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Unified CUSUM Charts for Monitoring Process Mean and Variability

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Abstract: In this paper, we propose two new CUSUM control charts which are based on the probability integral transformation. The CUSUM M-chart is specifically designed for detecting small shifts in process mean, while the CUSUM V-chart is specifically designed for detecting small changes in process variability. They can be thought of as the CUSUM version of the $\bar{X}$-chart and S-chart. Besides that the proposed control charts can be readily extended to non-normal distributions, a special feature is that both charts can be easily plotted on a single chart to provide simultaneous monitoring of the process mean and variability. An example and simulations are used to compare the existing optimal CUSUM control charts with the proposed CUSUM control charts. It is demonstrated that the CUSUM M-chart, the CUSUM V-chart and the combined CUSUM M- and V-charts are comparable to the existing optimal CUSUM procedures.

Keywords: Cumulative sum, probability integral transformation, uniform distribution.

1. Introduction

The development of cumulative sum (CUSUM) control charts dates back to the work by Page [20]. Over the years, CUSUM control charts have proven (see, for example, Lowry *et al.* [12], Lucas [14] [15], and Hawkins and Olwell [8]) to be superior to the classical Shewhart control charts in the sense that the CUSUM control charts tend to have smaller Average Run Lengths (ARL's) particularly when small changes in the population parameters of the process have occurred. The existing CUSUM control charts can be categorized into two schemes. The first scheme computes the CUSUM of individual samples and then evaluates the CUSUM either based on some reference value or some test statistic derived from the CUSUM. Included in this group is the classical CUSUM control chart. The second scheme computes a certain statistic of individual samples, then calculates the CUSUM based on the statistic. Included in this group are the existing CUSUM control charts for monitoring the process variability such as those found in Johnson and Leone [9], Page [21], Hawkins [6] [7], Chang and Gan [3], Liu [11], Lowry *et al.* [12], Hawkins and Olwell [8] and Acosta-Mejia [1]. A good review of the existing procedures can be found in the paper by Zacks [29]. It is worth mentioning that the approach used in Liu's CUSUM control chart is non-parametric and thus does not require the normality assumption of the underlying distribution. Other existing CUSUM control charts for monitoring non-normal processes include, among others, the works by Ramalhoto and Morais [22] [23] and by Gordon and Pollak [5], which is based on the Shiryayev-Roberts procedure (Shiryayev [25] and Roberts [24]).

In the univariate case the optimal CUSUM procedures are the ones constructed based on the likelihood ratio (Moustakides [19]), in the sense that the likelihood-based CUSUM procedure is optimal for a specified change of interest in a population parameter. In this regard, the classical CUSUM control chart and the so-called CUSUM S-chart (see, for example, Lowry *et al.* [12]) are considered to be the optimal CUSUM control charts for monitoring respectively the process mean and process variability. Extending the idea of Yeh *et al.* [26] in using the probability integral transformation, we propose two new CUSUM control charts: the CUSUM M-chart and the CUSUM V-chart. The CUSUM M-chart is specifically designed for detecting small shifts in process mean and the CUSUM V-chart is specifically designed for detecting small changes in process variability. The combined CUSUM M- and V-charts can be considered as the CUSUM version of the $\overline{X}$- and S-charts. The basic idea here is to transform the test statistics used into $U(0,1)$, a uniform random variable supported on $(0,1)$, based on the probability integral transformation. The CUSUM is then calculated based on $U(0,1)$. In essence, the proposed CUSUM M-chart and the CUSUM V-chart fall in the second scheme mentioned earlier.

The idea of calculating the CUSUM of $U(0,1)$'s is similar to the CUSUM control chart proposed in Liu [11], although in Liu's CUSUM procedure, the $U(0,1)$'s are obtained from a very different perspective. Since the probability integral transformation is not limited to normal distributions, the CUSUM M-chart and the CUSUM V-chart can be applied to non-normal processes provided that the distributions of the test statistics used to test process mean and variability are known. Also note that since both charts are CUSUM of $U(0,1)$'s when the process is in control, there is no need to develop separate sets of control limits for the CUSUM M-chart and the CUSUM V-chart. In fact, other test statistics can be properly transformed into $U(0,1)$'s and the CUSUM procedures developed based on the $U(0,1)$'s will have the same in-control distribution, thus making designing CUSUM control charts for different test statistics a relatively easier task. Moreover, they can be plotted on one single chart to provide a mechanism for simultaneous monitoring of the process mean and variability from a CUSUM perspective. Thus, the combined CUSUM M- and V-charts provide a unified scheme in monitoring the process mean and variability. This is particularly useful when there are multiple processes or multiple streams of processes to be monitored, since all quality characteristics can be monitored in one single chart.

The rest of the paper is organized as follows. In Section 2, we describe in detail the construction of the CUSUM M-chart and the CUSUM V-chart. We apply the proposed control charts to an example in Section 3. The performance of the CUSUM M-chart and the CUSUM V-chart is compared to the existing optimal CUSUM control charts. In Section 4, we study the ARL of the proposed control charts through simulations along with comparisons to the existing optimal CUSUM procedures. In Section 5, some related issues and potential future research are discussed.

2. CUSUM M-Chart and CUSUM V-Chart

Let X be the random variable which represents a univariate quality characteristic of interest obtained from a process. We assume that when the process is in control X has a normal distribution with mean μ_0 and standard deviation σ_0, where μ_0 and σ_0 are unknown. To monitor the quality of the process, we take repeatedly independent samples of size n, X_{j1}, X_{j2}, ..., X_{jn}, $j \geq 1$. We also assume that μ_0 and σ_0 can be estimated from a set of k samples each with n observations obtained from the initial start-up stage, and the process was in-control when those samples were taken. Here we assume that $n > 1$ so that the sample standard deviation used in our proposed charts can be computed. Let $\overline{\overline{X}} = \sum_{i=1}^{k} \overline{X}_i / k$ and $\overline{S} = (\sum_{i=1}^{k} S_i^2 / k)^{1/2}$ be the estimates of μ_0 and σ_0 respectively obtained from the initial samples, where $\overline{X}_i$ and S_i^2 are the sample mean and sample variance of the ith sample, $i = 1,2,...,k$. Let $\overline{X}_j = \sum_{h=1}^{n} X_{jh}/n$ and $S_j^2 = \sum_{h=1}^{n} (X_{jh} - \overline{X}_j)^2/(n-1)$, $j \geq 1$, be the sample mean and sample variance of the jth sample when the monitoring begins. Note that, under normality assumption, $\overline{X}_j - \overline{\overline{X}}/\sigma_0\sqrt{1/n + 1/N} \sim N(0,1)$, $(N-k)\overline{S}^2/\sigma_0{}^2 \sim \chi_{N-k}^2$, and $(n-1)S_j^2/\sigma_0{}^2 \sim \chi_{n-1}^2$ for $j \geq 1$, where χ_f^2 denotes the chi-square distribution with f degrees of freedom, and all three random variables are independent. It can be shown, as can be found in most introductory mathematical statistics textbook (e.g., Mood *et al.* [18]), that if μ_0 and σ_0 remain unchanged, for $j \geq 1$,

$$\frac{\overline{X}_j - \overline{\overline{X}}}{\overline{S}\sqrt{\frac{1}{n} + \frac{1}{N}}} \sim t_{N-k} \quad \text{and} \quad \frac{S_j^2}{\overline{S}^2} \sim F_{n-1,N-k},$$

where $N = n \times k$. Here t_{N-k} and $F_{n-1,N-k}$ denote the t-distribution with $N-k$ degrees of freedom and F-distribution with $n-1$ and $N-k$ degrees of freedom, respectively. Let $G_t(\cdot)$ and $G_F(\cdot)$ denote the distribution functions of t_{N-k} and $F_{n-1,N-k}$, respectively. Now define, for $j \geq 1$,

$$m_j = G_t\left(\frac{\overline{X}_j - \overline{\overline{X}}}{\overline{S}\sqrt{\frac{1}{n} + \frac{1}{N}}}\right) \quad \text{and} \tag{1}$$

$$v_j = G_F\left(\frac{S_j^2}{\overline{S}^2}\right). \tag{2}$$

Note that both m_j and v_j are random variables since, take m_j for example, the quantity, $\overline{X}_j - \overline{\overline{X}}/\overline{S}\sqrt{1/n + 1/N}$, of which the cumulative probability of a t-distribution with $N-k$ degrees of freedom is to be evaluated is random. It is shown in Appendix A that, given n and k, not only both m_j and v_j are $U(0,1)$, but m_j and v_j are independent under the normality assumption. This leads us to define the CUSUM control charts based on m_j's and v_j's as follows. Let, for $t \geq 1$,

$$S_m(t) = \sum_{j=1}^{t}(m_j - \tfrac{1}{2}) = (m_t - \tfrac{1}{2}) + S_m(t-1) \text{ and} \tag{3}$$

$$S_v(t) = \sum_{j=1}^{t}(v_j - \tfrac{1}{2}) = (v_t - \tfrac{1}{2}) + S_v(t-1), \tag{4}$$

where $S_m(0) = S_v(0) = 0$. Note that m_j and v_j are essentially the probability integral transformation. The fact that they are distributed as $U(0,1)$ is not limited to normal processes, provided that the distributions of the test statistics used to test process mean ($\bar{X}_j - \bar{\bar{X}}/\bar{S}\sqrt{1/n+1/N}$ in this case) and process variability ($S_j^2/\bar{S}^2$ in this case) are known.

Note that when the process is in control, given n and k, $S_m(t)$ and $S_v(t)$ have the same distribution and they will be independent if the normality assumption holds. Furthermore, $S_m(t)$ can be used to detect small shifts in process mean and $S_v(t)$ can be used to detect small changes in process variability. The idea is that if there is a shift in process mean, say an increase in mean, then $(m_j - 1/2)$ tends to be positive, which in turn will gradually increase the value of $S_m(t)$. Therefore, we would reject the process and conclude that the process mean has increased if $S_m(t) > H_u$, where H_u is some pre-determined upper control limit. Similarly, we would reject the process and conclude that the process mean has decreased if $S_m(t) < H_l$, where H_l is some pre-determined lower control limit. As for $S_v(t)$, when there is an increase in process variance, $(v_j - 1/2)$ tends to be positive, which will gradually increase the value of $S_v(t)$. On the other hand, when there is a decrease in process variance, $(v_j - 1/2)$ tends to be negative, which will in turn gradually decrease the value of $S_v(t)$. However, the latter should be interpreted more carefully in that when $S_v(t)$ falls below the lower control limit, it signals a possible decrease in process variance, and thus a possible improvement in the process.

As suggested by one referee, unlike the classical CUSUM charting procedures, formulae (3) and (4) represent untruncated cumulative sums. There may be problems associated with our proposed CUSUM control charts in detecting, for example, oscillating shifts in process mean. However, our goal in this paper is to design CUSUM control charts for detecting sustained changes in population parameters in the same direction. In such a case, there is no significant difference between truncated and untruncated cumulative sums.

Also note that when the process is in control $S_m(t)$ is symmetric about 0 (see Appendix B), therefore we can choose $|H_l| = H_u = H$. Furthermore, since $S_m(t)$ and $S_v(t)$ have the same distribution, the same control limit H can also be used in the $S_v(t)$-based control chart. Moreover, $S_m(t)$ and $S_v(t)$ can be plotted on one single chart. It should be noted that if the original observations are standardized to begin with, the distribution of m_j as $U(0,1)$, and thus $S_m(t)$, will not be affected. Therefore, we do not distinguish the control limit H between unstandardized and standardized observations. Also note that when the process is out of control, m_j and v_j are no longer distributed as $U(0,1)$. In fact, as explained earlier, it is this change in the behaviors of m_j and v_j that helps to signal the change in process parameters.

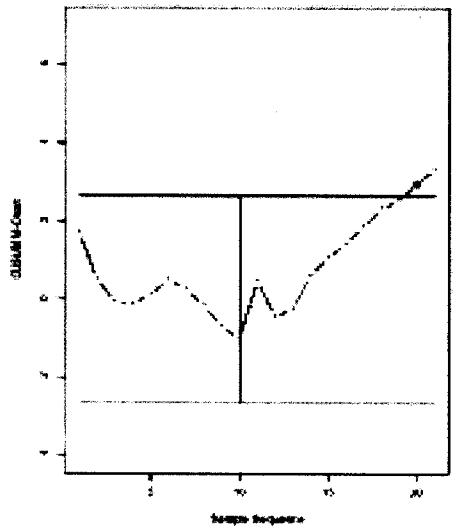

Figure 1a. The CUSUM M-chart with a mean shift starting at sample 11.

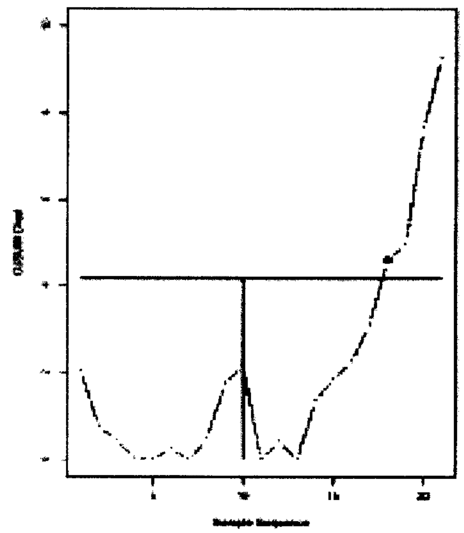

Figure 1b. The CUSUM chart with a mean shift starting at sample 11.

We shall call the $S_m(t)$-based control chart the CUSUM M-chart, the $S_v(t)$-based control chart the CUSUM V-chart, and the combined $S_m(t)$ and $S_v(t)$-based control charts the CUSUM M- and V-charts. Thus, the CUSUM M- and V-charts can be thought of as the CUSUM version of the $\overline{X}$- and S-charts. Note that when the subgroup size varies, we can easily modify the test statistics used in (1) and (2) to get the corresponding m_j's and v_j's, and consequently, $S_m(t)$ and $S_v(t)$ can be computed accordingly.

It is clear that, given n and k, when the process is in control $S_m(t)$ (likewise $S_v(t)$) is a sum of t independent and identically distributed (iid) $U(-1/2,1/2)$ random variables. When setting the control limits we suggest that $S_m(t)$ (likewise $S_v(t)$) be normalized by rescaling the statistic defined in (3) by a multiple of $\sqrt{12}$, i.e., $\sqrt{12} \times S_m(t)$, since the variance of $U(-1/2,1/2)$ is $1/12$.

The sensitivity of the CUSUM M-chart or the CUSUM V-chart in detecting changes in population parameters depends on H as well as the type of changes in population parameters the chart is designed to detect. For any given H and fixed changes in population parameters, the evaluation of the average run length (ARL) for either chart is theoretically complicated. However, the in-control ARL can be evaluated based on the Markov chain approximation proposed in Brook and Evans [2]. Here the run length is defined as the number of samples that must be taken until the first out-of-control signal is detected on a control chart. In Appendix C, we discuss how the Markov chain approximation is used to determine the in-control ARL's for various control limits. Table 1 gives the one-sided and two-sided in-control ARL's under various values of H for CUSUM M-chart or CUSUM V-chart. The two-sided in-control ARL's for the combined CUSUM M- and V-charts are given in the last column of Table 1. Note that, given n and k, the CUSUM M-chart and the CUSUM V-chart are independent and have the same distribution. Therefore, the in-control ARL of the combined CUSUM

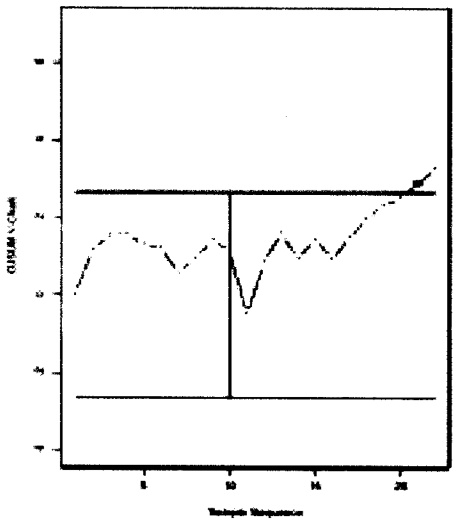

Figure 2a. The CUSUM V-chart with an increase in variability starting at sample 11.

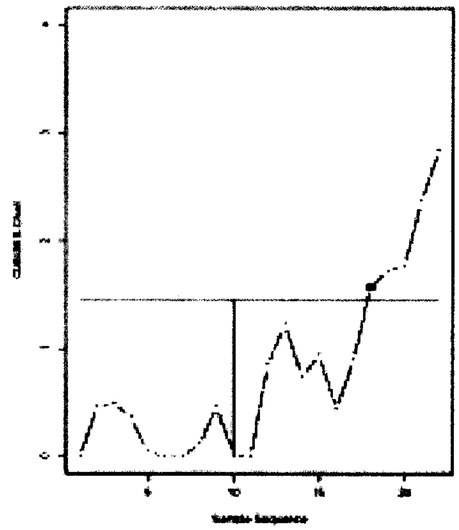

Figure 2b. The CUSUM S-chart with an increase in variability starting at sample 11.

M- and V-charts is approximately equal to half of the in-control ARL of individual charts. As for the in-control ARL of the two-sided chart, it is equal to half of the in-control ARL of the one-sided chart since $S_m(t)$, say, is symmetric for any given t.

It should also be noted that the Markov chain approach can not be applied to evaluating the out-of-control ARL of the CUSUM M-chart and CUSUM V-chart. This is due to the fact that, in our proposed setting, m_j and v_j are no longer distributed as $U(0,1)$ when the process is out of control, and further, the magnitude of change in population parameters can not be directly translated into a specific magnitude of change in either m_j or v_j, which is needed if the Markov chain approximation is to be used.

3. An Example

The example is taken from the data in Montgomery [17]. The measurements are related to the inside diameters of forged piston rings. We consider the original 25 samples each of size 5 as the samples used in setting up the control charts. Note that all 25 samples are in control with $\overline{\overline{X}}$ = 74.001 and $\overline{S}$ = 0.009.

Treating the in-control process as having a normal distribution with μ_0 = 74.001 and σ_0 = 0.009, we then generate repeated samples. The first ten samples are generated from $N(74.001, 0.009)$. Starting from sample 11, we apply different changes to the distribution. We first shift the mean from 74.001 to 74.0055 (equivalent to a mean shift of $0.5 \times \sigma_0$). Using H = 2.6566 which results in an in-control ARL of approximately 365, equivalent to the in-control ARL of a 3-σ Shewhart control chart, the corresponding CUSUM M-chart and CUSUM control chart are shown in Figure 1a and Figure 1b, respectively. The CUSUM control chart shown here is based on a graphical display of the tabular CUSUM. That is, we plot $\max(C_i^+, C_i^-)$, where $C_i^+ = \max(0,$

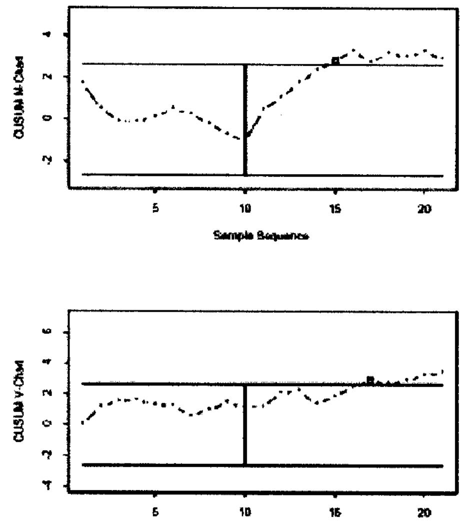

Figure 3a. The CUSUM M- and V-charts with both changes in mean and variability starting at sample 11.

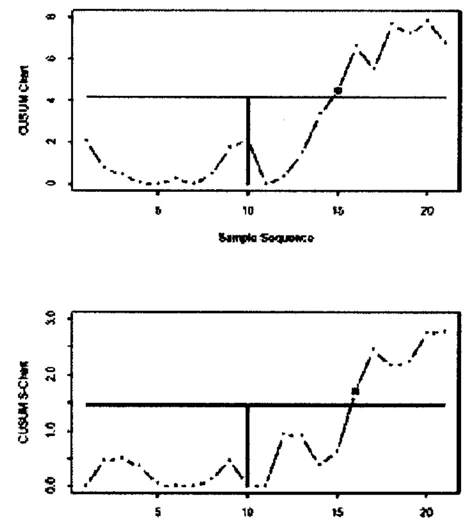

Figure 3b. The CUSUM and CUSUM S-charts with both changes in mean and variability starting at sample 11.

$Z_l - k_1 + C^+_{l-1})$, $C^-_l = \max(0, -Z_l - k_1 + C^-_{l-1})$, $Z_l = \sqrt{n}(\bar{X}_l - \mu_0)/\sigma_0$, k_1 is the reference value, and $C^+_0 = C^-_0 = 0$. The CUSUM control chart detects the shift on sample 18, while the CUSUM M-chart detects the shift on sample 20.

Next, while keeping the process mean at 74.001, we change, starting from sample 11, the standard deviation from 0.009 to 0.0135, which amounts to a 50% increase in process standard deviation. The CUSUM V-chart shown in Figure 2a indicates that an out-of-control signal is detected on sample 21. On the other hand, as indicated in Figure 2b, an out-of-control signal shows up on CUSUM S-chart on sample 18. Here the CUSUM S-chart is computed based on $Q_l = \max(0, S_l - k_2 + Q_{l-1})$, where S_l is the sample standard deviation of the lth sample, k_2 is the reference value, and $Q_0 = 0$.

Shown in Figure 3a and Figure 3b respectively are the corresponding CUSUM M- and V-charts and the combined CUSUM and CUSUM S-charts when the distribution of the process has been changed to $N(74.0055, 0.0135)$, again starting from sample 11. Note that the combined charts have an in-control ARL of approximately 183. Out-of-control signals are detected on samples 15 and 17 on the CUSUM M-chart and on the CUSUM V-chart, respectively. At the same time, out-of-control signals also appear on samples 15 and 16 respectively on the CUSUM chart and on the CUSUM S-chart. Both combined charts clearly show that both process mean and process variability are out-of-control. It should be noted that although in all three cases the classical CUSUM control charts provide earlier detection, the diagnostics resulted from the CUSUM M-chart and CUSUM V-chart are comparable.

Since the CUSUM M-chart and the CUSUM V-chart are on the same scale, they can be plotted on one single chart. Shown in Figure 4 is a graph redrawn from Figure 3a by combining the CUSUM M- and V-charts on a single chart. The symbols "m" and "v" represent, respectively, the process mean and process variability. The boxed "m" on

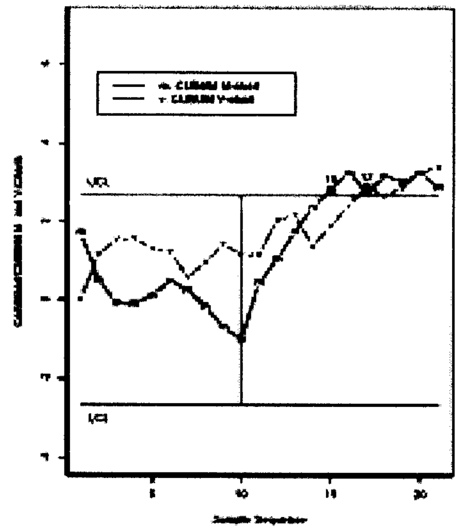

Figure 4. The combined single cusum control chart.

sample 15 indicates that the process mean is out-of-control on sample 15, and the circled "v" on sample 17 indicates that the process variability is also out-of-control on sample 17. Therefore, plotting CUSUM M- and V-charts on a single chart provides a mechanism, from a CUSUM perspective, for simultaneous monitoring of the process mean and process variability. Moreover, after proper probability integral transformation, other test statistics can be readily transformed into $U(0,1)$'s, and the CUSUM control charts developed accordingly can be plotted on the same plot. In the case when multiple processes or multiple streams of processes need to be monitored, there is a potential in using our approach to significantly reduce the total number of control charts to be monitored.

All the computer codes used to conduct the example and simulations (see Section 4) were written in S-plus. These codes can be made available by contacting the first author.

4. Numerical Studies

In this section, we compare the performance, defined in terms of the ARL, of the proposed CUSUM M-chart and CUSUM V-chart with the existing optimal CUSUM control charts for univariate processes. The comparison is carried out using Monte Carlo simulation. Here we define the run length as the number of samples that must be taken until the first out-of-control signal is detected on a control chart. The simulation run size is 5,000, and the subgroup sample sizes used are $n = 4$, 6, 8 and 10. Here we use H = 2.6566 and 2.6583 (Table 1) which result in in-control ARL's of approximately 365 and 500, respectively, for individual CUSUM M-chart or CUSUM V-chart. This leads to setting the probabilities of detecting an out-of-control signal for an in-control process to be approximately 0.0027 and 0.002, respectively. All the existing CUSUM control charts considered here are calibrated to have approximately the same in-control ARL's. For the CUSUM chart (see Section 3), the reference value is set at 0.5 and the control limits are set at 4.77 and 5.10 for in-control ARL's of approximately 365 and 500, respectively. As

Table 1. The in-control ARL's for various control limits *H*.

H	CUSUM M (V) (one-sided ARL)	CUSUM M (V) (two-sided ARL)	CUSUM M and V (two-sided ARL)
2.6178	100.44	50.22	25.11
2.6403	200.80	100.40	50.20
2.6476	296.93	148.47	74.24
2.6514	396.46	198.23	99.12
2.6538	504.04	252.02	126.01
2.6566	730.77	365.39	182.70
2.6583	1001.08	500.54	250.27

for the CUSUM S-chart (see Section 3), the reference value is set at 1.18 and the control limits are chosen to be 1.45 and 1.56, respectively, for in-control ARL's of approximately 365 and 500. In simulating the out-of-control ARL's, once a sample is generated it is used to evaluate all the competing CUSUM control charts.

Without loss of generality, we assume that the in-control process is distributed as $N(0,1)$. We then consider three possible changes in the process: mean shift, change in process variability, and both changes in process mean and process variability. Note that in practice when μ_0 and σ_0 are estimated from k samples taken when the process was in control, the out-of-control ARL will depend on k. Summarized in Table 2 and Table 3 are the simulated out-of-control ARL's for CUSUM M-chart and the classical CUSUM control chart. The numbers that appear in the parentheses are the standard errors of the simulated ARL's. Note that the classical CUSUM control chart is expected to have smaller ARL's since it is based on the sequential probability ratio test (Moustakides [19]). Nevertheless, judging from the results in Tables 2 and 3 and using $3\times$(standard error) as a guideline, when $0<\mu<1$ the CUSUM control chart has better performance, and when $1<\mu<2$ the CUSUM M-chart has slightly better performance in most cases considered. However, except for a very small shift in process mean $(\mu=0.25)$, these two charts are very comparable from a practical standpoint. The difference in performance between the two control charts diminishes as the sample size increases. It should be noted, however, that subgroup sizes of 8 or 10 may be considered large for most control charts applications.

In the case of changes in process variability, the CUSUM V-chart is compared to the so-called CUSUM S-chart (e.g., see Lowry *et al.* [12]). The CUSUM S-chart, computed based on the CUSUM of the sample standard deviation, is considered to be the optimal CUSUM procedure for monitoring the process variability. Note that our main interest here is to evaluate the performance of two competing CUSUM control charts in detecting increases in process variability which correspond to deterioration of quality. Although from the standpoint of continuous process improvement, decreases in process variability should also be carefully investigated. The simulated results are summarized in Table 4 and Table 5. For small increases in σ (between 10% and 30% increase), the CUSUM S-chart has a clear advantage over the CUSUM V-chart. As the increase in σ widens (between 30% and 100% increase), even though the CUSUM S-chart performs

Table 2. Comparisons of ARL for mean shifts ($\sigma = 1$, in-control ARL = 365).

μ	CUSUM M-chart	CUSUM chart	μ	CUSUM M-chart	CUSUM chart
($n = 4$)			($n = 8$)		
0.25	33.58 (0.3152)	27.87 (0.3168)	0.25	18.23 (0.1686)	15.45 (0.1535)
0.50	9.88 (0.0816)	8.70 (0.0673)	0.50	5.68 (0.0429)	5.36 (0.0330)
0.75	5.13 (0.0371)	4.92 (0.0283)	0.75	3.27 (0.0195)	3.14 (0.0151)
1.00	3.37 (0.0216)	3.46 (0.0615)	1.00	2.21 (0.0112)	2.40 (0.0093)
1.25	2.55 (0.0144)	2.73 (0.0116)	1.25	1.77 (0.0081)	1.97 (0.0069)
1.50	2.08 (0.0098)	2.27 (0.0083)	1.50	1.44 (0.0073)	1.69 (0.0068)
1.75	1.78 (0.0080)	1.99 (0.0067)	1.75	1.19 (0.0056)	1.39 (0.0069)
2.00	1.55 (0.0075)	1.79 (0.0067)	2.00	1.05 (0.0032)	1.16 (0.0052)
($n = 6$)			($n = 10$)		
0.25	23.48 (0.2146)	19.69 (0.2045)	0.25	15.06 (0.1323)	12.83 (0.1186)
0.50	7.14 (0.0517)	6.52 (0.0435)	0.50	4.72 (0.0344)	4.60 (0.0259)
0.75	3.85 (0.0264)	3.49 (0.0196)	0.75	2.88 (0.0162)	2.71 (0.0127)
1.00	2.59 (0.0146)	2.78 (0.0118)	1.00	1.97 (0.0090)	2.17 (0.0077)
1.25	2.02 (0.0094)	2.22 (0.0079)	1.25	1.58 (0.0076)	1.81 (0.0071)
1.50	1.71 (0.0078)	1.92 (0.0068)	1.50	1.25 (0.0062)	1.49 (0.0065)
1.75	1.42 (0.0071)	1.67 (0.0068)	1.75	1.07 (0.0036)	1.21 (0.0057)
2.00	1.21 (0.0056)	1.43 (0.0070)	2.00	1.01 (0.0015)	1.05 (0.0031)

better, the CUSUM V-chart becomes practically comparable to the CUSUM S-chart. The difference in performance between the two control charts also becomes smaller as n increases. Moreover, judging from the results of Table 3 in Chang and Gan [3], the CUSUM V-chart is also comparable to the CUSUM control chart based on $\log(s^2)$ in detecting increases in process variability.

As for changes both in process mean and process variability, the combined CUSUM M- and V-charts are compared to the combined CUSUM and CUSUM S-charts. The results for sample sizes $n = 4$ and 8 are summarized in Tables 6 and 7, respectively. Except when changes in mean and variability are both small, in which case the combined CUSUM and CUSUM S-charts perform better, the two combined control charts have comparable ARL's. Again, the overall difference decreases as n increases.

The merits of the proposed CUSUM M-chart and CUSUM V-chart rest on two features. One is that, after proper probability integral transformation, different test statistics can all be transformed into $U(0,1)$. The CUSUM procedures developed accordingly will have the same in-control distribution, thus making designing CUSUM control charts for different test statistics a relatively easier task. Secondly, these CUSUM charts can be plotted on a single plot (see Figure 4), thus making it possible the simultaneous monitoring, from the CUSUM perspective, of different population parameters on a single control chart. In this regard, there is a potential to significantly reduce the total number of control charts that need to be monitored, especially when there are multiple processes or multiple streams of processes to be monitored.

Table 3. Comparisons of ARL for mean shifts ($\sigma = 1$, in-control ARL = 500).

μ	CUSUM M-chart	CUSUM chart	μ	CUSUM M-chart	CUSUM chart
($n = 4$)			($n = 8$)		
0.25	35.89 (0.3272)	31.39 (0.3641)	0.25	19.08 (0.1692)	16.24 (0.1575)
0.50	10.36 (0.0837)	9.21 (0.0696)	0.50	5.92 (0.0437)	5.58 (0.0328)
0.75	5.44 (0.0401)	5.22 (0.0303)	0.75	3.41 (0.0207)	3.28 (0.0158)
1.00	3.51 (0.0221)	3.60 (0.0166)	1.00	2.35 (0.0102)	2.50 (0.0097)
1.25	2.67 (0.0137)	2.85 (0.0117)	1.25	2.02 (0.0062)	2.06 (0.0065)
1.50	2.23 (0.0085)	2.35 (0.0086)	1.50	1.78 (0.0060)	1.81 (0.0063)
1.75	2.03 (0.0061)	2.07 (0.0066)	1.75	1.53 (0.0071)	1.50 (0.0070)
2.00	1.87 (0.0057)	1.88 (0.0061)	2.00	1.27 (0.0063)	1.24 (0.0060)
($n = 6$)			($n = 10$)		
0.25	25.26 (0.2262)	21.82 (0.2326)	0.25	15.83 (0.1355)	13.54 (0.1220)
0.50	7.55 (0.0590)	6.88 (0.0462)	0.50	5.08 (0.0359)	4.88 (0.0268)
0.75	4.04 (0.0266)	3.99 (0.0201)	0.75	2.99 (0.0157)	2.83 (0.0125)
1.00	2.73 (0.0145)	2.91 (0.0121)	1.00	2.16 (0.0076)	2.25 (0.0078)
1.25	2.21 (0.0084)	2.31 (0.0082)	1.25	1.88 (0.0056)	1.88 (0.0061)
1.50	1.97 (0.0057)	1.99 (0.0062)	1.50	1.61 (0.0069)	1.58 (0.0070)
1.75	1.79 (0.0062)	1.77 (0.0065)	1.75	1.29 (0.0066)	1.27 (0.0063)
2.00	1.56 (0.0071)	1.52 (0.0070)	2.00	1.11 (0.0044)	1.09 (0.0040)

The proposed CUSUM control charts act very much like a CUSUM scheme with the reference value k set to be equal to 0. Take, for example, the CUSUM V-chart defined in (4). The CUSUM in (4) can be rewritten as

$$S_v(t) = (v_t - \frac{1}{2}) - k + S_v(t-1),$$

where $S_v(0) = 0$ and $k = 0$. Not only does the CUSUM with $k = 0$ tend to require a larger decision value H than other CUSUM's with positive k values in order to achieve the same in-control ARL, it also tends to have larger out-of-control ARL's. For instance, in order to achieve an in-control ARL of 2000, $H = 2.6617$ is required for the CUSUM V-chart with $k = 0$, whereas $H = 1.9103$ is needed for the CUSUM V-chart with $k = 0.1$. Here both H values are calculated based on the Markov chain approximation outlined in Appendix C. Note that for the case when $k = 0.1$, it is tantamount to shifting the uniform distribution to the left by 0.1. The simulated out-of-control ARL's ($n = 4$) of the two corresponding CUSUM V-charts are summarized in Table 8. It is evident from Table 8 that the CUSUM V-chart with $k = 0.1$ has better performance than the one with $k = 0$.

It should also be pointed out that, in the present paper, we are not able to provide specific guidance on how the value of k should be chosen, particularly in relation to the change in parameter of interest. This is due to the fact that we are not able to convert the changes in population parameters to the corresponding changes in the transformed

Table 4. Comparisons of ARL for increases in variability ($\mu = 0$, in-control ARL = 365).

σ	CUSUM V-chart	CUSUM S-chart	σ	CUSUM V-chart	CUSUM S-chart
($n = 4$)			($n = 8$)		
1.10	155.04 (1.7114)	77.76 (1.0518)	1.10	66.16 (0.7059)	49.91 (0.7252)
1.20	46.54 (0.4966)	29.49 (0.3765)	1.20	19.76 (0.2006)	14.18 (0.1928)
1.30	23.26 (0.2441)	15.18 (0.1774)	1.30	10.47 (0.1035)	7.32 (0.0844)
1.40	14.83 (0.1587)	9.71 (0.1057)	1.40	6.54 (0.0645)	4.56 (0.0480)
1.50	10.55 (0.1139)	7.02 (0.0701)	1.50	4.73 (0.0486)	3.43 (0.0323)
1.60	7.98 (0.0857)	5.52 (0.0516)	1.60	3.67 (0.0376)	2.66 (0.0237)
1.80	5.38 (0.0587)	3.89 (0.0330)	1.80	2.47 (0.0253)	1.79 (0.0159)
2.00	4.11 (0.0457)	3.09 (0.0250)	2.00	1.90 (0.0188)	1.33 (0.0118)
($n = 6$)			($n = 10$)		
1.10	92.13 (1.0115)	61.39 (0.8401)	1.10	50.41 (0.5091)	44.14 (0.5928)
1.20	27.63 (0.2920)	20.37 (0.2532)	1.20	15.67 (0.1557)	13.05 (0.1603)
1.30	13.96 (0.1447)	10.18 (0.1128)	1.30	8.07 (0.0791)	6.47 (0.0674)
1.40	8.98 (0.0913)	6.60 (0.0649)	1.40	5.23 (0.0507)	4.07 (0.0359)
1.50	6.35 (0.0641)	4.73 (0.0430)	1.50	3.81 (0.0367)	3.08 (0.0256)
1.60	5.01 (0.0523)	2.32 (0.0329)	1.60	2.89 (0.0281)	2.44 (0.0180)
1.80	3.35 (0.0357)	2.74 (0.0212)	1.80	2.03 (0.0196)	1.83 (0.0129)
2.00	2.49 (0.0269)	2.17 (0.0156)	2.00	1.55 (0.0137)	1.48 (0.0094)

uniform distributions. Furthermore, for the classical CUSUM control chart, if two one-sided CUSUM charts are used in a two-sided scheme designed to detect small mean shifts, using a reference value larger than half the shift gives better performance than the optimal one-sided procedure (with reference value equal to half the shift). However, we do not know whether our proposed CUSUM procedures will have similar performance. Future research along these lines would be worthwhile.

Throughout the discussion, we assume that the process starts at the target value after it is being reset. However, this is not always the case in practice. One way to increase the sensitivity of the proposed CUSUM V-chart, particularly at process start-up is to combine the CUSUM V-chart with the Shewhart-type S-chart. Although the combined control charts, if not adjusted, will result in a different in-control ARL than those of individual control charts. The other approach is to incorporate the fast initial response (FIR) feature (Lucas and Crosier [16]) in designing the CUSUM V-chart. Based on the Markov chain approximation, if we set $S_v(0) = 0.05$ and H = 2.6583, for instance, the in-control ARL is approximately 500, since such a small starting value 0.05 does not alter the starting state in the Markov chain. Summarized in Table 9 are the corresponding out-of-control ARL's (n = 4) for the CUSUM V-chart with a headstart at $S_v(0) = 0.05$. The ARL's in Table 9, as compared to those summarized in Table 5, indicate that even with such a small headstart the performance of the CUSUM V-chart is improved. Note that due to very much the same reason mentioned earlier, we are not able to provide insights on choosing the headstart values. Further studies are needed.

Table 5. Comparisons of ARL for increases in variability ($\mu = 0$, in-control ARL = 500).

σ	CUSUM V-chart	CUSUM S-chart	σ	CUSUM V-chart	CUSUM S-chart
($n = 4$)			($n = 8$)		
1.10	168.69 (1.8139)	93.74 (1.2526)	1.10	70.83 (0.7025)	59.14 (0.8016)
1.20	50.34 (0.5199)	33.12 (0.4141)	1.20	20.98 (0.2267)	17.63 (0.2137)
1.30	25.52 (0.2612)	16.52 (0.1903)	1.30	10.66 (0.1041)	8.28 (0.0861)
1.40	15.74 (0.1611)	10.43 (0.1122)	1.40	6.95 (0.0690)	5.27 (0.0496)
1.50	11.26 (0.1182)	7.48 (0.0721)	1.50	4.99 (0.0496)	3.92 (0.0341)
1.60	8.51 (0.0912)	5.78 (0.0539)	1.60	3.87 (0.0395)	3.09 (0.0245)
1.80	5.67 (0.0619)	4.12 (0.0350)	1.80	2.62 (0.0269)	2.27 (0.0162)
2.00	4.28 (0.0481)	3.22 (0.0260)	2.00	1.96 (0.0199)	1.81 (0.0120)
($n = 6$)			($n = 10$)		
1.10	100.05 (1.0450)	74.45 (1.0033)	1.10	55.68 (0.5497)	52.88 (0.7106)
1.20	29.52 (0.2989)	22.59 (0.2915)	1.20	16.55 (0.1594)	14.68 (0.1764)
1.30	15.21 (0.1534)	10.93 (0.1206)	1.30	8.48 (0.0801)	6.92 (0.0711)
1.40	9.52 (0.0933)	7.06 (0.0692)	1.40	5.58 (0.0541)	4.41 (0.0401)
1.50	6.72 (0.0680)	4.95 (0.0438)	1.50	3.98 (0.0397)	3.21 (0.0264)
1.60	5.19 (0.0526)	3.96 (0.0328)	1.60	3.13 (0.0309)	2.60 (0.0195)
1.80	3.48 (0.0375)	2.83 (0.0222)	1.80	2.08 (0.0202)	1.90 (0.0130)
2.00	2.62 (0.0282)	2.26 (0.0164)	2.00	1.59 (0.0144)	1.52 (0.0096)

Before closing this Section, it should be noted that performance measures other than ARL for comparing different control charting procedures exist and have been studied in the literature. One such alternative measure, whose numerical evaluations are easy to carry out despite its complex analytical properties, is the conditional expected delay (Kenett and Zacks [10]). It would be worthwhile to further investigate whether the proposed CUSUM M-chart and CUSUM V-chart will prove to be more advantageous under conditional expected delay evaluation criterion.

5. Conclusions and Discussion

Based on the probability integral transformation, we have proposed and studied two new CUSUM control charts. As compared to the classical Shewhart-type control charts, the CUSUM M-chart is designed to detect small shifts in process mean and the CUSUM V-chart is designed to detect small changes in process variability. The combined CUSUM M- and V-charts are considered to be the CUSUM version of the $\overline{X}$ - and S -charts. We have demonstrated through an example and simulations that except in the case of very small changes in population parameters, the CUSUM M-chart, the CUSUM V-chart and the combined CUSUM M- and V-charts are comparable to the existing optimal CUSUM control charts. To better understand the causes of any out-of-control signal, we suggest that other existing control charts be used in conjunction with the CUSUM M- and V-charts to provide a mechanism that offers faster detection and better diagnostics of any out-of-control process.

Table 6. Comparisons of ARL for changes both in mean and variability (in-control ARL = 183).

μ / σ	CUSUM M- and V-charts ($n = 4$)	($n = 8$)	CUSUM M- and S-charts ($n = 4$)	($n = 8$)
0.25/1.10	32.44 (0.3465)	17.14 (0.1683)	21.52 (0.2443)	13.04 (0.1274)
0.25/1.30	17.38 (0.1925)	8.23 (0.0827)	10.36 (0.1046)	6.15 (0.0558)
0.25/1.60	7.03 (0.0779)	3.35 (0.0340)	4.80 (0.0402)	2.81 (0.0210)
0.25/2.00	3.59 (0.0422)	1.80 (0.0173)	2.82 (0.0218)	1.71 (0.0113)
0.75/1.10	5.55 (0.0518)	3.26 (0.0265)	5.12 (0.0316)	3.40 (0.0171)
0.75/1.30	5.44 (0.0442)	3.20 (0.0231)	4.66 (0.0317)	3.11 (0.0176)
0.75/1.60	4.50 (0.0496)	2.42 (0.0221)	3.59 (0.0258)	2.32 (0.0144)
0.75/2.00	2.97 (0.0341)	1.58 (0.0138)	2.52 (0.0181)	1.61 (0.0098)
1.25/1.10	2.75 (0.0247)	1.78 (0.0089)	2.85 (0.0150)	2.05 (0.0075)
1.25/1.30	2.68 (0.0232)	1.78 (0.0106)	2.84 (0.0144)	2.03 (0.0085)
1.25/1.60	2.63 (0.0215)	1.64 (0.0117)	2.61 (0.0138)	1.81 (0.0094)
1.25/2.00	1.96 (0.0168)	1.37 (0.0097)	2.13 (0.0130)	1.46 (0.0082)
1.50/1.10	2.17 (0.0195)	1.46 (0.0075)	2.39 (0.0122)	1.77 (0.0068)
1.50/1.30	2.16 (0.0186)	1.46 (0.0082)	2.36 (0.0121)	1.71 (0.0075)
1.50/1.60	2.14 (0.0143)	1.42 (0.0091)	2.27 (0.0107)	1.61 (0.0083)
1.50/2.00	1.96 (0.0117)	1.25 (0.0076)	1.98 (0.0094)	1.36 (0.0074)
2.00/1.10	1.55 (0.0079)	1.07 (0.0036)	1.85 (0.0068)	1.24 (0.0061)
2.00/1.30	1.56 (0.0089)	1.10 (0.0043)	1.84 (0.0078)	1.28 (0.0064)
2.00/1.60	1.56 (0.0110)	1.12 (0.0048)	1.77 (0.0089)	1.27 (0.0063)
2.00/2.00	1.51 (0.0120)	1.11 (0.0048)	1.60 (0.0095)	1.20 (0.0057)

Indeed, the proposed CUSUM M-chart and CUSUM V-chart do not outperform the classical CUSUM and EWMA control charts. However, the proposed control charts have certain advantages (apart from the ARL assessment):

(i) Since the CUSUM M-chart and CUSUM V-chart have the same distribution when the process is in-control, there is no need to develop separate control limits. Furthermore, they can be plotted onto the same control chart, thus providing a mechanism for simultaneous monitoring, on a single control chart, of the process mean and process variability from the CUSUM perspective. This can significantly reduce the number of control charts that need to be monitored when there exist multiple streams of process.

(ii) The extension of the proposed control charts to the case of non-normal processes should also be noted. Under non-normal processes, if a proper probability integral transformation can be derived, then a similar CUSUM chart, i.e., CUSUM of properly normalized uniform random variables, can readily be defined.

Table 7. Comparisons of ARL for changes both in mean and variability (in-control ARL = 250).

μ / σ	CUSUM M- and V-charts ($n = 4$)	CUSUM M- and V-charts ($n = 8$)	CUSUM M- and S-charts ($n = 4$)	CUSUM M- and S-charts ($n = 8$)
0.25/1.10	34.33 (0.3360)	18.35 (0.1674)	22.72 (0.2460)	13.60 (0.1312)
0.25/1.30	18.53 (0.2009)	8.80 (0.0867)	10.67 (0.1090)	6.40 (0.0564)
0.25/1.60	7.62 (0.0841)	3.62 (0.0363)	5.08 (0.0438)	2.97 (0.0222)
0.25/2.00	3.88 (0.0439)	1.89 (0.0183)	2.97 (0.0225)	1.77 (0.0117)
0.75/1.10	5.97 (0.0551)	3.45 (0.0278)	5.19 (0.0333)	3.41 (0.0180)
0.75/1.30	5.75 (0.0515)	3.44 (0.0233)	4.83 (0.0324)	3.17 (0.0169)
0.75/1.60	4.93 (0.0453)	2.65 (0.0231)	3.74 (0.0261)	2.40 (0.0144)
0.75/2.00	3.29 (0.0374)	1.71 (0.0151)	2.63 (0.0186)	1.67 (0.0102)
1.25/1.10	2.95 (0.0251)	2.03 (0.0073)	2.87 (0.0152)	2.06 (0.0073)
1.25/1.30	2.94 (0.0249)	2.02 (0.0096)	2.83 (0.0138)	2.02 (0.0085)
1.25/1.60	2.77 (0.0210)	1.85 (0.0122)	2.68 (0.0140)	1.84 (0.0095)
1.25/2.00	2.48 (0.0162)	1.48 (0.0101)	2.24 (0.0127)	1.51 (0.0086)
1.50/1.10	2.41 (0.0194)	1.80 (0.0094)	2.39 (0.0121)	1.78 (0.0082)
1.50/1.30	2.41 (0.0183)	1.74 (0.0084)	2.39 (0.0119)	1.73 (0.0076)
1.50/1.60	2.32 (0.0143)	1.62 (0.0079)	2.29 (0.0109)	1.63 (0.0074)
1.50/2.00	2.15 (0.0109)	1.35 (0.0064)	2.03 (0.0096)	1.39 (0.0067)
2.00/1.10	1.87 (0.0065)	1.29 (0.0064)	1.85 (0.0067)	1.26 (0.0062)
2.00/1.30	1.87 (0.0086)	1.31 (0.0066)	1.86 (0.0078)	1.29 (0.0063)
2.00/1.60	1.85 (0.0117)	1.29 (0.0066)	1.82 (0.0091)	1.29 (0.0065)
2.00/2.00	1.73 (0.0134)	1.18 (0.0059)	1.71 (0.0097)	1.21 (0.0059)

(iii) The same methodology can also be extended to multivariate processes, provided that one can derive the proper probability integral transformation. Regardless of the case being considered, the CUSUM is essentially calculated based on a sequence of properly normalized uniform random variables.

The idea of using the probability integral transformation is very flexible in that it can be generalized to multivariate processes as well as to other control charts. One such appealing generalization is the exponentially weighted moving average (EWMA) control chart. Recently, Crowder and Hamilton [4] have proposed a univariate EWMA control chart to monitor the process variability, and Lowry *et al.* [13] have generalized the univariate EWMA control chart for monitoring process mean to multivariate cases. We can easily define the EWMA based on the proposed probability integral transformations m_j and v_j discussed in Section 2. For instance, one can define the EWMA based on the m_j's by calculating, say, $W_m(t) = \lambda(m_t - 1/2) + (1-\lambda)W_m(t-1)$, where $W_m(0) = 0$ and $0 < \lambda < 1$ is a smoothing constant. All the advantages and flexibility that CUSUM M-chart and CUSUM V-chart enjoy should hold for the EWMA extension. Similarly, we can define the EWMA M-and V-charts, the EWMA version of the $\overline{X}$-and S-charts.

Table 8. Comparisons of ARL for CUSUM V-chart with different k's ($n = 4$, $\mu = 0$, in-control ARL = 2000).

σ	$k = 0$	$k = 0.1$
($n = 4$)		
1.10	253.36 (2.2406)	204.65 (2.2363)
1.20	70.63 (0.6317)	57.16 (0.6472)
1.30	35.14 (0.3180)	26.39 (0.3232)
1.40	22.23 (0.2079)	15.93 (0.2055)
1.50	15.35 (0.1459)	10.85 (0.1466)
1.60	11.35 (0.1125)	7.99 (0.1127)
1.80	7.47 (0.0778)	4.88 (0.0738)
2.00	5.48 (0.0596)	3.49 (0.0541)

Table 9. ARL for CUSUM V-chart with a headstart at 0.05 ($n = 4$, $\mu = 0$, in-control ARL = 500).

σ	CUSUM V-chart ($S_v(0) \equiv 0.05$)
1.10	150.45 (1.9331)
1.20	42.40 (0.5454)
1.30	19.98 (0.2687)
1.40	12.67 (0.1718)
1.50	8.82 (0.1235)
1.60	6.41 (0.0934)
1.80	4.11 (0.0612)
2.00	3.08 (0.0447)

We are currently working on the EWMA extension based on the probability integral transformation. The results of our investigation will hopefully be reported in a follow-up paper. As for the possible multivariate extension of the proposed CUSUM M-chart and CUSUM V-chart, assuming that the in-control process follows $N_p(\mu_0, \Sigma_0)$, a p-dimensional normal distribution with mean vector μ_0 and variance-covariance matrix Σ_0, one can define m_j, for instance, based on $m_j = P(\chi_p^2 \le (\bar{X} - \mu_0)' \times \Sigma_0^{-1}(\bar{X} - \mu_0))$, where $\bar{X}$ is the sample mean vector. Consequently, a multivariate version of the CUSUM M-chart can be developed accordingly based on the m_j's (also see Yeh and Lin [27] and Yeh *et al.* [28]). We intend to pursue this line of work in future endeavor.

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Appendix A

First we show that the statistic described in (1) is $U(0,1)$ (the proof for (2) is very similar). For any $0<y<1$, we have

$$P(m_j \le y) = P\left(P\left(t_{N-k} \le \frac{\bar{X}_j - \bar{\bar{X}}}{\bar{S}\sqrt{\frac{1}{n}+\frac{1}{N}}}\right) \le y\right)$$

$$= P\left(\frac{\bar{X}_j - \bar{\bar{X}}}{\bar{S}\sqrt{\frac{1}{n}+\frac{1}{N}}} \le G_t^{-1}(y)\right)$$

$$= G_t(G_t^{-1}(y)) = y,$$

where $G_t(\cdot)$ is the cumulative distribution function of t_{N-k} and $G_t^{-1}(\cdot)$ is its inverse function. Since, for $0<y<1$, $P(m_j \le y) = y$, this shows that m_j is distributed as $U(0,1)$. Note that m_j is essentially the probability integral transformation. The fact that m_j is distributed as $U(0,1)$ will also hold for non-normal processes, provided that the distribution of the test statistic used to test process mean ($\bar{X}_j - \bar{\bar{X}}/\bar{S}\sqrt{1/n+1/N}$ in this case) is known.

Next we show that the statistics described in (1) and (2) are independent. Write

$$W_1 = \frac{\bar{X}_l - \bar{\bar{X}}}{\bar{S}\sqrt{\frac{1}{n}+\frac{1}{N}}} \text{ and } W_2 = \frac{S_l^2}{\bar{S}^2}.$$

Consider the conditional joint distributions of W_1 and W_2 given $\bar{\bar{X}}$ and $\bar{S}^2$, $f(w_1, w_2 \mid \bar{\bar{X}}, \bar{S}^2)$. Given $\bar{\bar{X}}$ and $\bar{S}^2$, W_1 and W_2 are independent so that the conditional joint distribution can be written as the product of marginal conditional distributions

$$f(w_1, w_2 \mid \bar{\bar{X}}, \bar{S}^2) = f(w_1 \mid \bar{\bar{X}}, \bar{S}^2) \times f(w_2 \mid \bar{\bar{X}}, \bar{S}^2).$$

However, neither of the distributions of W_1 and W_2 depends on $\bar{\bar{X}}$ or $\bar{S}^2$, therefore the two marginal conditional distributions are also marginal unconditional distributions. This implies that the conditional joint distribution of W_1 and W_2 is the same as the unconditional distribution, and furthermore, it is equal to the product of the two marginal distributions of W_1 and W_2. This proves the independence of (1) and (2).

Appendix B

First recall that $S_m(t)$ and $S_v(t)$ (normalized versions) are defined as

$$S_m(t) = \Sigma_{j=1}^{t}\sqrt{12}\left(m_j - \tfrac{1}{2}\right) \text{ and}$$

$$S_v(t) = \Sigma_{j=1}^{t}\sqrt{12}\left(v_j - \tfrac{1}{2}\right).$$

Note that $\sqrt{1/12}S_m(t) + t/2 = \Sigma_{j=1}^{t} m_j$, a sum of t iid $U(0,1)$ random variables given n and k.

It is easy to see that for any $x > 0$ and any given $t \geq 1$,

$$P(S_m(t) \leq -x) = P\left(-\Sigma_{j=1}^{t}\sqrt{12}\left(m_j - \tfrac{1}{2}\right) \geq x\right)$$

$$= P\left(\Sigma_{j=1}^{t}\sqrt{12}\left(m_j - \tfrac{1}{2}\right) \geq x\right)$$

$$= 1 - P(S_m(t) \leq x),$$

since $-(m_j - 1/2)$ and $(m_j - 1/2)$ have the same distribution $U(-1/2, 1/2)$. Therefore, $S_m(t)$ is symmetric about 0 for any given t.

Appendix C

Here we discuss briefly the Markov chain approach pertaining to our proposed CUSUM control charts. For a more detailed discussion on using the approach for general distributions, please refer to Brook and Evans [2]. For simplicity of discussion, we focus our discussion on CUSUM M-chart with one-sided control limit H, i.e., we plot

$$S_m(t) = \sum_{j=1}^{t}(m_j - \tfrac{1}{2}) = \sum_{j=1}^{t} D_j$$

against the sample number t, $t \geq 1$, and an out-of-control signal is detected if $S_m(t) > H$.

For a given H, we wish to compute the ARL of CUSUM M-chart. Note that the operation of the decision making scheme forms a Markov process with a continuous state space. Good approximations to the various characteristics of the run length distribution can be obtained by discretizing the distribution of D, $U(-1/2,1/2)$, so that the cumulative sum $S_m(t)$ is restricted to a finite set of values. More specifically, assume that we represent the continuous scheme by a Markov chain having $k+1$ states denoted by $E_0, E_1, \ldots, E_k$, where E_k is absorbing. The probability that the Markov chain remains in the same state at the next step should correspond to the case where $S_m(t)$ does not change in value by more than a small amount, say $1/2w$. That is, the next value of D should not be larger than $|1/2w|$. The quantity w is equal to the width of the grouping interval involved in the discretization of the distribution of D.

As a further restriction, it is required that the probability of a jump from E_i to the absorbing state E_k should be equal to the probability that $S_m(t)$ jumps beyond the point H from a position in $(-1/2, H)$ which corresponds approximately with the state E_i. Theses requirements lead to

$$w = \frac{2H}{2k-1}.$$

The transition probabilities for the Markov chain are then as follows, for $i = 0,1,2,\ldots,k-1$:

$$P_{i0} = P(E_i \to E_0) = P(D \leq -iw + \tfrac{1}{2}w) = -iw + \tfrac{1}{2}w + \tfrac{1}{2},$$

$$P_{ij} = P(E_i \to E_j) = P((j-i)w - \tfrac{1}{2}w < D \leq (j-i)w + \tfrac{1}{2}w) = w,$$

$$P_{ik} = P(E_i \to E_k) = P((k-i)w - \tfrac{1}{2}w < D) = -(k-i)w + \tfrac{1}{2}w + \tfrac{1}{2}.$$

Note that

$$P_{0k} = P(E_0 \to E_k) = P(D > H)$$

for any choice of w satisfying $w = 2H/(2k-1)$. In general, for any given H and k, the transition probability matrix has the following form:

$$\mathbf{P} = \begin{bmatrix} F_0 & p_1 & p_2 & \cdots & p_j & \cdots & p_{k-1} & 1-F_{k-1} \\ F_{-1} & p_0 & p_1 & \cdots & p_{j-1} & \cdots & p_{k-2} & 1-F_{k-2} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ F_{-i} & p_{1-i} & p_{2-i} & \cdots & p_{j-i} & \cdots & p_{k-1-i} & 1-F_{k-1-i} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ F_{1-k} & p_{2-k} & p_{3-k} & \cdots & p_{j-(k-1)} & \cdots & p_0 & 1-F_0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R & P_k \\ 0^T & 1 \end{bmatrix},$$

where

$$p_r = P(rw - \tfrac{1}{2}w < D \le rw + \tfrac{1}{2}w) = w,$$

$$F_r = P(D \le rw + \tfrac{1}{2}w) = rw + \tfrac{1}{2}w + \tfrac{1}{2}.$$

Setting

$$\mu = (I - R)^{-1}1,$$

where $1 = (1, 1, \ldots, 1)^T$, then the ith element of μ gives the average run length of the CUSUM scheme when starting from state E_i, $i = 1, 2, \ldots, k-1$.

Note that for our purposes, we choose k to be an even number since $S_m(0) = 0$, which means that the starting point is in state $E_{(k+2)/2}$. If k is chosen to be an odd number, then the starting point is on the boundary point of states $E_{(k+1)/2}$ and $E_{(k+3)/2}$. For practical applications, we choose $k = 10$ which provides a good approximation to the calculation of ARL's.

Given a fixed H, we calculate p_r's and F_r's, and, consequently, R. Then the 6th element of μ gives the one-sided ARL. Furthermore, since $S_m(t)$ is symmetric about 0, the two-sided ARL is equal to half of the one-sided ARL. Summarized in Table 1 are the ARL's for various values of H calculated under $k = 10$.