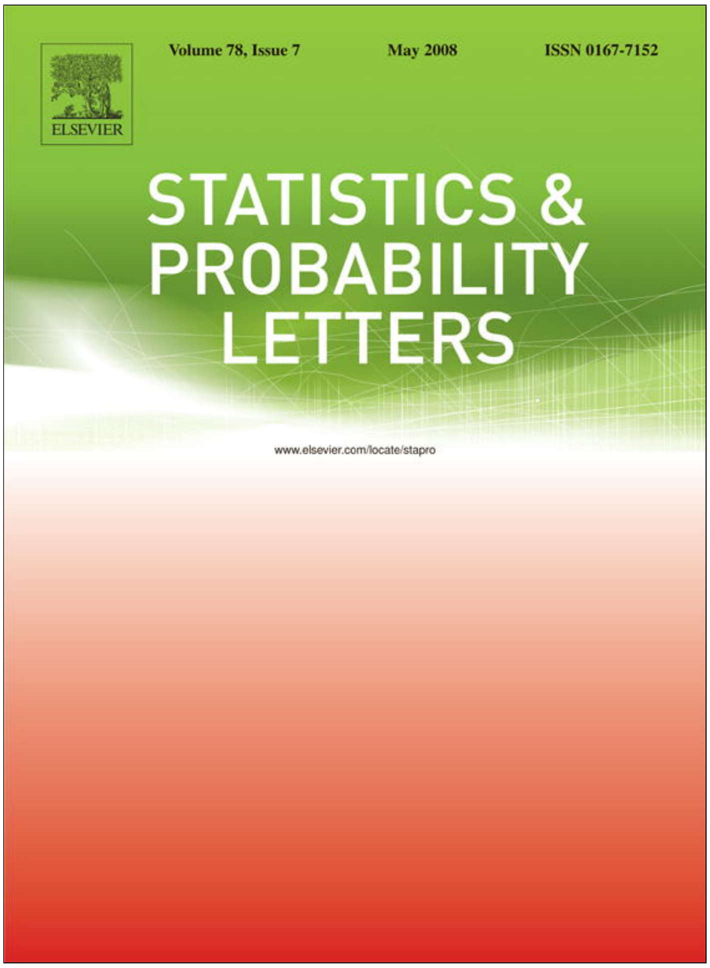
Volume 78, Issue 7
May 2008
ISSN 0167-7152
ELSEVIER
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

Available online at www.sciencedirect.com

ScienceDirect

Statistics and Probability Letters 78 (2008) 896–903

STATISTICS & PROBABILITY LETTERS

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Optimal foldover plans for regular s-level fractional factorial designs

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Received 8 December 2006; received in revised form 28 April 2007; accepted 7 September 2007
Available online 1 November 2007

Abstract

This article introduces a general decomposition structure of the foldover plan. While all the previous work is limited to two-level designs, our results here are good for general regular s-level fractional factorial designs, where s is any prime or prime power. The relationships between an initial design and its combined designs are studied. This is done for both with and without consideration of the blocking factor. For illustration of the usage of our theorems, a complete collection of foldover plans for regular three-level designs with 27 runs is given that is optimal for aberration and clear effect numbers.

MSC: primary 62K15; secondary 62K05

1. Introduction

A full factorial design requires s^k runs to be performed for s levels and k factors. Because of the large run sizes, such designs are rarely used in practice for large k. Fractional factorial (FF) designs, which consist of a subset or fraction of full factorial designs, are no doubt the most widely used designs in the experimental investigations due to their efficient use of experimental runs to study many factors simultaneously. One consequence of using a FF design is the aliasing of factorial effects. A standard follow-up strategy to break the aliasing in two-level designs is to add a foldover design by reversing the signs of one or more columns of the initial design. This idea has been extensively discussed in the literature, including Box and Hunter (1961), Box et al. (1978), Montgomery (2001), Neter et al. (1996), and Wu and Hamada (2000). Montgomery and Runger (1996) considered foldover for resolution IV designs by reversing the signs of one or two factors. Li and Lin (2003) gave a complete collection of all optimal foldover plans for regular two-level FF designs with 16 or 32 runs in terms of the aberration criterion (see also, Li and Mee (2002)). Subsequently, Li et al. (2003) further considered the optimal foldover plans for nonregular orthogonal designs. Fang et al. (2003) provided a theoretical justification for the optimal foldover designs based on a generalized discrepancy measure, proposed by Hickernell (1998). Ye and Li (2003) studied the theoretical properties of the foldover design and the resulting combined design obtained by augmenting a regular two-level FF design by its foldover.

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doi:10.1016/j.spl.2007.09.017

Table 1
Design matrix for 27-run three-level designs (The independent columns are numbered 1, 2 and 5)

1	**2**	3	4	**5**	6	7	8	9	10	11	12	13
1	0	1	1	0	1	0	1	1	1	0	1	1
0	1	1	2	0	0	1	1	2	0	1	1	2
0	0	0	0	1	1	1	1	1	2	2	2	2

All of the previous work is restricted to the case of two-level designs because of their simplicity. However, in practice larger numbers of levels may be used and so this is an interesting area. This article provides some general properties of a regular s-level FF design under different foldover plans, where s is any prime or prime power. To our knowledge, foldover plans for regular s-level FF is lacking in the literature. Our results here are apparently new.

This paper is organized as follows. Section 2 describes the general structure of a regular s-level foldover design. Section 3 generalizes the properties of foldover designs in Ye and Li (2003) to general regular s-level FF designs. As will be shown, such a generalization is not straightforward. The relationships between an initial design and its combined designs with or without consideration of effect of the blocking factor are studied. To illustrate the usefulness of our results, a complete collection of optimal foldover plans for regular three-level designs with 27 runs in terms of the aberrations and clear effect numbers of the combined designs is given in Section 4. Section 5 concludes this article with some remarks and future work.

2. General structure of a regular s-level foldover design

Some notations and definitions are first introduced below. For an s^k full factorial design, a factorial effect (or a word) can be denoted by a k-dimensional nonnull column vector z with elements from $GF(s)$, the finite field with s elements. For any $\lambda(\neq 0) \in GF(s)$, z and λz represent the same factorial effect. A factorial effect z represents a main effect if it has exactly one nonzero element, and a two-factor interaction if it has exactly two nonzero elements.

Consider a regular s^{k-p} fractional factorial design D with k s-level factors, whose levels are denoted by $0, 1, \dots, s-1$. The design matrix can be expressed as $D = R(C)$, where C is a $(k-p) \times k$ matrix over $GF(s)$ with full row rank in which no two columns are proportional to each other, and $R(\cdot)$ stands for the row space of a matrix over $GF(s)$, i.e., the set of all linear combinations of the row vectors of a matrix over $GF(s)$. Note that the matrix C is called the factor representation of design D in Chen and Hedayat (1996). A factorial effect z is a defining word, i.e., appears in the defining relation subgroup of design D, if and only if $Cz = 0$ over $GF(s)$. The length of a word is the number of its nonzero elements.

Now consider the general structure of a foldover design. A foldover plan can be denoted by a k-dimensional column vector ξ over $GF(s)$. Then the combined design consisting of the initial design and its foldover design is $R(C^*)$, where $C^* = (C^T, \xi)^T$. When ξ is a null vector, called the null foldover plan, $R(C^*)$ reduces to s replicates of the initial design $R(C)$.

Example 1. Consider the 3^{6-3} initial design D (denoted by 6–3.2 in Table 2), which consists of the independent columns 1, 2, 5 and the three additional columns 3, 6 and 7 in Table 1. It is known that $D = R(C)$ and

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Let A, B, E, F, G, H denote the six factors corresponding to the columns 1, 2, 5, 3, 6, 7. Design D has defining relations $F = AB$, $G = AE$ and $H = BE$. For a foldover plan $\xi = (0, 0, 0, 1, 1, 2)^T$, the combined design is $R(C^*)$ and

$$C^* = \begin{pmatrix} C \\ \xi^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

It can be shown that design $R(C^*)$ is a regular 3^{6-2} design with defining relations $G = B^2EF$ and $H = AB^2EF^2$.

One popular criterion for ordering the regular designs is the aberration criterion (Fries and Hunter, 1980), which is defined as follows. The vector $W(D) = (A_1(D), \ldots, A_k(D))$ is called the wordlength pattern of a design D, where $A_i(D)$ is the number of words of length i in the defining relation subgroup of design D. Specially, the three defining relation words of design D in Example 1 can be expressed as

$$U = \begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 2 \end{pmatrix},$$

its defining relation subgroup is just $R(U)$ and its wordlength pattern is $W(D) = (0, 0, 3, 6, 3, 1)$.

For simplicity, only $A_i(D)$'s ($i \geq 3$) of the wordlength pattern are displayed in this paper. For any two s^{k-p} designs D_1 and D_2, D_1 is said to have less aberration than D_2 if $A_r(D_1) < A_r(D_2)$, where r is the smallest integer such that $A_r(D_1) \neq A_r(D_2)$. If there exists no other design with less aberration than D_1, then D_1 is said to have minimum aberration. Obviously, a minimum aberration design D is the one sequentially minimizing the elements of the wordlength pattern $W(D)$.

As argued and demonstrated in Chen et al. (1993), when there is no design with resolution V or higher, the minimum aberration criterion does not always lead to the best designs. To explain this situation, Wu and Chen (1992) proposed the concept of clear main effect or clear two-factor interaction. A main effect or a two-factor interaction component is called clear if it is not aliased with any other main effects or two-factor interaction components. A two-factor interaction is called clear if all of its components are clear. It is known that clear main effects and clear two factors are estimable under the assumption that the three factor and higher order interactions are negligible. Thus, according to the estimation capacity, the designs containing most clear main effects and most clear two-factor interactions are preferred. A collection of these designs for 16, 32, 27, and 81 runs can be found in Ai and Zhang (2004c).

3. Some properties of foldover plans

This section generalizes the properties of foldover plans in Ye and Li (2003) to adapt to the general regular s-level FF designs. As pointed out at the end of Ye and Li's (2003) article, these generalizations are not trivial but interesting. Furthermore, the proofs of the properties are significantly simplified by the general structure of a foldover plan displayed in the previous section.

Without loss of generality, we assume that for a regular s^{k-p} design D, the first $(k-p)$ columns are independent columns and the remaining p columns are additional columns, i.e., the linear combinations of the first $(k-p)$ columns. In this way, the factor representation of design D can be expressed as $C = (\mathbf{I} \vdots C_1)$, where $\mathbf{I}$ is the identity matrix of order $(k-p)$ and C_1 is a $(k-p) \times p$ matrix. For a foldover plan $\xi = (\xi_1^T, \xi_2^T)^T$, where ξ_1 and ξ_2 are, respectively, $(k-p)$- and p-dimensional column vectors. Then the combined design is

$$R(C^*) = R\begin{pmatrix} \mathbf{I} & C_1 \\ \xi_1^T & \xi_2^T \end{pmatrix} = R\begin{pmatrix} \mathbf{I} & C_1 \\ \mathbf{0} & \xi_2^{*T} \end{pmatrix}, \tag{1}$$

where $\xi_2^* = \xi_2 - C_1^T \xi_1$ over $GF(s)$. It is thus sufficient to consider the foldover plans of the form $\xi = (\xi_1^T, \xi_2^T)^T = (0, \ldots, 0, x_{k-p+1}, \ldots, x_k)^T$ with the first nonzero element of $(x_{k-p+1}, \ldots, x_k)$ equal to 1. Following Li and Lin (2003), these foldover plans are called the core foldover plans. By noting that apart from the null foldover plan, the number of the vectors $\xi_2 = (x_{k-p+1}, \ldots, x_k)^T$ with the first nonzero element equal to 1 is $(s^p - 1)/(s - 1)$, the following Theorem can thus be obtained for any s-level fractional factorial. Theorems 2.1 and 2.2 in Ye and Li (2003) can be treated as special cases of Theorem 1 by taking $s = 2$.

Theorem 1. *For a regular s^{k-p} FF design, there are $1+(s^p-1)/(s-1)$ distinct combined designs under all foldover plans.*

Since a vector $z = (z_1, \ldots, z_k)^T$ is a defining word of an s^{k-p} design $R(C)$ if and only if $Cz = 0$ over $GF(s)$, so the following theorem can be straightforwardly obtained.

Theorem 2. *For a regular s^{k-p} FF design D, let $C = (\mathbf{I} \vdots C_1)$ be its factor representation, and $\xi = (0, \ldots, 0, x_{k-p+1}, \ldots, x_k)^T$ be a core foldover plan. Then $z = (z_1, \ldots, z_k)^T$ is a defining word of the combined design if and only if z is a defining word of design D and $\sum_{i=k-p+1}^{k} x_i z_i = 0$ over $GF(s)$.*

Note that for the case $p = 1$, there are only two distinct ways to generate a combined design, that is the null foldover plan and the foldover plan $(0, \ldots, 0, 1)$. Furthermore, the combined design under the foldover plan $(0, \ldots, 0, 1)$ is the s^k full factorial design, and so is an optimal foldover design.

Next, we consider the impact of inclusion of the blocking factor on foldover designs. As explained in Ye and Li (2003), because of the sequential nature of foldover designs, the combined design can be considered to be implicitly classified into s blocks. Without loss of generality, it is assumed that the blocking factor is arranged at the first column and takes the value 0 for the initial design D. Then the combined design including the blocking factor under the foldover plan ξ can be represented as

$$D_b = R(C_b) = R \begin{pmatrix} \mathbf{0} & C \\ 1 & \xi^T \end{pmatrix}, \tag{2}$$

where $\mathbf{0}$ is the $(k - p)$-dimensional null column vector and C is the factor representation of design D. Consider any nonnull $(k + 1)$-dimensional column vector $z_b = (z_0, z^T)^T$, where z is a k-dimensional column vector. Denote $C^* = (C^T, \xi^T)^T$ and under the operation over $GF(s)$. When $z_0 = 0$, we have $C_b z_b = 0$ if and only if $C^* z = 0$; while when $z_0 \neq 0$, we have $C_b z_b = 0$ if and only if $Cz = 0$ and $C^* z \neq 0$.

For the blocked combined design D_b, let $W_t(D_b) = \{A_1^t(D_b), \ldots, A_k^t(D_b)\}$ and $W_b(D_b) = \{A_1^b(D_b), \ldots, A_k^b(D_b)\}$ be the treatment and block wordlength patterns, respectively. Also denote $W(D) = \{A_1(D), \ldots, A_k(D)\}$ and $W(D^*) = \{A_1(D^*), \ldots, A_k(D^*)\}$ be the wordlength patterns of the initial design D and the combined design D^*, respectively. Note that the defining relation subgroup of the combined design D^* is a subset of that of the initial design D. Then $A_i^t(D_b) = A_i(D^*)$ and $A_i^b(D_b) = A_i(D) - A_i(D^*)$ for $i = 1, \ldots, k$. Several methods for combining the treatment and block wordlength patterns into one proper sequence have been proposed. This includes Sitter et al. (1997), Chen and Cheng (1999), Zhang and Park (2000), Cheng and Wu (2002) and Ai and Zhang (2004a). For detailed comparisons and reviews on the prevailing criteria refer to Ai and Zhang (2004b). Denote the combined wordlength pattern by $W(D_b) = \{C_1(D_b), \ldots, C_{2k}(D_b)\}$. In essence, a minimum aberration blocked design sequentially minimizes $C_i(D_b)$ for $i = 1, \ldots, 2k$.

Although the orderings of the treatment and block wordlength patterns in the combined sequences are different according to different criteria, it is worth noting that $A_i^b(D_b)$ is always ordered after $A_{i+1}^t(D_b)$ for any i in sequence $W(D_b)$. So for a fixed initial design D, sequentially minimizing $C_i(D_b)$ for $i = 1, \ldots, 2k$ is equivalent to sequentially minimizing $A_i(D^*)$ for $i = 1, \ldots, k$. Thus the following theorem can be adapted to all previous aberration criteria. The conjecture in Ye and Li (2003, p. 407) which indicated that their theorem for two-level may not hold for other criteria for blocking designs is thus disapproved.

Theorem 3. *For a given regular s^{k-p} FF design D, let $D^*(\xi^{(i)})$ and $D_b(\xi^{(i)})$ respectively be the unblocked and blocked combined designs under the two foldover plans $\xi^{(i)} (i = 1, 2)$. Then $D_b(\xi^{(1)})$ has less aberration than $D_b(\xi^{(2)})$ if and only if $D^*(\xi^{(1)})$ has less aberration than $D^*(\xi^{(2)})$.*

This theorem shows that the inclusion of the blocking factor does not affect the optimality of foldover designs based on the minimum aberration criterion. So an optimal foldover plan in terms of aberration of the combined design remains optimal with consideration of the blocking factor.

4. An illustrative example: Optimal foldover plans for regular three-level designs with 27 runs

As an illustration of the previous properties of foldover designs, we next give a complete collection of optimal foldover plans of the regular three-level designs with 27 runs in terms of the aberrations and the numbers of clear effects of the combined designs, based on the complete catalogue of three-level designs with 27 runs displayed in Table 6 of Chen et al. (1993).

Firstly, we put the column set of the saturated design with 27 runs in Table 1 in Yates order. All the initial 3^{k-p} designs are listed and numbered according to the order of their aberrations. They are given by a subset of k columns,

Table 2
Optimal foldover plans for 27-run 3^{k-p} designs

Design	Additional columns	Optimal foldover plans ξ^T	$(A_3(D^*), A_4(D^*), \ldots)$	$C_1(D^*)$	$C_2(D^*)$
5–2.1	3 9	(1, 0), (1, 1), (1, 2)	0 1 0	5	4
		(1)	1 0 0	2	7
5–2.2	3 6	(1, 2)	0 0 1	5	10
5–2.3	3 4	(1), (1, 0), (1, 1), (1, 2)	1 0 0	2	7
6–3.1	3 9 13	(1, 0, 1), (1, 0, 2), (1, 1, 0), (1, 1, 2), (1, 2, 0), (1, 2, 1)	0 3 0 1	6	0
		(1, 1)	2 0 0 2	0	9
6–3.2	3 6 7	(1, 1, 2), (1, 2, 1), (1, 2, 2)	0 2 2 0	6	4
		(1, 2), (1, 0, 2), (1, 2, 0)	1 1 1 1	3	6
6–3.3	3 6 11	(1, 1, 1), (1, 1, 2), (1, 2, 1)	0 2 2 0	6	4
		(1, 1), (1, 0, 2), (1, 2, 0), (1, 2, 2)	1 0 3 0	3	12
6–3.4	3 4 6	(1, 1, 2)	1 0 3 0	3	12
7–4.1	3 10 11 13	(1, 1, 2, 2), (1, 2, 1, 0), (1, 2, 2, 1)	0 5 6 1 1	7	0
		(1, 0, 1, 2)	2 9 0 2 0	1	6
7–4.2	4 8 9 11	(1, 0, 2, 2), (1, 1, 0, 1), (1, 1, 2, 2), (1, 2, 0, 1)	0 5 6 1 1	7	0
		(1, 1), (1, 0, 0, 2), (1, 0, 2, 0), (1, 1, 2, 1), (1, 2, 0, 2), (1, 2, 1, 1)	1 4 6 0 2	4	6
7–4.3	4 8 10 11	(1, 0, 1, 1), (1, 0, 1, 2), (1, 0, 2, 2), (1, 1, 2, 1), (1, 2, 2, 1)	1 3 6 3 0	4	3
		(1, 0, 2, 1)	4 1 3 3 2	3	9
7–4.4	3 4 9 13	(1, 2, 2)	2 0 9 2 0	1	15
		(1, 0, 0, 1), (1, 0, 0, 2), (1, 0, 1, 0), (1, 0, 1, 2), (1, 0, 2, 0), (1, 0, 2, 1), (1, 1, 0, 1), . . .	2 3 3 5 0	2	2
8–5.1	3 8 9 10 11	(1, 0, 2, 2, 2), (1, 2, 0, 1, 2), (1, 2, 1, 2, 1)	0 10 16 4 8 2	8	0
		(1), (1, 0), (1, 0, 0), (1, 0, 0, 0), (1, 1, 2, 2), (1, 0, 0, 0, 0), (1, 1, 1, 1, 0), (1, 1, 2, 0, 1)	5 15 9 8 3 0	1	7
8–5.2	4 8 9 10 11	(1, 0, 2, 1, 2), (1, 0, 2, 2, 2), (1, 1, 0, 2, 1), (1, 1, 2, 1, 2), (1, 1, 2, 2, 2), (1, 2, 0, 1, 2), . . .	1 9 13 9 6 2	5	0
		(1), (1, 0, 0, 0, 0)	5 15 9 8 3 0	1	7
8–5.2	4 8 9 10 11	(1, 0, 2, 1, 2), (1, 0, 2, 2, 2), (1, 1, 0, 2, 1), (1, 1, 2, 1, 2), (1, 1, 2, 2, 2), (1, 2, 0, 1, 2), . . .	1 9 13 9 6 2	5	0
		(1), (1, 0, 0, 0, 0)	5 15 9 8 3 0	1	7
8–5.3	3 4 9 11 13	(1, 0, 1, 1, 0), (1, 0, 2, 2, 1), (1, 1, 0, 2, 2), (1, 1, 1, 1, 0), (1, 2, 0, 2, 2), (1, 2, 2, 2, 1)	2 6 15 11 3 3	3	0
		(1, 0, 0, 1, 1), (1, 0, 0, 2, 1), (1, 1, 0, 1, 1), (1, 1, 0, 2, 1), (1, 2, 0, 1, 1), (1, 2, 0, 2, 1)	2 8 10 14 4 2	3	1
		(1, 0, 0, 0)	5 15 9 8 3 0	1	7
9–6.1	3 8 9 10 11 13	(1, 0, 2, 2, 2, 1), (1, 2, 0, 1, 2, 2), (1, 2, 1, 2, 1, 0)	0 18 36 12 36 18 1	9	0
		(1), (1, 0), (1, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0, 0), (1, 1, 2, 2, 2), (1, 0, 0, 0, 0, 0), . . .	8 30 24 32 24 3 0	1	8
9–6.2	3 4 8 9 10 11	(1, 0, 0, 2, 2, 2), (1, 0, 2, 1, 2, 1), (1, 1, 0, 2, 2, 2), (1, 1, 2, 0, 1, 2), (1, 2, 2, 0, 1, 2), . . .	2 15 30 26 30 15 3	4	0
		(1, 0, 0, 0, 0)	8 30 24 32 24 3 0	1	8
9–6.3	4 9 10 11 12 13	(1, 0, 1, 2, 1, 0), (1, 0, 1, 2, 1, 2), (1, 0, 1, 2, 2, 0),	3 13 28 34 23 17 3	2	0

Table 2 (*continued*)

Design	Additional columns	Optimal foldover plans ξ^T	$(A_3(D^*), A_4(D^*), \ldots)$	$C_1(D^*)$	$C_2(D^*)$
		(1, 0, 1, 2, 2, 2), (1, 0, 2, 2, 0, 0), . . .			
		(1, 0, 1, 1, 0, 0), (1, 0, 1, 1, 0, 2), (1, 0, 1, 1, 1, 2), (1, 0, 1, 1, 2, 2), (1, 0, 1, 2, 0, 0), . . .	3 17 20 35 28 16 2	3	0
		(1), (1, 0), (1, 0, 0, 0, 0)	10 23 32 30 22 4 0	1	8
10–7.1	3 6 7 8 10 11 12	(1, 0, 0, 2, 2, 2, 1), (1, 0, 1, 2, 1, 0, 2), (1, 0, 2, 1, 1, 2, 0), (1, 1, 0, 2, 0, 1, 2), . . .	3 24 63 60 105 72 31 6	3	0
		(1, 1, 1, 2, 2, 2)	12 54 54 96 108 27 13 0	1	9
10–7.2	3 4 6 7 8 10 11	(1, 1, 1, 2, 0, 1), (1, 2, 2, 1, 2, 0), (1, 0, 0, 1, 2, 1, 2), (1, 0, 1, 0, 2, 2, 1), . . .	4 24 51 85 90 66 41 3	1	0
		(1, 1, 2, 0, 0, 0), (1, 0, 0, 2, 0, 2, 1), (1, 0, 1, 1, 0, 2, 2), (1, 0, 2, 0, 0, 0, 1), . . .	5 30 39 83 90 87 24 6	3	0
		(1), (1, 0), (1, 1, 1, 2, 2), (1, 0, 0, 0, 0, 0), (1, 1, 1, 0, 1, 1, 0), (1, 2, 0, 1, 1, 0, 1)	15 42 69 96 93 39 10 0	1	9
11–8	3 4 6 7 8 9 10 11	(1, 1, 1, 2, 2, 0, 0), (1, 1, 1, 2, 2, 1, 0), (1, 1, 1, 2, 2, 2, 0), (1, 1, 2, 1, 0, 0, 1), . . .	6 36 102 162 270 264 169 72 12	1	0
		(1, 1, 1, 0, 0, 0), (1, 1, 1, 1, 0, 2, 2), (1, 1, 1, 1, 2, 2, 2), (1, 1, 2, 0, 0, 0, 0), . . .	8 51 67 179 261 274 177 66 10	3	0
		(1), (1, 0), (1, 2, 0, 1, 1, 2, 0, 1)	22 68 138 250 290 213 92 20 0	1	10
12–9	3 4 6 7 8 9 10 11 12	(1, 0, 0, 1, 2, 2, 2, 1), (1, 0, 0, 2, 2, 2, 2, 1), (1, 0, 1, 0, 2, 1, 1, 2), . . .	9 54 162 348 648 756 706 . . .	0	0
		(1, 1, 1, 1, 0, 2, 2, 2), (1, 1, 1, 1, 2, 2, 2, 2), (1, 2, 2, 2, 0, 1, 1, 1), . . .	12 81 117 357 621 792 706 . . .	3	0
		(1), (1, 0), (1, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0, 0) (1, 1, 1, 1, 2, 2, 2), (1, 0, 0, 0, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 0, 1, 1, 1, 0, 1)	30 108 252 546 810 765 517 . . .	1	11

consisting of three independent columns – Columns 1, 2 and 5 – and p additional columns. Only the latter are specified in Table 2. Note that the wordlength patterns of two pairs of designs, that is 6–3.2, 6–3.3 and 7–4.2, 7–4.3, are mistakenly exchanged in Chen et al. (1993). The orders of these two pairs of designs have been corrected in Table 2 here.

A foldover plan is given by a k-dimensional row vector. To save space, however, all the zero elements ahead of the first nonzero element in a foldover plan are omitted in Table 2. To implement these results in practice, a foldover plan must be augmented to a k-dimensional vector by adding adequate number of zero elements ahead of the vector. For every foldover plan, the wordlength pattern and the numbers of clear main effects and clear two-factor interactions of the corresponding combined design, denoted by $W(D^*) = (A_3(D^*), A_4(D^*), \ldots)$, $C_1(D^*)$ and $C_2(D^*)$, are listed in the columns 4–6, respectively.

For every initial design, the optimal foldover plans can be classified into three types in terms of the combination of minimum aberration and maximum numbers of clear effects. The Type-I foldover plan generates a minimum aberration combined design; the Type-II foldover plan generates a combined design which firstly maximizes the number of clear main effects $C_1(D^*)$, maximizes the number of clear two-factor interactions $C_2(D^*)$ and then finally sequentially minimizes the wordlength pattern $W(D^*)$; and finally the Type-III foldover plan generates a combined design in the same way as the Type-II except for the order of maximizing $C_1(D^*)$ and $C_2(D^*)$ being exchanged. Note that the Type-II and Type-III foldover plans are omitted from the table, if they are identical to the Type-I foldover plan. To save space, only a few optimal foldover plans in each type are displayed in Table 2. The complete catalogue of all foldover plans is available upon request. Usage of Table 2 is illustrated by the following example.

Example 2. Consider the initial design 6–3.2, which consists of the independent columns 1, 2, 5 and the three additional columns 3, 6 and 7. From the relationships among these six columns in Table 1, we can conclude that its wordlength pattern is (3, 6, 3, 1) and this design has neither clear main effect nor clear two-factor interaction. From Table 2, it can be seen that the first set of optimal foldover plans, {(0, 0, 0, 1, 1, 2), (0, 0, 0, 1, 2, 1), (0, 0, 0, 1, 2, 2)}, generates minimum aberration 3^{6-2} combined design, and is optimal foldover plan of both Type-I and Type-II, while the other set, {(0, 0, 0, 0, 1, 2), (0, 0, 0, 1, 0, 2), (0, 0, 0, 1, 2, 0)}, is optimal foldover plans of Type-III.

Li and Lin (2003) also introduced a concept of combined-optimal design, which can be adopted for selecting an initial design. An s^{k-p} initial design is called a combined-optimal design if the resulting optimal combined design has minimum aberration among all optimal combined designs. Certainly, the combined-optimal design can also be defined according to other optimal criteria. Obviously, the designs 5–2.2, 6–3.2, 6–3.3, 7–4.1, 7–4.2, 8–5.1, 9–6.1 and 10–7.1 are combined-optimal designs of both Type-I and Type-II. Moreover, the optimal combined designs of Type-I resulting from the first seven initial designs have minimum aberration among all designs with the same parameters.

5. Concluding remarks and further work

In this paper, we provide a general structure of the foldover plan study. Unlike all previous work on foldover, our results are good for any s-level fractional factorial. The properties between an initial design and its combined designs are discussed for both with and without blocking factor. The special case of $s = 3$ is used for illustration. In this case, Table 2 provides a powerful and efficient way for searching the optimal foldover plans. These results are apparently new.

Since the number of the level combinations of the regular combined design under a foldover plan is s times that of the initial design, it may not be attractive to continue adopting the regular foldover plans when s is greater than 3. Alternatively, one can consider the case of an $(s-1)^{-1}$ fraction of the foldover design in which the number of level combinations of the combined design is double that of the original design as discussed for a two-level design. Note that at this time the combined designs no longer be the regular designs. So the generalized minimum aberration criterion suggested by Xu and Wu (2001) and Ma and Fang (2001) may be adopted to rank the nonregular combined designs for selecting a best foldover plan of a regular initial design.

It is known from Theorem 1 that for a fixed s^{k-p} initial design, there are in total $(s^p-1)/(s-1)$ distinct ways to generate a foldover design apart from the null foldover plan. Although the forms of these $(s^p-1)/(s-1)$ combined designs are completely different, some of them may be equivalent or isomorphic. It is very meaningful to clarify when the combined designs resulting from distinct foldover plans are equivalent or isomorphic. This is an open question, which is worth further investigation.

Acknowledgements

The authors are grateful to Prof. K.-T. Fang for his valuable comments and suggestions. The first author's work was partially supported by NNSF of China grant Nos. 10671007, 10571093. The second author's work was partially supported by Hong Kong Research Grants Council grant RGC/HKBU/2030/99P and by Hong Kong Baptist University grant FRG/00-01/II-62.

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