Optimal Foldover Plans for Two-Level Nonregular Orthogonal Designs

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This article considers optimal foldover plans for nonregular designs. By using the indicator function, we define words with fractional lengths. The extended word-length pattern is then used to select among nonregular foldover designs. Some general properties of foldover designs are obtained using the indicator function. We prove that the full-foldover plan that reverses the signs of all factors is optimal for all 12-run and 20-run orthogonal designs. The optimal foldover plans for all 16-run (regular and nonregular) orthogonal designs are constructed and tabulated for practical use. Optimal foldover plans for higher-order orthogonal designs can be constructed in a similar manner.

KEY WORDS: Extended word length pattern; Foldover design; Generalized resolution; Indicator function; Orthogonal design.

1. INTRODUCTION

Foldover is a classic technique used to create a follow-up experiment. The design is obtained by reversing the signs of one or more factors (columns) in an initial design. We call a set of factors whose signs are reversed in the foldover design a *foldover plan*. Conventional wisdom is to reverse the signs of all factors (called a *full-foldover plan*). The foldover plan that reverses the sign of only one factor was discussed by Box, Hunter, and Hunter (1978) as a way to dealias the most important factor with other factors. Montgomery and Runger (1996) considered reversing the signs of one or two factors. Li and Mee (2002) and Li and Lin (2003) studied optimal foldover plans for regular orthogonal designs with respect to the minimum aberration criterion of the combined designs. Ye and Li (2003) gave some theoretical results on the blocking effect on the foldover of regular two-level designs. In this article we consider optimal foldover plans for nonregular two-level orthogonal designs.

A nonregular design is one whose columns do not form an elementary Abelian group. In nonregular orthogonal designs, factorial effects may be partially aliased, that is, neither orthogonal nor fully aliased. Compared with regular designs, nonregular designs have more complicated aliasing structure, but they can provide more flexibility in run sizes and entertain more distinct models.

Several authors have discussed foldover plans for nonregular designs. Foldover of Plackett–Burman designs was briefly mentioned by Box and Wilson (1951), Box and Hunter (1961) and Box et al. (1978). Recently, Diamond (1995) studied the projection properties of the foldover of the 12-run Plackett–Burman design. Miller and Sitter (2001) discussed applications of this foldover design and proposed a data analysis strategy for such designs. In this article we investigate foldover plans for general nonregular designs in terms of the generalized resolution and minimum aberration criteria proposed by Deng and Tang (1999). A mathematical tool that we use here is the *indicator function*, which is essential in describing properties of foldover designs, as shown in later sections. We extend the previous work to defining *fractional* word lengths and *extended word length patterns.*

Consider an experiment of 8 2-level factors in 16 runs, constructed by choosing the columns 1, 2, 4, 7, 8, 10, 12, and 15 of the third Hadamard matrix of Hall (1961). This design has 12 partially aliased 3-factor effects and 25 aliased 4-factor effects (1 fully aliased and 24 partially aliased). If a foldover design is used to break as many aliased effects as possible, then the key question arises: Should we use a full-foldover plan? Reversing the signs of all 8 columns would eliminate all 12 partially aliased 3-factor effects, but leave the 25 aliased 4-factor effects intact in the combined 32-run design. As we show in Section 4, a better foldover plan is to reverse the signs of only the second, third, and fourth columns. Doing so will produce only 12 aliased 4-factor effects in the combined designs, all of which are only *partially* aliased.

The remainder of the article is organized as follows. Section 2 discusses the indicator function and generalized minimum aberration criterion. Section 3 discusses some general properties of foldover designs are studied. Section 4 gives a catalog of optimal foldover plans for 16-run orthogonal designs. It is proved that the optimal foldover plan for 12-run and 20-run orthogonal designs is the full-foldover plan. Finally, Section 5 gives concluding remarks.

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2. INDICATOR FUNCTIONS AND FRACTIONAL LENGTH WORDS

Deng and Tang (1999) first generalized resolution and minimum aberration criteria to compare nonregular designs. To facilitate our investigation on foldover designs, we present these criteria through a different approach. Our approach is based on *indicator functions*, which relate a factorial design to a polynomial whose coefficients reveal aliasing structure of the design. In this section we start with a brief introduction on indicator functions, then extend the notion of "words" to accommodate *fractional length words* and *extended word length pattern*, concepts that play key roles in the discussion of foldover designs.

Consider a fractional factorial design with k factors. Denote its design space $\mathcal{D}$ as a collection of 2^k points $\{(d_1,\ldots,d_k), d_i=\pm 1,\ i=1,\ldots,k\}$. Then any k-factor design $\mathcal{A}$ can be seen as a collection of points in $\mathcal{D}$. The *indicator function* of $\mathcal{A}$ is a function defined on $\mathcal{D}$ such that $F(x)=r_x$ for $x\in\mathcal{A}$ and 0 otherwise, where r_x is the number of replicates of point x in design $\mathcal{A}$. Define $X_I(\mathbf{x})=\prod_{i\in I}x_i$ on $\mathcal{D}$ for $I\in\mathcal{P}$, where $\mathcal{P}$ is the collection of all subsets of $\{1,\ldots,k\}$. Then the indicator function of $\mathcal{A}$ has a polynomial form $F(\mathbf{x})=\sum_{I\in\mathcal{P}}b_I X_I(\mathbf{x})$, where the coefficients $\{b_I\ I\in\mathcal{P}\}$ are uniquely determined by

$$b_I = 1/2^k\sum_{\mathbf{x}\in\mathcal{A}} X_I(\mathbf{x}). \qquad (1)$$

In particular, $b_\emptyset = n/2^k$. (For more details, see Fontana et al. 2000; Ye 2003.)

For a regular 2^{k-p} design, the polynomial form of its indicator function can be easily obtained by its generators. For example, the indicator function of a 2^{5-2} design $\mathcal{A}$ with generators $1=x_1x_2x_4$ and $1=x_1x_3x_5$ is $F(x_1,x_2,x_3,x_4,x_5)=\frac{1}{4}(1+x_1x_2x_4)(1+x_1x_3x_5)=\frac{1}{4}+\frac{1}{4}x_1x_2x_4+\frac{1}{4}x_1x_3x_5+\frac{1}{4}x_2x_3x_4x_5$. Furthermore, it can be observed that terms with nonzero coefficients (except the constant) in the polynomial are exactly *defining words* of the defining contrast subgroup. This is true for all regular designs.

For nonregular factorial designs, their indicator functions are more complicated because of their complicated aliasing structure. Consider a 12×5 projection of the 12-run Plackett–Burman design listed in Table 1, denoted by $\mathcal{A}_{PB}$ throughout this article. Using (1), one can easily compute all coefficients $\{b_I, I\in\mathcal{P}\}$ and obtain the polynomial form of its indicator function, $F_{\mathcal{A}_{PB}}(x_1,x_2,\ldots,x_5)=\frac{1}{2^5}(12+4x_1x_2x_3-4x_1x_2x_4+4x_1x_2x_5+4x_1x_3x_4-4x_1x_3x_5+4x_1x_4x_5+4x_2x_3x_4-4x_2x_3x_5+4x_2x_4x_5-4x_3x_4x_5-4x_1x_2x_3x_4+4x_1x_2x_3x_5-4x_1x_2x_4x_5+4x_1x_3x_4x_5+4x_2x_3x_4x_5+8x_1x_2x_3x_4x_5)$. The relationship between the indicator function and defining words of a regular design allows us to extend the notion of *words* to nonregular designs. It is natural to call a term with nonzero coefficient (except the constant) in the indicator function of a design a *word*. For example, the design $\mathcal{A}_{PB}$ has the following 15 words: $x_1x_2x_3$, $x_1x_2x_4,\ldots,x_1x_2x_3x_4x_5$.

Table 1. A Projection of a 12-Run Plackett–Burman Design

Run	X_1	X_2	X_3	X_4	X_5
1	1	1	1	1	1
2	−1	1	−1	1	1
3	−1	−1	1	−1	1
4	1	−1	−1	1	1
5	−1	1	−1	−1	−1
6	−1	−1	1	−1	1
7	−1	−1	−1	1	−1
8	1	−1	−1	−1	−1
9	1	1	−1	−1	1
10	1	1	1	−1	−1
11	−1	1	1	1	−1
12	1	−1	1	1	−1

In a regular design, word length is defined to be the number of letters of the word. (We call each x_i in a word a *letter*.) Each word implies full aliasing among associated factorial effects. In a nonregular design, however, partial aliasing exists. Thus two words with the same numbers of letters may indicate different degrees of aliasing. Note that $|b_I/b_\emptyset|$, which is always between 0 and 1, measures the degree of aliasing associated with the word X_I. We define the *generalized word length* of X_I as the number of letters of the word plus $(1-|b_I/b_\emptyset|)$. For a regular design, word length equals the number of letters of the word, because each word X_I has $|b_I/b_\emptyset|=1$, whereas the length of words for a nonregular design may be fractional. Consider the word $x_1x_2x_3$ in design $\mathcal{A}_{PB}$, based on the definition, its length is $3+(1-4/12)=3\frac{2}{3}$. In fact, this design has 10 length-$(3\frac{2}{3})$ words, 5 length-$(4\frac{2}{3})$ words, and 1 length-$(5\frac{1}{3})$ word. This new definition keeps the desirability of *words*. The shorter a word, the less desired it is. Let $f_{i+j/n}$ be the number of length-$(i+j/n)$ words. We define *extended word length pattern* (EWLP) of $\mathcal{A}$ to be $(f_1,\ldots,f_{1+(n-1)/n},f_2,\ldots,f_{2+(n-1)/n},\ldots,f_k,\ldots,f_{k+(n-1)/n})$. For example, the extend word length pattern of design $\mathcal{A}_{PB}$ has $f_{3\frac{2}{3}}=10, f_{4\frac{2}{3}}=5, f_{5\frac{1}{3}}=1$. Its resolution is $3\frac{2}{3}$.

3. PROPERTIES OF FOLDOVER DESIGNS

Denote a foldover plan by a set γ, in which each element represents a factor whose sign is reversed in the foldover design. From the definition of the indicator function, we can easily obtain the following three properties, which are essential for studying the foldover design:

1. Let $F(x_1,x_2,\ldots,x_k)$ be the indicator function of a design. If the sign of factor X_i is reversed, then the indicator function of the new design is $F(x_1,\ldots,-x_i,\ldots,x_k)$.
2. Let $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ be indicator functions of two designs, $\mathcal{A}$ and $\mathcal{B}$. Then the indicator function of the combined design $\mathcal{A}\cup\mathcal{B}$ is given by $F_{\mathcal{A}\cup\mathcal{B}}=F_{\mathcal{A}}+F_{\mathcal{B}}$.
3. Denote the initial design and its foldover resulting from a foldover plan γ by $\mathcal{A}$ and $\mathcal{A}_\gamma$. Let the combined design of $\mathcal{A}$ and $\mathcal{A}_\gamma$ be denoted by $\mathcal{S}=\mathcal{A}\cup\mathcal{A}_\gamma$. Then the indicator functions of $\mathcal{A}_\gamma$ and $\mathcal{S}$ are given by $F((-1)^{\delta_1}x_1,\ldots,(-1)^{\delta_k}x_k)$ and $F(x_1,\ldots,x_k)+F((-1)^{\delta_1}x_1,\ldots,(-1)^{\delta_k}x_k)$, where $\delta_i=1$ if $X_i\in\gamma$ and 0 otherwise, $i=1,\ldots,k$.

The proofs are straightforward and thus are omitted.

In general, let $\mathcal{A}$ be a two-level factorial design with the indicator function $F_{\mathcal{A}}(x_1,\ldots,x_k)=\sum_{I\in\mathcal{P}}b_I X_I$. Let $\mathcal{A}_\gamma$ be its foldover with the indicator function $F_{\mathcal{A}_\gamma(x_1,\ldots,x_k)}=\sum_{I\in\mathcal{P}}b'_I X_I$. The indicator function of the combined design $\mathcal{S}=\mathcal{A}\cup\mathcal{A}_\gamma$ is given by $F_{\mathcal{S}}(x_1,\ldots,x_k)=\sum_{I\in\mathcal{P}}c_I X_I$, where $c_I=b_I+b'_I$ from Property 3. Here we call a word an *even-letter* word if it has an even number of letters. That is, the *integer* part of its length is an even number. The odd-letter word is defined similarly. From

Property 3, it is not difficult to see that a word in the initial design will be cancelled in the combined design if and only if the signs of an odd number of factors are reversed in the foldover design; that is,

$$c_I = \begin{cases} 0 & \text{if } \gamma \cap I \text{ has an odd number of elements} \\ 2b_I & \text{if } \gamma \cap I \text{ has an even number of elements.} \end{cases} \quad (2)$$

Condition (2) shows that it is not necessary to reverse the signs of all factors to eliminate all 3-letter words. Specifically, for designs with resolution at least 4 but less than 5, there exists a better foldover plan than the full-foldover plan. This is because the shortest word must be a 4-letter word, which can be eliminated by simply reversing the sign of one factor involved in the word.

Three additional important properties of foldover designs are given next. For simplicity of presentation, their proofs are provided in Appendix A.

4. Let $\mathcal{A}$ be a two-level factorial design with the generalized resolution $3 \le R_{\mathcal{A}} < 4$. Let $\mathcal{S}$ be the combined design of $\mathcal{A}$ and its full foldover. Then the resolution of $\mathcal{S}$ is at least 4. Moreover, the design has no odd-letter word.
5. Let $\mathcal{A}$ be a two-level factorial design with the generalized resolution $4 \le R_{\mathcal{A}} < 5$. The combined design after full foldover has the same resolution as the initial design. Moreover, all even-letter words of the initial design have the same lengths in the combined design.
6. There exists a foldover plan that can either increase the resolution or reduce the number of shortest-length words of the initial design.

Note that Properties 4 and 5, regarding the full-foldover plan, are direct extensions of the well known fact that the full-foldover plan results in a combined design of resolution IV for any regular 2^{k-p} design of resolution III or IV.

4. OPTIMAL FOLDOVER DESIGNS

Consider a combined design $\mathcal{S} = \mathcal{A} \cup \mathcal{A}_\gamma$. For a given design $\mathcal{A}$, its optimal foldover plan is given by γ^* such that $EWLP(\mathcal{A} \cup \mathcal{A}_{\gamma^*}) = \min_\gamma EWLP(\mathcal{A} \cup \mathcal{A}_\gamma)$. In this section we discuss the optimal foldover plans for 12-run, 16-run, and 20-run designs. We focus on 16-run designs, because they are most commonly used in practice.

4.1 Foldover of Minimum Aberration 16-Run Designs

For $n = 16$, there are 5 nonisomorphic Hadamard matrices. Lin and Draper (1992) first studied the geometric projection properties of these Hadamard matrices onto dimensions 3, 4, and 5. Using a computer search method, Sun (1993) obtained all nonisomorphic $16 \times m$ designs. Li and Lin (2003) studied optimal foldover plans (in terms of the aberration criterion) for the regular 16-run designs.

Here we focus on the optimal foldover plans for nonregular 16-run designs (see, e.g., Lin and Draper 1995). By using the generalized aberration criterion discussed in Section 2, we identify the best designs among regular and nonregular designs for all nonisomorphic $16 \times m$ ($5 \le m \le 14$) designs given by Sun (1993). Table 2 shows that when $m \le 8$, the minimum aberration regular designs are preferred. They have a resolution of 4 or higher, whereas the best nonregular designs have a generalized resolution of 3.5. For $m \ge 9$, however, the minimum aberration nonregular designs may be preferable, because they have a resolution of 3.5, compared with the minimum aberration designs with a resolution of 3. In the former designs, no main effects are fully confounded with the two-factor interactions. The EWLP of each design is also displayed in the table, because it provides further information on the design considered. For example, for $m = 9$, there are 23 resolution-3.5 nonregular designs. Among them, the number of length-3.5 words ranges from 16 to 28. The EWLPs of all of the $16 \times m$ designs are provided in Tables 5–14 as are the optimal foldover plans for all 16-run designs and the results. (To save space, these tables are not included in this article. An electronic copy can be obtained at *http://legacy.csom.umn.edu/wwwpages/faculty/wli.)*

Table 3 summarizes the optimal foldover plans for all minimum aberration designs. The design index in the table corresponds to the index number used by Sun (1993). As indicated in Table 3, for $m \le 8$, the minimum aberration designs are regular designs; the others are nonregular designs. For $m = 12$, there are two minimum aberration designs, designs 12.29 and 12.36. Note that nonisomorphic designs may have the same EWLPs as shown here. Table 3 demonstrates that in all cases, the combined designs have resolution 4. The conventional foldover plan—the full-foldover plan—is optimal in some cases, especially for large values of m.

4.2 Combined-Optimal Designs

The experimenter usually prefers the minimum aberration design, because it minimizes the aliasing among effects generally. However, the foldover of a minimum aberration de-

Table 2. Summary of Best 16-Run Designs Among Regular and Nonregular Designs

m	Total number of designs	Number of regular designs	Minimum aberration regular design: Resolution	Minimum aberration regular design: WLP	Minimum aberration nonregular design: Resolution	Minimum aberration nonregular design: WLP
5	11	5	5.0	(0 0, 0 0, 1 0)	3.5	(0 1, 0 2, 0 1)
6	27	5	4.0	(0 0, 3 0, 0 0, 0 0)	3.5	(0 2, 1 4, 0 2, 0 0)
7	55	6	4.0	(0 0, 7 0, 0 0, 0 0)	3.5	(0 4, 3 8, 0 4, 0 0)
8	80	6	4.0	(0 0, 14 0, 0 0, 0 0)	3.5	(0 12, 1 24, 0 16, 0 0)
9	87	5	3.0	(6 0, 9 0, 9 0, 0 0)	3.5	(0 16, 14 0, 0 32, 0 0)
10	78	4	3.0	(8 0, 18 0, 16 0, 8 0)	3.5	(0 32, 10 32, 0 64, 0 32)
11	58	3	3.0	(12 0, 26 0, 28 0, 24 0)	3.5	(0 48, 14 48, 0 112, 8 64)
12	36	2	3.0	(16 0, 39 0, 48 0, 48 0)	3.5	(0 64, 15 96, 0 192, 0 192)
13	18	1	3.0	(22 0, 55 0, 72 0, 96 0)	3.5	(0 88, 15 160, 0 288, 0 384)
14	10	1	3.0	(28 0, 77 0, 112 0, 168 0)	3.5	(0 112, 21 224, 0 448, 0 672)

NOTE: For 16-run designs, each word length is either k or $k + .5$ $(2 < k < m+1)$.

Table 3. Minimum Aberration 16× m Designs and Their Optimal Foldover Plans

m	Design index	Initial design Resolution	Initial design WLP	Optimal foldover plan	Combined design Resolution	Combined design WLP
6	6.5	4.0	(0 0, 3 0, 0 0, 0 0)	{1}	4.0	(1 0, 0 0, 0 0)
7	7.6	4.0	(0 0, 7 0, 0 0, 0 0)	{1, 6}	4.0	(3 0, 0 0, 0 0)
8	8.6	4.0	(0 0, 14 0, 0 0, 0 0)	{1, 2}	4.0	(6 0, 0 0, 0 0)
9	9.25	3.5	(0 16, 14 0, 0 32, 0 0)	{9}	4.0	(14 0, 0 0, 0 0)
10	10.48	3.5	(0 32, 10 32, 0 64, 0 32)	{1, 2, 3, 4}	4.0	(10 32, 0 0, 0 32)
11	11.37	3.5	(0 48, 14 48, 0 112, 8 64)	{1, 2, 3, 4}	4.0	(14 48, 0 0, 8 64)
12	12.29	3.5	(0 64, 15 96, 0 192, 0 192)	Full-foldover	4.0	(15 96, 0 0, 0 192)
	12.36	3.5	(0 64, 15 96, 0 192, 0 192)	{1, 2, 3, 4}	4.0	(15 96, 0 0, 0 192)
13	13.15	3.5	(0 88, 15 160, 0 288, 0 384)	Full-foldover	4.0	(15 160, 0 0, 0 384)
14	14.8	3.5	(0 112, 21 224, 0 448, 0 672)	Full-foldover	4.0	(21 224, 0 0, 0 672)

NOTE: For both initial and combined designs, each word length is either k or $k+.5$ $(2<k<m+1)$.

sign may not produce a good design in terms of the aberration of the *combined design.* Thus Li and Lin (2003) proposed the *combined-optimal design.* Recall that for a given design $\mathcal{A}$, its optimal foldover plan is given by γ^* such that $EWLP(\mathcal{A}\cup\mathcal{A}_{\gamma^*})=\min_\gamma EWLP(\mathcal{A}\cup\mathcal{A}_\gamma)$. Then a design $\mathcal{A}^c$ is called a combined-optimal design if $EWLP(\mathcal{A}^c\cup\mathcal{A}^c_{\gamma^*})=\min_{\mathcal{A}} EWLP(\mathcal{A}\cup\mathcal{A}_{\gamma^*})$.

Table 4 summarizes the optimal foldover plans for all combined-optimal designs. We first note that only one combined-optimal design—design 6.3—is a regular design. All the other designs are nonregular designs. For $m\le 9$, the combined-optimal designs have a nice property: their combined designs have a resolution of 4.5 or higher. In these cases, no two-factor interaction is fully confounded with a main effect or other two-factor interactions. For $m\ge 10$, the combined-optimal designs are generally not recommended; these initial designs are resolution-3 designs, whereas the minimum aberration designs have a resolution of 3.5. In both cases, their combined designs have a resolution of 4.

In Section 1, we described a 16 × 8 design in which there are 12 partially aliased three-factor effects. This design corresponds to design 8.42 of Table 4. As shown in the table, the optimal foldover plan $\gamma^*=\{2,3,4\}$ results in the $EWLP=(0,12;1,12;0,0)$ for the combined design. Had a full foldover plan been used, the resulting combined design would have had 1 length-4 word and 24 length-4.5 words. Obviously, γ^* is a much better plan.

4.3 Foldover of 12-Run and 20-Run Designs

All 12-run orthogonal designs are projection of the unique nonisomorphic 12-run Hadamard matrix. It can be shown that for all 12-run orthogonal designs, the full-foldover plan is the optimal foldover plan. Furthermore, it is the only foldover plan that generates resolution IV combined designs. The proof is given in Appendix B.

In a recent report, Sun, Li, and Ye (2002) presented a complete catalog of all these nonisomorphic 20-run 2-level designs. We investigate the optimal foldover plans of all these 20-run designs and find that all optimal foldover plans are full-foldover plans. The proof of this is also given in Appendix B.

5. CONCLUDING REMARKS

In this article we have investigated optimal foldover plans for nonregular designs. We used the indicator function to define the fractional length word pattern and the EWLP. After providing some theoretical results on foldover designs, we discussed the optimal foldovers of 12-run, 16-run, and 20-run designs.

Tables 2, 3, and 4 focus on the use of aberration criterion based on the EWLP. We also obtained optimal foldover plans according to the G_2 criterion proposed by Tang and Deng (2000). The G_2 criterion aims to compare the average degree of confounding of words with same number of letters. Using the indicator function, the G_2 aberration criterion can be defined as $\{\alpha_1(\mathcal{A}),\dots,\alpha_s(\mathcal{A})\}$, where $\alpha_j(\mathcal{A})=\sum_{\|I\|=j}(b_I/b_\emptyset)^2$ (Ye 2003).

Table 4. Combined-Optimal 16× m Designs and Their Optimal Foldover Plans

m	Design index	Initial design Resolution	Initial design WLP	Optimal foldover plan	Combined design Resolution	Combined design WLP
6	6.3	3.0	(0 8, 0 0, 0 0, 1 0)	{1, 4}	6.0	(0 0, 0 0, 1 0)
	6.18	3.5	(0 8, 0 0, 0 0, 1 0)	{1, 2}	6.0	(0 0, 0 0, 1 0)
7	7.11	3.0	(1 4, 1 8, 1 4, 0 0)	{1, 7}	4.5	(0 4, 1 4, 0 0)
	7.32	3.5	(0 8, 0 12, 1 4, 0 0)	{2, 3}	4.5	(0 4, 1 4, 0 0)
8	8.39	3.0	(1 12, 0 24, 1 12, 0 0)	{2, 3, 4}	4.5	(0 12, 1 12, 0 0)
	8.42	3.5	(0 12, 1 24, 1 12, 0 0)	{2, 3, 4}	4.5	(0 12, 1 12, 0 0)
9	9.71	3.0	(1 20, 0 42, 0 30, 0 14)	Full-foldover	4.5	(0 42, 0 0, 0 14)
10	10.62	3.0	(2 28, 1 62, 0 58, 0 44)	Full-foldover	4.0	(1 62, 0 0, 0 44)
11	11.45	3.0	(3 39, 3 90, 0 102, 0 102)	Full-foldover	4.0	(3 90, 0 0, 0 102)
	11.46	3.0	(3 38, 3 90, 0 106, 0 102)	Full-foldover	4.0	(3 90, 0 0, 0 102)
12	12.27	3.0	(4 52, 6 128, 0 176, 0 208)	Full-foldover	4.0	(6 128, 0 0, 0 208)
	12.33	3.0	(4 52, 6 128, 0 176, 0 208)	{1, 2, 3, 4, 5, 6}	4.0	(6 128, 0 0, 0 208)
13	13.14	3.0	(5 68, 10 180, 0 288, 0 384)	Full-foldover	4.0	(10 180, 0 0, 0 384)
	13.16	3.0	(7 60, 10 180, 0 288, 0 384)	{1, 2, 3, 4, 5, 6, 7}	4.0	(10 180, 0 0, 0 384)
	13.17	3.0	(4 72, 10 180, 0 288, 0 384)	{1, 2, 3, 4, 5, 6}	4.0	(10 180, 0 0, 0 384)
14	14.9	3.0	(7 84, 14 252, 0 448, 0 672)	{1, 2, 3, 4, 5, 6, 7}	4.0	(14 252, 0 0, 0 672)

NOTE: For both initial and combined designs, each word length is either k or $k+.5$ $(2<k<m+1)$.

Tang and Deng (2000) proposed a family of G_p aberration criteria defined as $\alpha_j(\mathcal{A}) = \sum_{\|I\|=j} |b_I/b_\emptyset|^p$. Ye (2003) showed a justification for using G_2 over other G_p criteria. That is, the sum of the G_2 word length pattern vector of a design $\mathcal{A}$ equals $\frac{2^k}{n} - 1$, regardless of whether it is regular or nonregular if it has no replicates. This property does not hold for other G_p criteria. In general, the EWLP criteria and the G_2 criteria do not necessarily produce the same optimal foldover designs. However, for all 16-run two-level designs, we find that the optimal foldover plans resulting from the G_2 criterion are the same as those resulting from the EWLP criterion. The G_2 criterion values of the initial designs and the combined designs are given in Tables 5–14 (available at *http://legacy.csom.umn.edu/wwwpages/faculty/wli)*.

During the final stage of this project, we were made aware that table 2 of Deng and Tang (2002) provides the minimum aberration designs for the cases of $16 \times p$ ($p = 3, \ldots, 14$). The difference is that in our Table 2, we further distinguish between minimum aberration regular design and minimum aberration nonregular design for each p.

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APPENDIX A: PROOFS OF PROPERTIES 4, 5, AND 6 OF SECTION 3

Proof of Property 4

Because the initial design has a resolution of at least 3 but less than 4, the word with the least length in its indication function has 3 letters. When the full-foldover plan is used, the signs of all factors are reversed in the foldover design. Thus, all 3-letter words and other odd-letter words will not appear in the indicator function of the foldover design.

Proof of Property 5

Because the resolution of the initial design is at least 4 and less than 5, the term with the least length in the indicator function has 4 letters. For a full-foldover plan, all even-letter words stay in the combined design and their coefficients $c_I = 2b_I$. The constant term $c_\emptyset = 2b_\emptyset$ is also doubled. Therefore, for an even-letter word I, the word length $\|I\| - c_I/c_\emptyset$ is unchanged. From the definition, the combined design has the same resolution as $\mathcal{A}$.

Proof of Property 6

Let X_I be the shortest word. Then for a foldover plan γ that contains an odd number of factors in I, the word X_I is then eliminated in the combined design.

APPENDIX B: OPTIMAL FOLDOVER PLANS FOR 12-RUN AND 20-RUN DESIGNS

In the 12-run Plackett–Burman design, all main effects are partially confounded with all other two-factor interactions not containing this effect. That is, all 3-letter words are present in the indicator function. Without loss of generality, consider a projection to the first l columns. For $I \subset \{1, 2, \ldots, l\}$, it is easy to see that $\sum_{\mathcal{A}} X_I = n(b_I/b_\emptyset)$ are the same. Thus all 3-letter words of the projection are still in its indicator function. Because the full-foldover plan is the only foldover plan to make all 3-letter words disappear in the combined design, the full-foldover plan is the optimal foldover plan.

By examining all nonisomorphic 20-run orthogonal arrays presented by Sun et al. (2002), we find that each design has all possible 3-letter words in its indicator function. Therefore, following the same arguments for 12-run designs, the optimal foldover plan of 20-run designs is the full-foldover plan.

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