_{\mathbf{x}})_{01} = \frac{\sigma^2}{N}\left[\frac{1}{\{\delta_2+(n-1)\gamma_2+n(m-1)\delta_3\}} + \frac{r^2}{\delta_2-\gamma_2}\right] \quad \text{if } (\gamma_2-\delta_3) < 0, \tag{4.3}$$

$$V(\hat{y}_{\mathbf{x}})_{02} = \frac{\sigma^2}{N}\left[\frac{1}{\{\delta_2+(n-1)\gamma_2+n(m-1)\delta_3\}} + \frac{r^2}{(\delta_2-\gamma_2)+(\gamma_2-\delta_3)n}\right] \quad \text{if } (\gamma_2-\delta_3) > 0, \tag{4.4}$$

where $r^2 = \sum_{i=1}^{k} x_i^2$.

Next, we develop two methods of construction of *D*-optimal robust first order regression designs under the structure $\sigma^2 W(\rho,\rho_1)$: Method I gives *D*-ORFODs for $(\gamma_2-\delta_3) < 0$, and Method II gives *D*-ORFODs for $(\gamma_2-\delta_3) > 0$.

*Method* I: For the values of correlation coefficient $\rho$ and $\rho_1$ satisfying $(\gamma_2-\delta_3) < 0$, a *D*-ORFOD of $k$ factors under the variance–covariance structure $\sigma^2 W(\rho,\rho_1)$ as in (4.1) can be constructed by selecting any $k$ columns from $H_m \otimes U_n$, where $\otimes$ denotes Kronecker product, $H_m$ is a Hadamard matrix of order $m\times m$, and $U_n$ is a column vector of order $n\times 1$ with elements $+1$ or $-1$ such that sum of elements of $U_n$ is zero. Note that the resulting designs satisfy (i)–(iii) of (4.2), and its variance function is as (4.3).

*Method* II: A *D*-ORFOD of $k$ factors under the variance–covariance structure $\sigma^2 W(\rho,\rho_1)$, whatever be the values of correlation coefficient $\rho$ and $\rho_1$ satisfying $(\gamma_2-\delta_3) > 0$, can be constructed by selecting any $k$ columns barring the first column from $H_m \otimes J_n$, where $\otimes$ denotes Kronecker product, $H_m$ is the Standard Hadamard matrix of order $m\times m$ (having all the elements of the first row and the first column as 1's), and $J_n$ is a column vector of order $n \times 1$ of all 1's. It can be readily verified that (i), (ii) and (iv) of (4.2) hold for these designs and its variance function is as (4.4).

**Examples.** *D*-ORFODs with two factors, four groups, each of four observations, (i.e., all examples consist of $k=2$, $m=4$, $n=4$, $N=mn=16$) are constructed below. The designs (*D*-ORFODs, denoted by $d_i$ from Method $i$; $i=1$ and 2) are displayed below (row being factors and column being runs). We construct *D*-ORFODs via

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad H_4 \otimes U_4 = \begin{pmatrix} U_4 & U_4 & U_4 & U_4 \\ U_4 & -U_4 & U_4 & -U_4 \\ U_4 & U_4 & -U_4 & -U_4 \\ U_4 & -U_4 & -U_4 & U_4 \end{pmatrix}, \quad H_4 \otimes J_4 = \begin{pmatrix} J_4 & J_4 & J_4 & J_4 \\ J_4 & -J_4 & J_4 & -J_4 \\ J_4 & J_4 & -J_4 & -J_4 \\ J_4 & -J_4 & -J_4 & J_4 \end{pmatrix},$$

where $U_4=(1,1,-1,-1)'$ and $J_4=(1,1,1,1)'$. Then the designs $d_1$ and $d_2$ can be constructed by using the first two columns of $H_4 \otimes U_4$, and the second and third columns of $H_4 \otimes J_4$, respectively. Also note that the first four runs are associated to the first group, the second four runs are associated to the second group.

| d1 | : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 |
| | $x_2$ | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |

| d2 | : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 |
| | $x_2$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |

### 4.2. Inter-class correlation structure

It is an extension of intra-class structure. This situation is observed if the observations are grouped into some groups such that within each group there is the same intra-class structure and between groups there is no correlation. Inter-class structure is actually a particular case of compound symmetry structure which is obtained from $W_1(\rho^*,\rho_1^*)$ as in (4.1), assuming $\rho_1^*=0$ and is given below.

$$D(\mathbf{e}) = \sigma_1^2[I_m \otimes A] = \sigma_1^2 W_1(\rho^*,0), \tag{4.5}$$

where $\otimes$ denotes Kronecker product, $A$ is as in (4.1) and $N=mn$. Similarly as the variance–covariance structure $\sigma^2 W(\rho,\rho_1)$ as in (4.1), $\sigma^2 W$ under the inter-class variance–covariance structure $\sigma_1^2 W_1(\rho^*,0)$ as in (4.5) will be reduced to $\sigma^2 W(\rho,0)$, where $\rho=\sigma_1^2/(\sigma_1^2+\delta^2)\rho^*$.

The necessary and sufficient conditions for *D*-optimal robust first order regression design under the inter-class variance–covariance structure $\sigma^2 W(\rho,0)$ can be simplified to Theorem 4.2 below.

**Theorem 4.2.** *A set of necessary and sufficient conditions for a D-optimal robust first order regression design under the inter-class variance–covariance structure $\sigma^2 W(\rho,0)$, over the design space $\mathcal{X}$ are*

(i) $v_{0j}=0, i.e. \sum_{u=1}^{N} x_{uj}=0$;

(ii) $v_{ij}=0; i \neq j$;

(iii) $$\sum_{u=1}^{N} x_{ui}^2 = N; \sum_{u=1}^{n} x_{ui} = \sum_{u=(n+1)}^{2n} x_{ui} = \cdots = \sum_{u=n(m-1)+1}^{mn} x_{ui} = 0, \text{ if } \rho>0; \text{ and}$$

(iv) $$\sum_{u=1}^{N} x_{ui}^2 = N; \sum_{u=1}^{n} x_{ui} = \sum_{u=(n+1)}^{2n} x_{ui} = \cdots = \sum_{u=n(m-1)+1}^{mn} x_{ui} = \pm n, \text{ if } \rho<0.$$

Note that Methods I and II provide *D*-ORFODs for $\rho>0$ and $\rho<0$, respectively, under the inter-class variance–covariance structure $\sigma^2 W(\rho,0)$.

### 4.3. Intra-class correlation structure

Intra-class structure is the simplest variance–covariance structure which arises when errors of any two observations have the same correlation and each has the same variance. It is also known as an uniform correlation structure. This happens when all the observations studied are from the same batch or from the same run in a furnace. This is actually a special case of inter-class structure when $m=1$, as given below.

$$D(\mathbf{e}) = \sigma_1^2[(1-\rho^*)I_N + \rho^* E_{N\times N}] = \sigma_1^2 W_1(\rho^*), \tag{4.6}$$

where $\sigma_1>0$, and $-(N-1)^{-1}<\rho^*<1$. Similarly as the variance–covariance structure in (4.1) and (4.5), $\sigma^2 W$ under the intra-class variance–covariance structure $\sigma_1^2 W_1(\rho^*)$ as in (4.6) will be reduced to $\sigma^2 W(\rho)$, which is also an intra-class structure as in (4.6), where $\rho=\sigma_1^2/(\sigma_1^2+\delta^2)\rho^*$.

Following Theorem 3.1, the necessary and sufficient conditions for *D*-optimal robust first order regression design under the intra-class variance–covariance structure $\sigma^2 W(\rho)$ can be simplified to $v_{0j}=\sum_{u=1}^N x_{uj}=0$; $v_{ij}=\sum_{u=1}^N x_{ui}x_{uj}=0$, $i\neq j$; and $v_{ii}=\sum_{u=1}^N x_{ui}^2$ = constant = $N$ = $\mu$.

**Theorem 4.3.** *A design is a D-optimal robust first order under the intra-class structure if and only if it is a D-optimal first order rotatable design in the usual model* (*i.e., when errors are uncorrelated and homoscedastic*), *whatever be the value of the intra-class correlation coefficient* $\rho\in W(\rho)$.

## 5. Concluding remarks

In this paper, we consider *D*-optimal designs for log-normal lifetime distribution: the mean lifetime $g(\mathbf{x},\boldsymbol{\beta})$ of a component is a reasonable function of the explanatory variables. This article derives the model $\ln T=g(\mathbf{x},\boldsymbol{\beta})+\delta\tau$, where $\tau\sim N(0,1)$. Assuming additive error component *e*, which arises mainly due to experimentation, the final model $\ln T=g(\mathbf{x},\boldsymbol{\beta})+\delta\tau+e$ has been derived. From practical point of view, this model is more appropriate for a lifetime improvement experiment.

Recently, Myers et al. (2002) analyzed "The Worsted Yarn Data" (Myers et al., 2002, Table 2.7, p. 36) using a conventional (errors are uncorrelated and homoscedastic) second order response surface design, and treating response ($y=T$) as the cycles to failure ($T$). They noticed that variance is not constant and the analysis is inappropriate. Using the log transformation of the cycles to failure (i.e., $y=\ln T$), they analyzed the data, and found that log model, overall, is an improvement on the original quadratic fit. There remains evidences of heterogeneity in variances, however. Myers et al. (2002, p. 128) also noticed that in *industrial applications* experimental units are not independent at times *by design*. This leads to correlation among observations via a *repeated measures* scenario as in *split plot* design. Das and Lee (2009) showed that simple log transformation is insufficient to reduce the variance constant, and they found that log-normal distribution is much more appropriate. Thus, it is reasonable to consider $y=\ln T$ as the response with correlated errors for the model of a lifetime distribution. If errors are indeed uncorrelated and homoscedastic, the usual response surface designs are appropriate for the response $y=\ln T$. The *D*-optimal first order designs for some well-known correlation structures have been developed. It is crucial to use proper optimal operating conditions of the process parameters in a lifetime improvement experiment.

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## References

Aitkin, M., Clayton, D.G., 1980. The fitting of exponential, Weibull and extreme value distributions to complex censored survival data using GLIM. Appl. Statist. 29, 156–163.
Anderson, T.W., 1984. In Introduction to Multivariate Statistical Analysis, second ed. Wiley and Sons, New York.
Bischoff, W., 1992. On exact D-optimal designs for regression models with correlated observations. Ann. Inst. Statist. Math. 44, 229–238.
Bischoff, W., 1995. Determinant formulas with applications to designing when the observations are correlated. Ann. Inst. Statist. Math. 47, 385–399.
Bischoff, W., 1996. On maximin designs for correlated observations. Statist. Probab. Lett. 26, 357–363.
Box, G.E.P., Draper, R.N., 2007. Response Surfaces, Mixtures, and Ridge Analyses, second ed. Wiley and Sons, New York.
Box, G.E.P., Hunter, J.S., 1957. Multifactor experimental designs for exploring response surfaces. Ann. Math. Statist. 28, 195–241.
Condra, L.W., 1993. Reliability Improvement of Experiments. Marcel Dekker, New York.
Das, R.N., 1997. Robust second order rotatable designs: Part-I. Calcutta Statist. Assoc. Bull. 47, 199–214.
Das, R.N., 2003. Robust second order rotatable designs: Part-III. J. Indian Soc. Agricultural Statist. 56, 117–130.
Das, R.N., 2004. Construction and analysis of robust second order rotatable designs. J. Statist. Theory Appl. 3, 325–343.
Das, R.N., Lee, Y.J., 2009. Log normal versus gamma models for analyzing data from quality-improvement experiments. Qual. Eng. 21 (1), 79–87.
Das, R.N., Park, S.H., 2008a. On efficient robust rotatable designs with autocorrelated errors. J. Korean Statist. Soc. 37 (2), 97–106.
Das, R.N., Park, S.H., 2008b. On exact D-optimal robust designs with tri-diagonal errors. J. Adv. Appl. Statist. 9 (1), 37–51.
Draper, N.R., Lin, D.K.J., 1996. Response surface designs. In: Ghosh, S., Rao, C.R. (Eds.), Handbook of Statistics, vol. 13. North-Holland, New York, pp. 323–375 (Chapter 11).
Gennings, C., Chinchilli, V.M., Carter, W.H.J.R., 1989. Response surface analysis with correlated data: a non-linear model approach. J. Amer. Statist. Assoc. 84, 805–809.
Kiefer, J., Wynn, H.P., 1981. Optimum balanced block and latin square designs for correlated observations. Ann. Statist. 9, 737–757.
Kiefer, J., Wynn, H.P., 1984. Optimum and minimax exact treatment designs for one-dimensional autoregressive error processes. Ann. Statist. 12, 431–449.
Khuri, A.I., Cornell, J.A., 1996. Responses Surfaces. Marcel Dekker, New York.
Meeker, W.O., Escobar, L.A., 1998. Statistical Methods for Reliability Data. John Wiley and Sons, New York.
Mukhopadhyay, A.C., Bagchi, S.B., Das, R.N., 2002. Improvement of quality of a system using regression designs. Calcutta Statist. Assoc. Bull. 53, 225–233.
Myers, R.H., Montgomery, D.C., Vining, G.G., 2002. Generalized Linear Models with Applications in Engineering and the Sciences. John Wiley and Sons, New York.
Nelson, W.B., 1972. Graphical analysis of accelerated life test data with the inverse power law model. IEEE Trans. Reliab. R 21, 2–11.
Panda, R.N., Das, R.N., 1994. First order rotatable designs with correlated errors. Calcutta Statist. Assoc. Bull. 44, 83–101.
Peto, R., Lee, P., 1973. Weibull distributions for continuous carcinogenesis experiments. Biometrics 29, 457–470.

Pike, M.C., 1966. A method of analysis of a certain class of experiments in carcinogenesis. Biometrics 22, 142–161.
Pukelsheim, F., 1993. Optimal Experimental Design. Wiley and Sons, New York.
Taguchi, G., 1986. Introduction to Quality Engineering Asian Productivity Organization. UNIPUB, White Plains, NY.
Taguchi, G., 1987. System of Experimental Design: Engineering Methods to Optimize Quality and Minimize Cost. UNIPUB/Kraus International, White Plains, NY.