Optimal Foldover Plans for Two-Level Fractional Factorial Designs

William Li

Department of Operations
and Management Science
University of Minnesota
Minneapolis, MN 55455
(*wli@csom.umn.edu*)

Dennis K. J. Lin

Department of Supply Chain
and Information Systems
Pennsylvania State University
University Park, PA 16802-1913
(*DKL5@psu.edu*)

A commonly used follow-up experiment strategy involves the use of a foldover design by reversing the signs of one or more columns of the initial design. Defining a foldover plan as the collection of columns whose signs are to be reversed in the foldover design, this article answers the following question: Given a 2^{k-p} design with k factors and p generators, what is its optimal foldover plan? We obtain optimal foldover plans for 16 and 32 runs and tabulate the results for practical use. Most of these plans differ from traditional foldover plans that involve reversing the signs of one or all columns. There are several equivalent ways to generate a particular foldover design. We demonstrate that any foldover plan of a 2^{k-p} fractional factorial design is equivalent to a core foldover plan consisting only of the p out of k factors. Furthermore, we prove that there are exactly 2^{k-p} foldover plans that are equivalent to any core foldover plan of a 2^{k-p} design and demonstrate how these foldover plans can be constructed. A new class of designs called combined-optimal designs is introduced. An n-run combined-optimal 2^{k-p} design is the one such that the combined 2^{k-p+1} design consisting of the initial design and its optimal foldover has the minimum aberration among all 2^{k-p} designs.

KEY WORDS: Foldover design; Minimum aberration design; Optimal foldover; Word length pattern.

1. INTRODUCTION

Practitioners often use two-level factorial designs to investigate the effects of several factors simultaneously. The number of runs n required by a full 2^k factorial design increases geometrically as the number of factors k increases. This makes it more desirable to use fractional factorial designs to reduce the number of runs. One consequence of using a fractional factorial design is the aliasing of factorial effects. Consider a project conducted at a major automotive company. The team used a designed experiment to determine the effect of postcrimp stresses on the crimp resistance. The study involved six two-level factors: crimp height (factor 1), preconditioning thermal shock (factor 2), dry heat soak (factor 3), fixture material (factor 4), thermal shock life test (factor 5), and discoloration (factor 6). The team decided to conduct a 16-run design. The company—which was experiencing high warranty cost because of the crimp failure symptoms—had sufficient funds to conduct another 16-run design when necessary. The first 16 experiments were conducted according to a 2^{6-2} design, in which $5 = 123$ and $6 = 124$. In this article we follow the notation used by Box, Hunter, and Hunter (1978) and call the factors 1–4, whose columns constitute a full 2^4 factorial design, *basic factors*. Factors 5 and 6—which are involved in the two *generators* 1235 and 1246—are called *generated factors*. A generator is also called a *defining word*. The analysis of the initial 16-run design showed that all main effects and several two-factor interactions were significant. Because in the initial design, three pairs of two-factor interactions are fully aliased, the team decided to conduct another 16-run design to dealias as many two-factor interactions as possible.

A standard follow-up strategy discussed in many textbooks involves adding a second fraction, called a *foldover design* (or simply *foldover*), by reversing the signs of one or more columns of the *initial design* (e.g., Box et al. 1978; Montgomery 2001; Neter, Kutner, Nachtsheim, and Wasserman 1996; Wu and Hamada 2000). For the crimp project, the team decided to use a foldover design, and the question was which foldover design should be used. Note that there are $2^6 = 64$ ways to generate a foldover design. Denote a *foldover plan*, γ, as the collection of columns whose signs are to be reversed in the foldover design; then each foldover is generated by a foldover plan. For example, $\gamma = 456$ produces a foldover design by reversing the signs of factors 4, 5, and 6. (Note that we use “factors” and “columns” interchangeably in this article.)

In this article we develop optimal foldover plans for commonly used fractional factorial designs. The criterion that we use is the *aberration* (Fries and Hunter 1980) of the combined design, as defined in Section 2. (A *combined design* refers to the combination of the initial design and its foldover.) Note that foldovers may be constructed for various reasons. If the analysis of the initial design reveals a particular set of main and interaction effects that are significant, then the foldover design should be chosen to resolve confounding problems with these significant effects. For example, if one particular factor is very important and should not be confounded with other factors, then a foldover based on reversing the sign of this factor is appropriate. On the other hand, if the goals are to dealias all (or as many as possible) main effects from two-factor interactions, and to dealias as many as possible

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two-factor interactions from each other, then the aberration criterion appears to be a good choice.

Note that the aberration criterion has been used (sometimes implicitly) among the existing foldover strategies. For example, a commonly used foldover strategy for a resolution III design involves reversing the signs of all factors. This is usually considered a good strategy because the resulting combined design has resolution IV, which is higher than the resolution of the initial design. This article demonstrates that the use of the aberration criterion can lead to further improvement. The combined design may have a higher resolution or the same resolution with fewer numbers of two-factor interactions that are confounded with each other. Note that the initial design and the foldover are usually conducted at different stages; thus a blocking factor should be considered. Ye and Li (2003) demonstrated that all optimal foldover plans are still optimal in terms of the aberration of the combined designs in the presence of a blocking variable.

This article is organized as follows. Section 2 introduces notation and existing work. Section 3 considers equivalent foldover plans. Section 4 presents an algorithm for searching optimal foldover plans. In particular, we obtain optimal foldover plans for commonly used 16-run and 32-run fractional factorial designs. Section 5 explores properties of these foldovers. We then introduce a class of designs of size n whose optimal foldover plans result in minimum-aberration $2n$-run designs. Conclusions and future work are given in Section 6.

2. NOTATION AND EXISTING WORK

Let w_i denote the number of words of length i in the defining relation of a design $\mathbf{d}$. The vector $W(\mathbf{d}) = (w_1, w_2, w_3, \ldots, w_k)$ is called the *word-length pattern* (WLP) of the design. [For simplicity, only $(w_3, \ldots, w_7)$ of WLPs are displayed in this article.] The *resolution* of $\mathbf{d}$ is defined as the smallest r such that $w_r \geq 1$. For any two designs $\mathbf{d}_1$ and $\mathbf{d}_2$, let s be the smallest integer such that $w_s(\mathbf{d}_1) \neq w_s(\mathbf{d}_2)$. Then $\mathbf{d}_1$ is said to have *less aberration* than $\mathbf{d}_2$, denoted by $W(\mathbf{d}_1) < W(\mathbf{d}_2)$, if $w_s(\mathbf{d}_1) < w_s(\mathbf{d}_2)$. When there is no design with less aberration than $\mathbf{d}_1$, $\mathbf{d}_1$ has *minimum aberration.*

A classic approach to constructing a foldover design is to reverse the signs of all k factors. We call this type of foldover plan a *full-foldover plan* and denote it by $\gamma^f = 1 \cdots k$. The corresponding foldover design $\mathbf{d}'(\gamma^f)$ is called a *full-foldover.* Most popular statistical software packages (e.g., SAS) take this approach. However, the combined design generated by this foldover plan may not be optimal with respect to its WLP. Consider a fractional factorial 2_{IV}^{7-2} design generated by $6 = 1234$ and $7 = 1245$. When the signs of all seven factors are reversed, the combined design has $W = (0, 1, 0, 0, 0)$. This is a resolution IV design that has three pairs of fully aliased two-factor interactions. A quick search reveals that $\gamma = 6$ produces a resolution V design with $W = (0, 0, 1, 0, 0)$—namely, all two-factor interactions are clear. A two-factor interaction is called *clear* if it is not aliased with any main effects or other two-factor interactions (Wu and Chen 1992).

Other foldover plans have also been proposed in the literature. Sign reversal of one factor was considered by, for example, Box et al. (1978) and Wu and Hamada (2000). Montgomery and Runger (1996) considered foldovers generated by reversing the signs of one or two factors. For resolution IV designs, the rule of reversing signs of all factors is not directly applicable, because the resulting combined design will have the same number of length-4 words. Some software packages consider different foldover strategies for these designs; for example, in Design Expert V6 it is suggested that the sign of a single column be reversed. Another software package, RS/Discover, suggests reversing the sign of the factor if the generator in which this factor is involved is an odd-length word. However, it is not clear whether any of these previously given foldovers is optimal (with respect to the WLP of the combined design). To our knowledge, the optimality of foldover designs has not been addressed in the literature.

3. EQUIVALENT FOLDOVER PLANS

Denote the initial design by $\mathbf{d}$, the foldover by $\mathbf{d}'$, and the combined design by $\mathbf{D}$. The optimal foldover plan, γ^*, is the one such that $W(\mathbf{D}(\gamma^*)) = \min_{\gamma\in\Gamma} W(\mathbf{D}(\gamma))$, where $\Gamma = \{\gamma_1, \ldots, \gamma_q\}$ is the foldover plan space and q is the total number of possible foldover plans. The resulting foldover, $\mathbf{d}'(\gamma^*)$, and combined design, $\mathbf{D}(\gamma^*)$, are called the *optimal foldover* and *optimal combined design.*

Given a 2^{k-p} design, the optimal foldover plan γ^* can be found by searching all $q = 2^k$ possible foldover plans. Note, however, that many of these q foldover plans produce the same foldover design. We call these *equivalent foldover plans.* Consider, for example, a 2_V^{5-1} design defined by a generating relation $5 = 1234$. Obviously, the foldover plans $\gamma_i = i$ $(i = 1, 2, 3, 4)$ are equivalent to each other, and they are all equivalent to a foldover plan $\gamma_c = 5$, which involves only the generated factor of the design—factor 5. (Without loss of generality, we use $1, \ldots, k-p$ to denote the basic factors and $k-p+1, \ldots, k$ to denote the generated factors.) In general, if a foldover plan γ_c consists only of the generated factors, then we call it a *core foldover plan.* An important property is that every foldover plan is equivalent to a specific core foldover plan.

Theorem 1. For a 2^{k-p} design with p generators $G_1, \ldots, G_p$, any foldover plan is equivalent to a core foldover plan. Moreover, for every core foldover plan, there are 2^{k-p} foldover plans that are equivalent to it.

The proof is given in the Appendix. We now use an example to illustrate how equivalent foldover plans can be constructed using the foregoing theorem. Consider a 2^{5-2} design generated by $4 = 12$ and $5 = 13$. Table 1 indicates how the $2^5 = 32$

Table 1. Four Equivalent Foldover Plan Groups for a 2^{5-2} Design with 4 = 12 and 5 = 13

Core foldover plans	Foldover plans consisting only of basic factors							
	0	1	2	3	12	13	23	123
0	0	145	24	35	125	134	2345	123
4	4	15	2	345	1245	13	235	1234
5	5	14	245	3	12	1345	234	1235
45	45	1	25	34	124	135	23	12345

NOTE: Each row displays the eight foldover plans that are equivalent to the core foldover plan listed in the first column.

foldover plans can be classified into four groups. The four core foldover plans—0, 4, 5 and 45—are listed in the first column, where 0 means that no variable is chosen for the sign reversal in the foldover design and 0 is defined as a core foldover plan. Note that when $\gamma = 0$, the foldover design is identical to the initial design. Then each foldover plan in an equivalent group is a union of a unique combination of the $k-p$ basic factors $1, 2, \ldots, k-p$, and a unique combination of the generated factors. For example, to find the eight foldover plans that are equivalent to $\gamma_c = 4$, we first list all combinations of the three basic factors: 0, 1, 2, 3, 12, 13, 23, and 123. We then add the generated factors 4 or 5, or both, or neither, to each of them, based on the rules (A1) and (A2) stated in the proof of the theorem (see the Appendix). Consider, for instance, $\gamma = 123$, which means that the signs of 1, 2, and 3 are reversed. Then factor 4 satisfies (A1) and should be added, and factor 5 does not satisfy either (A1) or (A2) and should not be included. Thus $\gamma_c = 4$ is equivalent to 1234. The other equivalent foldover plans can be found similarly.

The foldover plans in each group produce the same foldover design $\mathbf{d}'$. Thus it is sufficient to use one representative foldover plan in each group. We recommend the use of core foldover plans presented in this article.

4. CONSTRUCTION OF OPTIMAL FOLDOVERS

Based on Theorem 1, we present an algorithm to search for optimal foldover plans. The algorithm is an exhaustive search method based on a specific set of the p generated factors. Note that the number of candidate foldover plans is only 2^p, a fraction of the total number of candidate foldover plans 2^k. The computer program comprises the following steps:

1. Input n, k, and p of the initial design $\mathbf{d}$.
2. Generate all $2^p - 1$ defining words of $\mathbf{d}$ for a given set of $G_1, \ldots, G_p$.
3. For each core foldover plan γ_i, $(i = 1, \ldots, 2^p)$:

   a. Consider all defining words of $\mathbf{d}$. For each word, if there is an even number of factors whose signs are reversed by γ_i, then this word is retained in the defining relation of the combined design $\mathbf{D}(\gamma_i)$; otherwise, the word is deleted.

   b. Compare $W(\mathbf{D}(\gamma_i))$ with $W(\mathbf{D}(\gamma^*))$, where γ^* is the best core foldover plan among those that are considered before γ_i. Update γ^* when $W(\mathbf{D}(\gamma_i)) < W(\mathbf{D}(\gamma^*))$.

4. Output γ^* and $W(\mathbf{D}(\gamma^*))$.

We use this algorithm to construct the optimal foldovers for 16- and 32-run designs. Although fractional factorial designs with the minimum aberration are commonly used in practice, in some situations other designs can better meet practical needs. For example, as argued by Wu and Chen (1992), there are practical situations in which certain interactions can be identified a priori as being potentially important and should be estimated clear of each other. Then one may have to choose a design with higher aberration. Chen, Sun, and Wu (1993) presented a catalog of complete 16-run designs and selected 32-run designs. Finding optimal foldovers of these designs would be useful for practitioners. Thus, by using the computer search method described in this section, we constructed optimal foldovers of all of these designs. The methodology described in this article is applicable to any 2^{k-p} fractional factorial design. We have focused here on 16-run and 32-run designs with $k \leq 11$, because most standard textbooks give designs of up to 11 factors and foldovers of designs with $n \geq 64$ are rarely used in practice. However, foldovers of other (larger) designs can be constructed in a straightforward manner.

Tables 2 and 3 provide optimal foldover plans for all 16-run designs and selected 32-run designs. The 32-run designs are selected according to the minimum aberration criterion and the number of clear two-factor interactions. A complete catalog of 32-run designs and their optimal foldover plans are available on request. In the first column of each table, designs are recorded as $k - p{\cdot}i$ $(i = 1, 2, \ldots)$, which correspond to those designs presented by Chen et al. (1993). Design $5-1{\cdot}1$, for example, refers to the first of 2^{5-1} designs. The first design in each group of (k, p) is the minimum aberration 2^{k-p} design. The generating relations and the WLP of each design $\mathbf{d}$ are given in the second and third columns. The optimal foldover plan γ^* and the corresponding full-foldover plan $\gamma^f = 1 \cdots k$ are given in the next column. In all cases, only the equivalent core foldover plans are reported in the tables. For each design, if there is more than one optimal core foldover plan, then all of them are reported as γ^*. If $\gamma^* = \gamma^f$, then an asterisk is put beside the optimal foldover plan. The WLPs of two foldover plans, $W(\mathbf{D}(\gamma^*))$ and $W(\mathbf{D}(\gamma^f))$, are reported in the last column. Note that all comparisons are made in terms of the aberration of the combined design.

5. MAJOR FINDINGS

5.1 Performance of Optimal Foldover Designs

From Tables 2 and 3 it can be seen that *for most designs, there exist better foldover plans than the classic full-foldover plans under the minimum aberration criterion.* In 52 out of 77 cases, we found better foldover plans than the corresponding full-foldover plans. Although some of these plans may be obtained by previously reported methods in the literature, most are new. Table 4 provides more detailed information, showing the number of optimal foldover designs that are better than the full-foldovers for each given set of (k, p).

For resolution III designs, the full-foldover plans produce combined resolution IV designs. An optimal foldover plan can further improve this desirable property by two means. First, it may further increase the resolution of the combined design $\mathbf{D}$. One example is design $7-2{\cdot}5$, for which the WLP of the combined design from full-foldover is $W(\mathbf{D}(\gamma^f)) = (0, 1, 0, 0, 0)$, whereas the optimal combined design has $W(\mathbf{D}(\gamma^*)) = (0, 0, 0, 0, 1)$. Second, it may lead to a resolution IV design with fewer length-4 words. Thus the optimal combined design can dealias more two-factor interactions from each other. Consider, for example, design $7-3{\cdot}2$, for which the full-foldover plan produces a combined design with three length-4 words, $I = 2356 = 2347 = 4567$, but the corresponding optimal combined design has only one length-4 word, $I = 2356$.

Although foldovers of resolution III designs are more common in practice, augmenting resolution IV designs can sometimes be important as well. Such examples were discussed by Montgomery and Runger (1996). The objective here is

*Table 2. Optimal Foldover Designs With Respect to W(**D**) (16 runs, Complete Catalog[1])*

$k-p$	Generating relations	Initial $W(\mathbf{d})$	Optimal foldover plan (γ^*) Full-foldover plan (γ^f)	$W(\mathbf{D}(\gamma^*))$ $W(\mathbf{D}(\gamma^f))$
5 − 1·1	5 = 1234[2]	(0 0 1)	—	— —
5 − 1·2	5 = 123	(0 1 0)	5 0	Full factorial (0 1 0 0 0)
5 − 1·3	5 = 12	(1 0 0)	5(*) 5	Full factorial Full factorial
6 − 2·1	5 = 123, 6 = 124	(0 3 0 0)	5, 56, 6[3] 0	(0 1 0 0 0) (0 3 0 0 0)
6 − 2·2	5 = 12, 6 = 134	(1 1 1 0)	56 5	(0 0 1 0 0) (0 1 0 0 0)
6 − 2·3	5 = 12, 6 = 34	(2 0 0 1)	56(*) 56	(0 0 0 1 0) (0 0 0 1 0)
7 − 3·1	5 = 123, 6 = 124, 7 = 134	(0 7 0 0 0)	5, 56, 567, 57, 6, 67, 7 0	(0 3 0 0 0) (0 7 0 0 0)
7 − 3·2	5 = 12, 6 = 13, 7 = 234	(2 3 2 0 0)	567 56	(0 1 2 0 0) (0 3 0 0 0)
7 − 3·3	5 = 12, 6 = 13, 7 = 24	(3 2 1 1 0)	567(*) 567	(0 2 0 1 0) (0 2 0 1 0)
7 − 3·4	5 = 12, 6 = 13, 7 = 14	(3 3 0 0 1)	567(*) 567	(0 3 0 0 0) (0 3 0 0 0)
7 − 3·5	5 = 12, 6 = 13, 7 = 23	(4 3 0 0 0)	567(*) 567	(0 3 0 0 0) (0 3 0 0 0)
8 − 4·1	5 = 123, 6 = 124, 7 = 134, 8 = 234	(0 14 0 0 0)	56, 5678, 57, 58, 67, 68, 78 0	(0 6 0 0 0) (0 14 0 0 0)
8 − 4·2	5 = 12, 6 = 13, 7 = 14, 8 = 234	(3 7 4 0 1)	5678 567	(0 3 4 0 0) (0 7 0 0 0)
8 − 4·3	5 = 12, 6 = 13, 7 = 24, 8 = 34	(4 5 4 2 0)	5678(*) 5678	(0 5 0 2 0) (0 5 0 2 0)
8 − 4·4	5 = 12, 6 = 13, 7 = 23, 8 = 1234	(4 6 4 0 0)	567 5678	(0 3 4 0 0) (0 6 0 0 0)
8 − 4·5	5 = 12, 6 = 13, 7 = 23, 8 = 14	(5 5 2 2 1)	5678(*) 5678	(0 5 0 2 0) (0 5 0 2 0)
8 − 4·6	5 = 12, 6 = 13, 7 = 23, 8 = 123	(7 7 0 0 1)	567(*) 567	(0 7 0 0 0) (0 7 0 0 0)
9 − 5·1	5 = 123, 6 = 124, 7 = 134, 8 = 234, 9 = 1234	(4 14 8 0 4)	5678 9	(0 6 8 0 0) (0 14 0 0 0)
9 − 5·2	5 = 12, 6 = 13, 7 = 24, 8 = 34, 9 = 1234	(6 9 9 6 0)	56789(*) 56789	(0 9 0 6 0) (0 9 0 6 0)
9 − 5·3	5 = 12, 6 = 13, 7 = 23, 8 = 14, 9 = 234	(6 10 8 4 2)	5678(*) 5678	(0 10 0 4 0) (0 10 0 4 0)
9 − 5·4	5 = 12, 6 = 13, 7 = 23, 8 = 14, 9 = 24	(7 9 6 6 3)	56789(*) 56789	(0 9 0 6 0) (0 9 0 6 0)
9 − 5·5	5 = 12, 6 = 13, 7 = 23, 8 = 123, 9 = 14	(8 10 4 4 4)	5679(*) 5679	(0 10 0 4 0) (0 10 0 4 0)
10 − 6·1	5 = 123, 6 = 124, 7 = 134, 8 = 234, 9 = 1234, $\underline{10}$ = 34	(8 18 16 8 8)	9$\underline{10}$(*) 9$\underline{10}$	(0 18 0 8 0) (0 18 0 8 0)
10 − 6·2	5 = 12, 6 = 13, 7 = 23, 8 = 14, 9 = 24, $\underline{10}$ = 134	(9 16 15 12 7)	56789(*) 56789	(0 16 0 12 0) (0 16 0 12 0)
10 − 6·3[4]	5 = 12, 6 = 13, 7 = 23, 8 = 14, 9 = 24, $\underline{10}$ = 34	(10 15 12 15 10)	56789$\underline{10}$(*) 56789$\underline{10}$	(0 15 0 15 0) (0 15 0 15 0)
10 − 6·4	5 = 12, 6 = 13, 7 = 23, 8 = 123, 9 = 14, $\underline{10}$ = 24	(10 16 12 12 10)	5679$\underline{10}$(*) 5679$\underline{10}$	(0 16 0 12 0) (0 16 0 12 0)
11 − 7·1	5 = 123, 6 = 124, 7 = 134, 8 = 234, 9 = 1234, $\underline{10}$ = 34, $\underline{11}$ = 24	(12 26 28 24 20)	9 $\underline{10}$ $\underline{11}$(*) 9 $\underline{10}$ $\underline{11}$	(0 26 0 24 0) (0 26 0 24 0)
11 − 7·2	5 = 12, 6 = 13, 7 = 23, 8 = 123, 9 = 14, $\underline{10}$ = 24, $\underline{11}$ = 34	(13 25 25 27 23)	5679 $\underline{10}$ $\underline{11}$(*) 5679 $\underline{10}$ $\underline{11}$	(0 25 0 27 0) (0 25 0 27 0)
11 − 7·3	5 = 12, 6 = 13, 7 = 23, 8 = 123, 9 = 14, $\underline{10}$ = 24, $\underline{11}$ = 124	(13 26 24 24 26)	5679$\underline{10}$(*) 5678$\underline{10}$	(0 26 0 24 0) (0 26 0 24 0)

NOTE: [1] There are two designs for $k = 12$, and designs for $k = 13, 14, 15$ are unique. Optimal foldovers for these designs are equivalent to the full foldovers.
[2] This design is of resolution V; thus its foldover is not considered.
[3] $\gamma = 56$ means that the signs of columns 5 and 6 are reversed in the foldover. When there is more than one optimal core foldover plan, they are separated by a comma.
[4] $W(\mathbf{D})$ of this design is not the minimum aberration 2^{10-5} design.

*Table 3. Optimal Foldover Designs With Respect to W(**D**) (32 Runs, Selected)*

$k-p$	Generating relations	Initial $W(\boldsymbol{d})$	Optimal foldover plan (γ^*) Full-foldover plan (γ^f)	$W(\boldsymbol{D}(\gamma^*))$ $W(\boldsymbol{D}(\gamma^f))$
7−2·1	6 = 1234, 7 = 1245	(0 1 2 0 0)	6, 7 67	(0 0 1 0 0) (0 1 0 0 0)
7−2·2	6 = 123, 7 = 145	(0 2 0 1 0)	67 0	(0 0 0 1 0) (0 2 0 1 0)
7−2·3	6 = 123, 7 = 124	(0 3 0 1 0)	6, 7, 67 0	(0 1 0 0 0) (0 3 0 0 0)
7−2·4	6 = 12, 7 = 1345	(1 0 1 1 0)	67(*) 67	(0 0 0 1 0) (0 0 0 1 0)
7−2·5	6 = 12, 7 = 345	(1 1 0 0 1)	67 6	(0 0 0 0 1) (0 1 0 0 0)
7−2·6	6 = 12, 7 = 134	(1 1 1 0 0)	67 6	(0 0 1 0 0) (0 1 0 0 0)
7−2·7	6 = 12, 7 = 34	(2 0 0 1 0)	67(*) 67	(0 0 0 1 0) (0 0 0 1 0)
7−2·8	6 = 12, 7 = 14	(2 1 0 0 0)	67(*) 67	(0 1 0 0 0) (0 1 0 0 0)
8−3·1	6 = 123, 7 = 124, 8 = 2345	(0 3 4 0 0)	6, 67, 678, 68, 7, 78 0	(0 1 2 0 0) (0 3 0 0 0)
8−3·2	6 = 123, 7 = 124, 8 = 135	(0 5 0 2 0)	78 0	(0 1 0 2 0) (0 5 0 2 0)
8−3·3	6 = 123, 7 = 124, 8 = 125	(0 6 0 0 0)	67, 68, 78 0	(0 2 0 0 0) (0 6 0 0 0)
8−3·4	6 = 123, 7 = 124, 8 = 134	(0 7 0 0 0)	6, 67, 678, 68, 7, 78, 8 0	(0 3 0 0 0) (0 7 0 0 0)
8−3·5	6 = 12, 7 = 134, 8 = 235	(1 2 3 1 0)	678 6	(0 0 2 1 0) (0 2 0 1 0)
8−3·6	6 = 12, 7 = 13, 8 = 2345	(2 1 2 2 0)	678(*) 678	(0 1 0 2 0) (0 1 0 2 0)
8−3·7	6 = 12, 7 = 134, 8 = 135	(1 3 2 0 1)	67, 68 6	(0 1 1 0 1) (0 3 0 0 0)
8−3·8	6 = 12, 7 = 34, 8 = 135	(2 1 2 2 0)	678 67	(0 0 2 1 0) (0 1 0 2 0)
8−3·9	6 = 12, 7 = 13, 8 = 245	(2 2 1 1 1)	678 67	(0 1 1 0 1) (0 2 0 1 0)
8−3·10	6 = 12, 7 = 13, 8 = 145	(2 2 2 0 0)	678 67	(0 1 2 0 0) (0 2 0 0 0)
9−4·1	6 = 2345, 7 = 1345, 8 = 1245, 9 = 1235	(0 6 8 0 0)	67, 68, 69, 78, 79, 89 6789	(0 2 4 0 0) (0 6 0 0 0)
9−4·2	6 = 123, 7 = 124, 8 = 134, 9 = 2345	(0 7 7 0 0)	67, 6789, 68, 69, 78, 79, 89 9	(0 3 3 0 0) (0 7 0 0 0)
9−4·3	6 = 123, 7 = 124, 8 = 135, 9 = 145	(0 9 0 6 0)	678, 679, 689, 69, 78, 789 0	(0 3 0 4 0) (0 9 0 6 0)
9−4·4	6 = 123, 7 = 124, 8 = 134, 9 = 125	(0 10 0 4 0)	89 0	(0 3 0 4 0) (0 9 0 6 0)
9−4·5	6 = 123, 7 = 124, 8 = 134, 9 = 234	(0 14 0 0 0)	67, 6789, 68, 69, 78, 79, 89 0	(0 6 0 0 0) (0 14 0 0 0)
9−4·6	6 = 12, 7 = 134, 8 = 135, 9 = 245	(1 5 6 2 1)	6789 6	(0 1 4 2 0) (0 5 0 2 0)
9−4·7	6 = 12, 7 = 134, 8 = 135, 9 = 145	(1 7 4 0 3)	67, 678, 679, 68, 689, 69 6	(0 3 2 0 2) (0 7 0 0 0)
9−4·8	6 = 12, 7 = 34, 8 = 135, 9 = 245	(2 3 6 4 0)	678, 6789, 679 67	(0 1 4 2 0) (0 3 0 4 0)
9−4·9	6 = 12, 7 = 13, 8 = 14, 9 = 2345	(3 3 4 4 1)	6789(*) 6789	(0 3 0 4 0) (0 3 0 4 0)
9−4·10	6 = 12, 7 = 13, 8 = 24, 9 = 345	(3 3 4 4 1)	6789 678	(0 2 3 1 1) (0 3 0 4 0)
10−5·1	6 = 1234, 7 = 1235, 8 = 1245, 9 = 1345, $\underline{10}$ = 2345	(0 10 16 0 0)	67, 6$\underline{10}$, 67$\underline{10}$, . . . , 9$\underline{10}$[1] 6789$\underline{10}$	(0 4 8 0 0) (0 10 0 0 0)
10−5·2	6 = 123, 7 = 124, 8 = 135, 9 = 145, $\underline{10}$ = 12345	(0 15 0 15 0)	678, 679, 689, 69$\underline{10}$, 789, 78$\underline{10}$ 0	(0 5 0 10 0) (0 15 0 15 0)

(*continued*)

Table 3. (continued)

$k-p$	*Generating relations*	*Initial* $W(\mathbf{d})$	*Optimal foldover plan* (γ^*) *Full-foldover plan* (γ^f)	$W(\mathbf{D}(\gamma^*))$ $W(\mathbf{D}(\gamma^f))$
$10-5{\cdot}3$	$6=123, 7=124, 8=134,$ $9=125, \underline{10}=135$	(0 16 0 12 0)	$7\underline{10}, 789, 78\underline{10}, 79\underline{10}, 89, 89\underline{10}$ 0	(0 6 0 8 0) (0 16 0 12 0)
$10-5{\cdot}4$	$6=123, 7=124, 8=134,$ $9=234, \underline{10}=125$	(0 18 0 8 0)	$89\underline{10}$ 0	(0 6 0 8 0) (0 18 0 8 0)
$10-5{\cdot}5$	$6=12, 7=134, 8=135,$ $9=145, \underline{10}=345$	(1 14 7 0 7)	$678, 67\underline{10}, 6789\underline{10}, 679, 68\underline{10}, 689, 69\underline{10}$ 6	(0 6 4 0 4) (0 14 0 0 0)
$10-5{\cdot}6$	$6=12, 7=134, 8=135,$ $9=145, \underline{10}=2345$	(1 10 11 4 3)	$6789\underline{10}$ $6\underline{10}$	(0 3 7 4 0) (0 10 0 4 0)
$10-5{\cdot}7$	$6=12, 7=34, 8=135, 9=245,$ $\underline{10}=12345$	(2 7 12 7 2)	$678, 6789\underline{10}, 679, 67\underline{10}$ 67	(0 3 6 4 2) (0 7 0 7 0)
$10-5{\cdot}8$	$6=12, 8=13, 8=234, 9=235,$ $\underline{10}=145$	(2 8 12 4 2)	$6789\underline{10}$ 67	(0 2 8 4 0) (0 8 0 4 0)
$10-5{\cdot}9$	$6=12, 8=13, 8=14, 9=234,$ $\underline{10}=245$	(2 9 9 6 4)	$6789\underline{10}$ 678	(0 3 6 4 2) (0 9 0 6 0)
$10-5{\cdot}10$	$6=12, 7=13, 8=14, 9=234,$ $\underline{10}=12345$	(3 8 11 4 1)	6789 678	(0 3 7 4 0) (0 8 0 4 0)
$11-6{\cdot}1$	$6=123, 7=124, 8=134,$ $9=125, \underline{10}=135, \underline{11}=145$	(0 25 0 27 0)	$67\underline{10}, 67\underline{11}, 689, \ldots, 89\underline{11}$[2] 0	(0 10 0 16 0) (0 25 0 27 0)
$11-6{\cdot}2$	$6=123, 7=124, 8=134,$ $9=234, \underline{10}=125, \underline{11}=135$	(0 26 0 24 0)	$78\underline{10}, 78\underline{11}, 79\underline{10}\,\underline{11}, 79\underline{11}, 89\underline{10}, 89\underline{10}\,\underline{11}$ 0	(0 10 0 16 0) (0 26 0 24 0)
$11-6{\cdot}3$	$6=12, 7=13, 8=234, 9=235,$ $\underline{10}=145, \underline{11}=12345$	(2 14 22 8 6)	$6789\underline{10}$ 67	(0 4 14 8 0) (0 14 0 8 0)
$11-6{\cdot}4$	$6=12, 7=13, 8=234, 9=235,$ $\underline{10}=245, \underline{11}=1345$	(2 16 16 12 10)	$6789, 6789\underline{10}\,\underline{11}$ $67\underline{11}$	(0 6 10 8 4) (0 16 0 12 0)
$11-6{\cdot}5$	$6=12, 7=13, 8=234, 9=235,$ $\underline{10}=245, \underline{11}=345$	(2 18 14 8 14)	$6789, 678\underline{10}, 678\underline{11}, 6789\underline{10}\,\underline{11}, 679\underline{10}, 679\underline{11}$ 67	(0 8 8 4 8) (0 18 0 8 0)
$11-6{\cdot}6$	$6=12, 7=13, 8=24, 9=1235,$ $\underline{10}=1245, \underline{11}=345$	(3 13 19 11 9)	678 $6789\underline{10}\,\underline{11}$	(0 5 12 7 4) (0 13 0 11 0)
$11-6{\cdot}7$	$6=12, 7=13, 8=14, 9=235,$ $\underline{10}=245, \underline{11}=1345$	(3 15 13 15 13)	$6789, 678\underline{10}, 6789\underline{10}\,\underline{11}$ $678\underline{11}$	(0 7 8 7 8) (0 15 0 15 0)
$11-6{\cdot}8$	$6=12, 7=13, 8=14, 9=235,$ $\underline{10}=245, \underline{11}=345$	(3 16 12 12 16)	$6789\underline{10}, 6789\underline{11}, 678\underline{10}\,\underline{11}$ 678	(0 8 8 4 8) (0 16 0 12 0)
$11-6{\cdot}9$	$6=12, 7=13, 8=14, 9=234,$ $\underline{10}=235, \underline{11}=245$	(3 16 13 12 13)	$89\underline{10}, 89\underline{10}\,\underline{11}, 89\underline{11}$ 9	(0 7 9 6 6) (0 16 0 12 0)
$11-6{\cdot}10$	$6=12, 7=13, 8=14, 9=234,$ $\underline{10}=25, \underline{11}=1345$	(4 12 18 12 8)	$6789\underline{10}\,\underline{11}$ $678\underline{10}\,\underline{11}$	(0 6 10 8 4) (0 12 0 12 0)

NOTE: [1]The complete set is: $67, 678, 689, 67\underline{10}, 68, 689, 68\underline{10}, 69, 69\underline{10}, 6\underline{10}, 78, 789, 78\underline{10}, 79, 79\underline{10}, 89, 89\underline{10}, 8\underline{10}, 9\underline{10}$.
[2]The complete set is: $67\underline{10}, 67\underline{11}, 689, 68\underline{11}, 69\underline{11}, 6\underline{10}\,\underline{11}, 789, 78\underline{10}, 79\underline{10}, 89\underline{10}, 89\underline{11}$.

to dealias two-factor interactions from each other. We find that the improvement over the full-foldover plans from the optimal foldover plans is usually substantial. For example, optimal foldovers of the minimum-aberration designs $8-3{\cdot}1$, $9-4{\cdot}1$, $10-5{\cdot}1$, and $11-6{\cdot}1$ dealias 2, 4, 6, and 15 out of 3, 6, 10, and 25 pairs of two-factor interactions. The percentages of dealiased pairs of two-factor interactions by optimal foldovers of non–minimum-aberration designs are also in the range of 60%–80%. Notable exceptions are designs $7-2{\cdot}1$ and $7-2{\cdot}2$, for which the optimal combined designs have resolution V and VI. This demonstrates that augmenting resolution IV designs may also produce designs with a higher resolution.

Table 4. Summary of Numbers of Designs for Which Optimal Foldovers Are Better Than Full Foldovers Under Minimum Aberration

k	5	6	7	8	9	10	11	*Total*
$n=16$	1/3	2/3	2/5	3/6	1/5	0/4	0/3	9/29
$n=32$			5/8	9/10	9/10	10/10	10/10	43/48

NOTE: For a given set of (k, n), the denominator is the total number of designs considered, and the numerator is the number of designs for which optimal foldovers are better than the full foldovers.

5.2 Combined-Optimal Designs

The initial designs are usually chosen according to the aberration criterion. However, as previously discussed, there are practical situations in which a practitioner would consider using non–minimum-aberration designs. Thus the other criteria (e.g., the number of clear two-factor interactions) may also be useful. In the context of the foldover design, one useful criterion for selecting among initial designs is the aberration of the optimal combined design. Consider the three designs for $k=6$ and $p=2$ in Table 2. Design $6-2{\cdot}1$ is the minimum-aberration design; however, its optimal combined design has $W=(0,1,0,0)$. In contrast, the optimal combined design resulting from design $6-2{\cdot}3$ has $W=(0,0,0,1)$. We call this design a *combined-optimal* design. In general, a 2^{k-p} design $\mathbf{d}^*$ is called a combined-optimal design if the resulting optimal combined design has the minimum aberration among all combined optimal designs, that is, $W([\begin{smallmatrix}\mathbf{d}^*\\(\mathbf{d}^*)'(\gamma^*)\end{smallmatrix}]) = \min_{\mathbf{d}} W([\begin{smallmatrix}\mathbf{d}\\\mathbf{d}'(\gamma^*)\end{smallmatrix}])$.

Combined-optimal designs can be useful when the experiment is conducted sequentially. For example, suppose that the experimenter is determined to conduct a 64-run design to investigate eight factors. The experimenter may prefer to conduct a 32-run design at the first stage and then use a foldover design to complete the experiment. The question is which 2^{8-3} design should be used if the goal is to optimize the *combined* design. The traditional approach is to use a minimum-aberration design of $8-3{\cdot}1$ in Table 3. It has the WLP of $(0, 3, 4, 0, 0)$, and the optimal combined design has $W = (0, 1, 2, 0, 0)$. Table 3 shows that the combined-optimal design is $8-3{\cdot}5$. This design has the WLP of $(1, 2, 3, 1, 0)$ for the initial design. However, the optimal combined design has $W(\mathbf{D}) = (0, 0, 2, 1, 0)$. This is not obtainable if the initial design has minimum aberration.

Interestingly, in the crimp project described earlier, the best combined design that the team could get was a resolution IV design (see design $6-2{\cdot}1$ in Table 2). Had the team chosen design $6-2{\cdot}3$ (the combined-optimal 2^{6-2} design), they would have been able to obtain a resolution VI combined design, in which all two-factor interactions are clear. As one referee pointed out, a similar idea was also used by Bullington, Hool, and Maghsoodloo (1990). They mentioned that an n-run two-level design from Taguchi's orthogonal arrays (Taguchi 1986) can be conducted sequentially; the first half is an orthogonal n-run array, and the second half is a foldover of the first design.

Tables 2 and 3 identify all combined-optimal designs by printing them in bold in the first column. (To find the 32-run combined-optimal designs, we have checked the complete set of 32-run designs.) A careful check on the $W(\mathbf{D})$'s of the combined designs reveals an interesting fact: Most combined-optimal designs with n runs, when combined by their optimal foldovers, result in combined designs that are minimum-aberration 2^{k-p+1} designs. For example, the optimal foldover of the combined-optimal 2^{7-3} design produces a 2^{7-2} design $\mathbf{D}$ with $W(\mathbf{D}) = (0, 1, 2, 0, 0)$. This is the corresponding 2^{7-2} minimum-aberration design. We call this type of combined-optimal design a *strong combined-optimal* design. The only combined-optimal design that is not a strong combined-optimal design in Tables 2 and 3 is design $10-6{\cdot}3$. The optimal combined design from this design has $W(\mathbf{D}) = (0, 15, 0, 15, 0)$, which is the second best to design $10-5{\cdot}1$. We are unable to explain such a special case. It will be an interesting research topic to find out under what conditions strong combined-optimal designs exist.

6. CONCLUSIONS AND FUTURE WORK

In this article we have obtained and tabulated the optimal foldover plans under the aberration criterion for all 16-run and selected 32-run designs. We have also proposed a computer-search method for constructing optimal foldover plans. We reduced the computations substantially by focusing on the core foldover plans, which constitute a much smaller subset of all foldover plans. It has come to our attention that Li and Mee (2002) have also recently considered the optimality of foldover designs. Their emphasis is on providing a sufficient condition for the situation where using an alternative foldover plan can be a better choice than reversing the signs of all factors. This provides a nice complement to this research.

We conclude the article with two remarks. First, one disadvantage of the foldover design is that the run size may become large in some situations. In these cases, partial foldovers proposed by Mee and Peralta (2000) can be considered. Optimal partial foldover plans are currently under study. Mee and Peralta (2000) pointed out that foldover designs are sometimes inefficient. Such an argument, however, is valid for conventional full-foldover, but may not apply to the optimal foldover plan given in this article. Second, as indicated in Section 1, there are various reasons for using a foldover design. Thus the aberration criterion should not be considered the only criterion. However, in the situation where the use of aberration criterion is justified, the optimal foldover plans presented in this article are recommended. We also note that the proposed approach can be applied to finding optimal foldovers with respect to other design criteria in a straightforward manner.

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APPENDIX: PROOF OF THEOREM 1

We prove the theorem by showing that all 2^k foldover plans can be classified into 2^p groups. In each group there are 2^{k-p} foldover plans, all of which are equivalent to a core foldover plan.

First, a 2^{k-p} design has p generated factors $(k-p+1), \ldots, k$, which are involved in the generators $G_1, \ldots, G_p$. Thus there are 2^p distinct core foldover plans. (We define 0 as a core foldover plan.)

Second, for any given core foldover plan γ_c, we can construct 2^{k-p} equivalent foldover plans as follows. A foldover plan consisting only of the $k-p$ basic factors $\gamma_b = a_1 \cdots a_l$ ($1 \le l \le k-p$) is chosen. Then we construct a new foldover plan by including all factors in γ_b and all of the generated factors, $(k-p+j)$'s, that satisfy one of the following two statements:

(A1) $k-p+j$ is in the core foldover plan γ_c, and there is an even number of a_i's in G_j.

(A2) $k-p+j$ is not in the core foldover plan γ_c, and there is an odd number of a_i's in G_j.

It can be easily seen that the resulting foldover plan γ consisting of columns in γ_b and the generated factors chosen by the rules of (A1) and (A2) is equivalent to γ_c. Note that two foldover plans are equivalent if the sign changes of all generators are the same according to both plans. (The sign of the generator is changed if there is an odd number of factors involved in this generator whose signs are reversed.) For a given generator $G_j = (a_1 \cdots a_l)(k-p+j)$, if its sign is changed according to γ_c, then the generated factor $k-p+j$ is in the core foldover plan. According to (A1), if there is an even number of a_i's in G_j, then $k-p+j$ should be added to γ_b; according to (A2), if there is an odd number of a_i's in G_j, then $k-p+j$ should not be added to γ_b. In both cases, the sign of G_j is also changed.

The foregoing step shows that there are 2^{k-p} distinct foldover plans in each of these 2^p groups. (The example displayed in Table 1 can be used for illustration purposes.) When the foregoing step is repeated for all core foldover plans, a total of $2^{k-p} \times 2^p = 2^k$ foldover plans are constructed by this method. If we can prove that all of these foldover plans are distinct, then the theorem is proved. To show that all of these foldover plans are distinct, consider two different core foldover plans, γ_c and $\gamma_{c'}$. The construction method described in the last step shows that for all γ_b consisting only of basic factors, there exists a foldover plan γ_2 such that $\gamma_c \equiv \gamma_b + \gamma_2$, where γ_2 consists of all generated factors $k-p+j$ $(j = 1, \ldots, p)$ satisfying (A1) and (A2) for γ_c. Similarly, $\gamma_{c'} \equiv \gamma_b + \gamma_{2'}$, where $\gamma_{2'}$ consists of all generated factors satisfying (A1) and (A2) for $\gamma_{c'}$. Because $\gamma_c \not\equiv \gamma_{c'}$, we have $\gamma_2 \not\equiv \gamma_{2'}$; that is, the two foldover plans are distinct.

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