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A NEW CLASS OF ORTHOGONAL ARRAYS AND ITS APPLICATIONS

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ABSTRACT : The paper presents two new non-isomorphic symmetric orthogonal arrays *OA* $(4^5, 41, 4, 3)$. Furthermore, based on these orthogonal arrays, we have obtained some new trend resistance orthogonal arrays and an asymmetric orthogonal array *OA* $(4^5, 40, (4^2) \times 4^{39}, 2)$

1. INTRODUCTION

Rao [13], [14] introduced the concept of orthogonal arrays in the context of fractional factorial experiments. Orthogonal arrays are related to combinatorics, finite fields, geometry and error-correcting codes. Asymmetric orthogonal arrays, also introduced by Rao [15] have received great attention in recent years. Symmetric and asymmetric orthogonal arrays have been used extensively by Taguchi [17] and his colleagues in industrial experiments for quality improvement. Their use in agricultural experiments has also been widespread. Orthogonal plans have been found useful in setting up many experiments in the physical and engineering sciences. Sometimes when the experimental runs are carried out in a time-ordered sequence, the response can depend on the run order. To avoid unwanted time effect, one may be interested in a run order which is trend resistant i.e. the experimental runs are arranged in such a way that all the effects (main or t-factor interaction) are independent of trend (linear, quadratic etc.)

An $N \times n$ array A with entries from S, where S is a set of q symbols or levels, is said to be an orthogonal array with q levels, strength g and index λ, if every $N \times g$ subarray of A contains each g-tuple based on S exactly λ times (as a row), and is denoted by *OA (N, n, q, g)*. Two orthogonal arrays are said to be isomorphic if one can be obtained from the other by a sequence of permutations of the columns, the rows, and the levels of each factor. It is well known that an orthogonal array of strength g is a fractional factorial design of resolution $g+1$.

Bose (1947) studied the following packing problem. Let $m_t(k,q)$ denote the maximum number of columns, that can be chosen in a k rowed matrix, whose elements belong to a Galois Field of order q, $GF(q)$ where $q = p^n$ and p is prime, and which has the property P_t that no t columns are linearly dependent. Equivalently $m_t(k,q)$ is the maximum number of factors which can be accommodated in a confounded q^m symmetric factorial experiment with q^{m-k} blocks, each of size q^k, such that no t-factor or lower order interaction is confounded.

Consider the $k-1$ dimensional projective space over the finite field F_q of order q denoted by $PG(k-1,q)$. An n-cap in $PG(k-1,q)$ is a set of n points, no three of which are linearly dependent. Finding value of $m_t(k,q)$ is known as the Packing problem. The problem of determining the values of $m_3(k,q)$, first considered by Bose [1], was quickly solved for $q = 2$ for all k. An excellent review of this problem is given in Hirschfield and Storme [10].

Recently Edel and Bierbrauer [7] have shown that 41 is the largest size cap in *PG* (4,4) i.e. $m_3(5,4) = 41$. In statistical terminology this implies that 41 is the maximum number of factors, each at 4 levels, which can be accommodated in a symmetrical 4^{41} factorial design with blocks of size 4^5, such that no three factor or lower order interaction is confounded. As remarked by Bose and Srivasatava [3] it can be interpreted that in a fractionally replicated 4^{41-36} design, consisting of a single block with 4^5 plots or experimental units, such that no main effect is aliased with any other main effect or with any 2-factor interaction.

An n-cap in $PG(k-1,q)$ is a set of n points, no three of which are linearly dependent. If we write the n points as columns of a matrix such that every set of three columns is linearly independent, then it is used as a generator matrix of a linear orthogonal array of strength 3.

The packing problem discussed by Bose [1] has equivalent formulations in finite projective spaces and coding theory. A close connection between orthogonal arrays and linear codes is discussed in Hadayat, Sloane and Stufken [9]. In this paper, two new orthogonal arrays and various statistical applications of these orthogonal arrays are discussed. Section 2 gives two non-isomorphic orthogonal arrays $OA(4^5,41,4,3)$ and they are distinguished using minimum aberration criteria. Trend resistance orthogonal arrays obtained from the generator matrix of this orthogonal array are discussed in Section 3. Asymmetric orthogonal array $OA(4^5,40,(4^2)\times 4^{39},2)$, obtainable from the above generator matrix is presented in Section 4.

2. ORTHOGONAL ARRAY $OA(4^5, 41, 4, 3)$

Let $q(\geq 2)$ be a prime or a prime power and consider $GF(q)$, the Galois field of order q. The following fundamental lemma due to Bose and Bush [2] is helpful.

Lemma 2.1 : Let there exist an $r \times n$ matrix C, with elements from $GF(q)$ such that every $r \times g$ submatrix of C has rank g. Further let ξ denote an $q^r \times r$ matrix whose rows are all possible r-tuples over $GF(q)$. Then an orthogonal array $OA(q^r, n, q, g)$ can be constructed,

i.e., $A = \xi C$ is an $OA(q^r, n, q, g)$.

Edel and Bierbrauer (1999) have given two essentially different 41-caps in $PG(4,4)$.

A : The columns of the following matrix M_1 form a 41-cap in $PG(4,4)$.

$$M_1 = \begin{bmatrix} 10000213010223333122103103230321021023032 \\ 01000132101013221322010121332022301101303 \\ 00100303223220123321330101023302112102012 \\ 00010032111103331223101030223133210010212 \\ 00001130331132032231021013303320332120102 \end{bmatrix}$$

B : The columns of the following matrix M_2 form another 41-cap in $PG(4,4)$

$$M_2 = \begin{bmatrix} 10000112213322333111333020022100311310012 \\ 01000100200210110110130300230321231311222 \\ 00100012002001101101103302003312213311222 \\ 00010110011100001111111111111111111101001 \\ 00001001111122222211133333300022222200113 \end{bmatrix}$$

Let the elements be $GF(4)$ of $\{0,1,x,x^2\}$ where $x^2 = x+1$. Here elements are denoted by 0,1,2,3 with 2+3=1 and 2.3=1.

In Lemma 2.1 if we put q=4, r=5, g=3 and n=41, then every 4×3 submatrix of M_1 (M_2) has rank 3. Hence $A = \xi M_1$ yields an $OA(4^5,41,4,3)$. Again with M_2, we get another $OA(4^5,41,4,3)$. These orthogonal arrays are non-isomorphic. The orthogonal array $OA(4^5,41,4,3)$ is apparently new in the literature, being saturated in the sense that it contains the maximum possible number of 4-level factors.

Minimum aberration is a criterion to distinguish between several non-isomorphic orthogonal arrays (see Fries and Hunter (8)). For any two fractional factorial designs d_1 and d_2 with the same parameters, let i be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$, where A_i denotes the number of words of length i in the defining contrast subgroup, then d_1 is said to have less aberration than d_2 if

$$A_r(d_1) < A_r(d_2).$$

Consider, for example, the following two 2^{7-2} designs.

$$d_1 : I = 4567 = 12346 = 12357$$

$$d_2 : I = 1236 = 1457 = 234567$$

the word length patterns are $W(d_1) = (0,0,0,1,2,0,0)$ and

$W(d_2) = (0,0,0,2,0,1,0)$. Here d_1 has minimum aberration.

The weight enumerator of an $[n,k,d]$ code C is $Wc(x,y) = \sum A_i x^{n-i} y^i$, where A_i is the number of codewords of weight i. The weight ennmerator of the dual code $C^{\perp}$ can be obtained using the formula given by Macwilliams and Sloane (11) as $Wc^{\perp}(x,y) = (1/N)Wc(x+(q-1)y, x-y)$ for $N = q^k$. For details see Macwilliams and Sloane (11). The weight distribution of dual code of a q-ary linear $[n,k,d]_q$ code is related to the word length pattern of q level fractional factorial design as given by Ma and Fang (12).

$$WLP = (1/q-1) \text{ weight distribution of the dual code.}$$

The orthogonal array generated above is a $[41,5,28]_4$ code. Edel and Bierbrauer [7] have given the weight distribution of $[41,5,28]_4$ code C. The weight distribution of the code generated by M_1 is $A_{28} = 120$, $A_{29} = 360$, $A_{31} = 288$, $A_{32} = 135$, $A_{37} = 120$ and the weight distribution of the code generated by M_2 is $A_{24} = 29$, $A_{26} = 12$, $A_{28} = 105$, $A_{30} = 660$, $A_{32} = 90$, $A_{34} = 36$, $A_{36} = 51$, $A_{38} = 60$.

Using Macwilliam's identity we have obtained the weight enumerator of the dual code $[41,36,4]_4$. We observe that the number of codewords with weight 4 in the code generated by M_1 are 9450 and in the code generated by M_2 are 9930. Hence the $OA(4^5,41,4,3)$ generated by M_1 has less aberration and thus is better.

3. Trend Resistance Orthogonal Arrays

In some industrial situations, the order of the treatment combinations is important such as, where the experiment is carried over a period of time and consecutive observations under any given plan are influenced by a time trend in addition to the factorial effects. One may be interested in a run order which is trend resistant.

For any $d \in D_N$, the class of N runs plan, let the successive observations correspond to equispaced points of time and suppose that these observations are influenced by a time trend that can be represented by a polynomial of degree $v(\leq 1 \leq v \leq N-1)$. Hence one may be interested to get a v-trend free plan, which is optimal.

A fractional factorial plan that ensures estimability of the general mean and complete sets of contrasts belonging to factorial effects involving at most f factors under the assumption of absence of factorial effects involving $(t+1)$ or more factors, where $1 \leq f \leq t \leq n-1$, is referred to as a resolution (f,t) plan. For details see Dey & Mukerjee [6].

An orthogonal array is called trend free of order (f,v) $(f,v \geq 1)$ if it has strength at least f and in addition the associated plan satisfies

$$\sum u^k \chi_{d_0}^{i_1 \ldots i_f}\left(u; j_{i_1} \cdots j_{i_f}\right) = \left(q_{i_1} \cdots q_{i_f}\right)^{-1} \sum_{u=1}^{N} u^k \qquad 1 \leq k \leq v$$

for every $i_1, \ldots, i_f$ and $j_{i_1}, \ldots, j_{i_f}$ $(1 \leq i_1 < \ldots < i_f \leq n;\ 0 \leq j_{i_1} \leq q_{i_1}-1, \ldots, 0 \leq j_{i_f} \leq q_{i_f}-1)$, where $\chi_d^{i_1 \ldots i_f}\left(u; j_{i_1} \cdots j_{i_f}\right)$ is an indicator function assuming value 1 if the observation at the time point u according to d corresponds to level $j_{i_1} \cdots j_{i_f}$ of the $i_i^{th}, \ldots, i_f^{th}$ factors respectively and value 0, otherwise.

Results due to Coster and Cheng (5) can be used to construct some trend resistance orthogonal arrays obtainable from the generator matrix of $OA(4^5,41,4,3)$.

Theorem 3.1 : Let $q(\geq 2)$ be a prime or prime power. Suppose that there exists an $r \times n$ matrix C, with elements from $GF(q)$, such that every $r \times g$

submatrix of C has rank g and every column of C has at least $\nu+1$ nonzero elements, then there exists a symmetric $OA(q^r, n, q, g)$ that is trend free of order $(1, \nu)$.

From the matrix M_1 given in Section 2, if we take all the columns of weight 5, we have a matrix

$$C_1 = \begin{bmatrix} 3 & 3 & 1 & 2 & 2 & 2 \\ 2 & 1 & 3 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 & 1 & 2 \\ 3 & 1 & 2 & 2 & 3 & 2 \\ 3 & 2 & 2 & 3 & 1 & 2 \end{bmatrix}$$

and from matrix M_2 we get

$$C_2 = \begin{bmatrix} 2 & 3 & 3 & 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 2 & 3 & 1 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 & 3 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 2 & 1 & 3 \end{bmatrix}$$

where, we define the weight of a column as the number of nonzero elements in a column. The matrices C_1 and C_2 satisfy the conditions of Theorem 3.1 with $q=4$, $r=5$, $g=3$, $\nu+1=5$, $n=6$ and $n=8$ respectively. Hence C_1 gives an $OA(4^5, 6, 4, 3)$ and C_2 gives an $OA(4^5, 8, 4, 3)$.

These orthogonal arrays are trend free of order (1,4). Similarly suppose from M_1 and M_2 we take all columns which have weight greater than or equal to 4. The two matrices obtained are

$$C_3 = \begin{bmatrix} 2 & 1 & 3 & 0 & 1 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 1 & 2 & 2 & 3 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 2 & 1 & 2 \\ 1 & 3 & 2 & 1 & 0 & 1 & 0 & 1 & 3 & 2 & 2 & 1 & 3 & 2 & 2 & 1 & 3 & 3 & 2 & 0 & 2 & 2 & 3 & 0 & 1 & 3 \\ 3 & 0 & 3 & 2 & 2 & 3 & 2 & 2 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 1 & 0 & 2 & 3 & 3 & 0 & 2 & 1 & 1 & 2 & 2 \\ 0 & 3 & 2 & 1 & 1 & 1 & 1 & 0 & 3 & 3 & 3 & 1 & 2 & 2 & 3 & 0 & 2 & 2 & 3 & 1 & 3 & 3 & 2 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 & 3 & 1 & 1 & 3 & 2 & 0 & 3 & 2 & 2 & 3 & 1 & 3 & 3 & 0 & 3 & 3 & 2 & 0 & 3 & 3 & 2 & 2 \end{bmatrix}.$$

which gives an $OA(4^5,26,4,3)$. This is trend free of order (1,3).

$$C_4 = \begin{bmatrix} 3\,3\,3\,3\,3\,2\,2\,2\,3\,3\,3\,0\,1\,0\,0\,3\,1\,1\,3\,1\,2 \\ 0\,2\,1\,1\,0\,1\,1\,0\,1\,3\,0\,3\,3\,2\,1\,2\,3\,1\,3\,2\,2 \\ 2\,0\,1\,0\,1\,1\,0\,1\,1\,0\,3\,3\,3\,1\,2\,2\,1\,3\,3\,2\,2 \\ 2\,0\,1\,0\,1\,1\,0\,1\,1\,0\,3\,3\,3\,1\,2\,2\,1\,3\,3\,2\,2 \\ 1\,1\,0\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,1 \end{bmatrix}$$

C_4 gives an $OA(4^5,21,4,3)$, which is trend free of order (1,3).

If all columns of weight greater than or equal to 3 are taken from M_1 matrix we get

$$C_5 = \begin{bmatrix} 213010223333122323032102121031002303 \\ 132101013221322133202230130101210130 \\ 303223220123321102330211223301010201 \\ 032111103331223022313321021010301021 \\ 130331132032231330332033220210112010 \end{bmatrix}$$

and from M_2 we get

$$C_6 = \begin{bmatrix} 333332223330100311312112212220022100 \\ 021101101303321231322100201000130112 \\ 201011011033312213322012000102003112 \\ 201011011033312213322110010011111010 \\ 110111111111111111111001112233300001 \end{bmatrix}$$

which generate $OA(4^5,36,4,3)$. These are trend free of order (1,2).

4. Asymmetric Orthogonal Array

Rao [15] introduced asymmetric orthogonal arrays having factors with mixed levels which are used for planning industrial experiments. A large number of techniques are known for constructing orthogonal arrays based on Galois field, finite Geometries, Difference schemes, Hadamard matrices, Latin squares and Error correcting codes. For details see Hadayat, Sloane and Stufken [9]. Suen, Das and Dey [16] gave a general method for constructing asymmetric orthogonal arrays of arbitrary strength based on finite fields.

An orthogonal array $OA(N, n, q_1 \times q_2 \times \ldots \times q_n, g)$ is $N \times n$ matrix with symbols in the i^{th} column from a finite set of $q_i (\geq 2)$ symbols, $1 \leq i \leq n$, such that in every $N \times g$ submatrix, all possible combinations of symbols appear equally often as a row. In particular, if $q_1 = \ldots \ldots = q_n$ $(= q$, say), then we get a symmetric orthogonal array which is denoted by $OA(N, n, q, g)$.

Consider an arbitrary $OA(N, n, q_1 \times q_2 \times \ldots \times q_n, g)$ where for $1 \leq i \leq n$, $q_i = s^{u_i}$, s is a prime or prime power, $N = s^r$, the u_i's and r are positive integers. Consider the columns of the array as factors and denote these factors by $F_1, F_2, \ldots, F_n$. For factor $F_1 (1 \leq i \leq n)$ define u_i columns, say $P_{i_1}, \ldots, P_{i_{u_i}}$, each of order $r \times 1$ with elements from $GF(s)$. We have $\sum_{i=1}^{n} u_i$ columns in all corresponding to n factors. The following theorem is helpful.

Theorem 4.1 : Let C be a $r \times \sum_{i=1}^{n} u_i$ matrix given by

$$C = \left[p_{11}, \ldots, p_{1u_1}, \ldots, p_{n_1}, \ldots, p_{nu_n}\right] = A\left[A_1 : A_2 : \ldots : A_n\right]$$

where $A_i = \left[p_{i_1}, \ldots, p_{i_{u_i}}\right]$ $1 \leq i \leq n$, such that for every choice of g matrices $A_{i_1}, \ldots, A_{i_g}$, out of $A_1, A_2, \ldots, A_n$, the $r \times \sum_{j=1}^{g} u_{1_g}$ matrix $A_{i_1 \ldots i_g}$ has full column rank over $GF(s)$. Then $A = \xi C$ leads to an $OA(s^r, n, (s)^{u_2} \times \ldots (s)^{u_n}, g)$, where ξ is an $s^r \times r$ matrix whose rows are all possible r-tuples over $GF(s)$.

Consider the matrix M_1 given in Section 3.

$$M_1 = \left[\begin{array}{ll} p_{11}p_{1u_1} & p_{21}p_{31} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots p_{40\,1} \\ \hline 10 & 000213010223333122103103130321021023032 \\ 01 & 000132101013221322010121332022301101303 \\ 00 & 100303223220123321330101023302112102012 \\ 00 & 010032111103331223101030223133210010212 \\ 00 & 001130331132032231021013303320332120102 \end{array}\right]$$

$$A_1 : A_2 : A_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A_{40}$$

where $A_1 = \left[p_{11}p_{1u_1}\right]$, $A_2 = \left[p_{21}\right]$, $A_3 = \left[p_{31}\right]$, ... $A_{40} = \left[p_{40\,1}\right]$

Any three columns of this matrix are linearly independent. Allocate first two columns to the first factor F_1 (with 16 levels) while the remaining columns correspond to the other 39 factors F_i (with 4 levels), $i = 2,3,\ldots,40$. With $g = 2$ the rank condition of theorem 4.1 is satisfied by the above matrix. Hence on computing ξM_1, we get an $OA(4^5, 40, 16\times(4)^{39}, 2)$. Thus two non-isomorphic asymmetric orthogonal arrays from symmetric orthogonal array are obtained.

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