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Construction of mixed-level supersaturated design

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Abstract. Supersaturated design is a form of fractional factorial design in which the number of columns is greater than the number of experimental runs. Construction methods of supersaturated design have been mainly focused on two levels cases. Much practical experience, however, indicates that two-level may sometimes be inadequate. This paper proposed a construction method of mixed-level supersaturated designs consisting of two-level and three-level columns. The χ^2 statistic is used for a measure of dependency of the design columns. The dependency properties for the newly constructed designs are derived and discussed. It is shown that these new designs have low dependencies and thus can be useful for practical uses.

Key words: Average and maximum dependency, χ^2-statistics, orthogonality

1 Introduction

Supersaturated design is a form of fractional factorial design in which the number of columns is greater than the number of experimental runs. In practice, it is used for screening the active factors, where the collected data are analyzed under an assumption of effect sparsity. Supersaturated designs were originated by Satterthwaite (1959) as a random balance design and formulated by Booth and Cox (1962) in a systematic manner. Many papers have considered the construction of two-level supersaturated designs and their properties (Lin (1991, 1993, 1995), Wu (1993), Iida (1994), Deng, Lin and Wang (1994, 1999), Nguyen (1996), Cheng (1997), Li and Wu (1997), Tang and Wu (1997), and Yamada and Lin (1997)). These are mainly focused on two-level designs.

For two-level designs, the dependency is measured by the squared inner product between two design columns because the dependency between two estimated effects of the design columns can be represented by a function of the inner products. Total evaluation of the constructed designs is performed by the average squared inner product over all pairs of columns, it is sometimes denoted by $E(s^2)$. The maximum value of the squared inner product over all pairs of columns is also applied in the overall evaluation of the goodness of a design.

Three-level supersaturated designs defined by Yamada and Lin (1999) are a natural extension of two-level supersaturated design. The measure for dependency (or non-orthogonality here) between two columns is defined by χ^2 statistic which is applied to the hypothesis test in a two-way contingency table. Note that $\chi^2 = 0$ implies perfect independence or full orthogonality. The overall evaluation is performed by the average χ^2 values and the maximum χ^2 values over all pairs of columns. A lower bound of χ^2 dependency for supersaturated design was studied by Yamada and Matsui (1998). Furthermore, Yamada, Ikebe, Hashiguchi and Niki (1999) have shown a construction method, which can be regarded as an extension of a construction method of two-level supersaturated design proposed by Tang and Wu (1997). Recently, Fang, Lin and Ma (2000) shows multi-level supersaturated design criteria and construction method by applying uniform designs.

One of the typical situations where supersaturated design works well may be screening the active factors out of many candidate factors. As pointed by one referee, two-level fractional factorials are the most reasonable choice. Many practical evidences, however, indicate that some factors may require more than two levels. For example, when we need to examine the effect of the three machines in a pre-production stage, the machines should be treated as a three-level factor. In other words, some factors are able to keep as two-level, while there are other factors which must be with higher level. This is especially true for qualitative factors. Such problem is called a mixed-level problem. We believe that there is a strong demand for mixed-level supersaturated design. Such designs are, however, lacking in the literature. This paper proposes a construction method of mixed-level supersaturated designs consisting of two-level and three-level columns, built on our previous work.

2 Mixed-level supersaturated design

Let $\boldsymbol{c}^l$ be an n-dimensional column vector consisting of equal numbers of 1's, 2's, ..., l's and let $\mathscr{C}_n^l$ be the set of all collections of $\boldsymbol{c}^l$. A column vector $\boldsymbol{c}^l \in \mathscr{C}_n^l$ and an $n \times p$ matrix $\boldsymbol{C}_n^l = [\boldsymbol{c}_1^l, \ldots, \boldsymbol{c}_p^l]$ are called an l-level column and an l-level design matrix respectively. Let $\boldsymbol{c}_i^{l_j} \in \mathscr{C}_n^{l_j}$ $(1 \le i \le p_j)$ and $\boldsymbol{C}_n^{l_j} = [\boldsymbol{c}_1^{l_j}, \boldsymbol{c}_2^{l_j}, \ldots, \boldsymbol{c}_{p_j}^{l_j}]$ $(1 \le j \le q)$. An $n \times \sum_{j=1}^{q} p_j$ matrix $\boldsymbol{C}_n = [\boldsymbol{C}_n^{l_1}, \boldsymbol{C}_n^{l_2}, \ldots, \boldsymbol{C}_n^{l_q}]$ is called a mixed-level design matrix consisting of $l_1, l_2, \ldots, l_q$-level columns. For example, a mixed-level design consisting of two-level and three-level columns is denoted by $\boldsymbol{C} = [\boldsymbol{C}_n^2, \boldsymbol{C}_n^3] = [\boldsymbol{c}_1^2, \ldots, \boldsymbol{c}_{p_1}^2, \boldsymbol{c}_1^3, \ldots, \boldsymbol{c}_{p_2}^3]$, where $q = 2$, $l_1 = 2$, $l_2 = 3$, $\boldsymbol{c}_i^2 \in \mathscr{C}_n^2$ $(1 \le i \le p_1)$ and $\boldsymbol{c}_i^3 \in \mathscr{C}_n^3$ $(1 \le i \le p_2)$. Yamada and Matsui (1998) defined the degree of saturation by

$$v = \frac{\sum_{j=1}^{q} (l_j - 1) p_j}{n - 1} \tag{1}$$

for any $n \times \sum_{j=1}^{q} p_j$ design matrix $C_n = [C_n^{l_1}, C_n^{l_2}, \ldots, C_n^{l_q}]$, where $c_i^{l_j} \in \mathscr{C}_n^{l_j}$ $(1 \le i \le p_j)$ and $C_n^{l_j} = [c_1^{l_j}, c_2^{l_j}, \ldots, c_{p_j}^{l_j}]$ $(1 \le j \le q)$. Thus, a design C_n is called a supersaturated design and a saturated design when $v > 1$ and $v = 1$, respectively.

Let $n^{rs}(c^{l_1}, c^{l_2})$ be the number of rows whose values are $[r, s]$ in the $n \times 2$ matrix $[c^{l_1}, c^{l_2}]$, where $c^{l_1} \in \mathscr{C}_n^{l_1}$ and $c^{l_2} \in \mathscr{C}_n^{l_2}$, so we have $\sum_{1 \le r \le l_1} \sum_{1 \le s \le l_2} \cdot n^{rs}(c^{l_1}, c^{l_2}) = n$. The dependency between $c^{l_1} \in \mathscr{C}_n^{l_1}$ and $c^{l_2} \in \mathscr{C}_n^{l_2}$ is measured by

$$\chi^2(c^{l_1}, c^{l_2}) = \sum_{1 \le r \le l_1} \sum_{1 \le s \le l_2} \frac{(n^{rs}(c^{l_1}, c^{l_2}) - n/(l_1 l_2))^2}{n/(l_1 l_2)}. \tag{2}$$

Following the basic idea of the popular design criterion $E(s^2)$, the criterion $\mathrm{ave}\,\chi^2_{l_i, l_j}$ is adapted here. Let $c_i^{l_j} \in \mathscr{C}_n^{l_j}$ $(1 \le i \le p_j)$, $C_n^{l_j} = [c_1^{l_j}, c_2^{l_j}, \ldots, c_{p_j}^{l_j}]$ $(1 \le j \le q)$ and $C_n = [C_n^{l_1}, C_n^{l_2}, \ldots, C_n^{l_q}]$. The average χ^2 value over all pairs of two l_i-level columns is

$$\mathrm{ave}\,\chi^2_{l_i l_i} = \sum_{1 \le k < m \le p_i} \chi^2(c_k^{l_i}, c_m^{l_i}) \Big/ \binom{p_i}{2} \tag{3}$$

and the average χ^2 value over all pairs of an l_i-level column and an l_j-level column is

$$\mathrm{ave}\,\chi^2_{l_i l_j} = \sum_{1 \le k \le p_i} \sum_{1 \le m \le p_j} \chi^2(c_k^{l_i}, c_m^{l_i})/(p_i p_j). \tag{4}$$

Tang and Wu (1997) and Nguyen (1996) had independently obtained a lower bound for $E(s^2)$ of an arbitrary two-level supersaturated design. A similar lower bound for the sum of χ^2 values of an arbitrary mixed-level supersaturated design is obtained by Yamada and Matsui (1998). Furthermore, the χ^2 efficiency is defined by

$$\frac{(1/2)n(n-1)v(v-1)}{\chi^2(C_n)}, \tag{5}$$

where

$$\chi^2(C_n) = \sum_{1 \le i < j \le q} \sum_{1 \le k \le p_i} \sum_{1 \le m \le p_j} \chi^2(c_k^{l_i}, c_m^{l_j}) + \sum_{1 \le i \le q} \sum_{1 \le k < m \le p_i} \chi^2(c_k^{l_i}, c_m^{l_i}) \tag{6}$$

for any $n \times \sum_{j=1}^{q} p_j$ design matrix $C_n = [C_n^{l_1}, C_n^{l_2}, \ldots, C_n^{l_q}]$, where $c_i^{l_j} \in \mathscr{C}_n^{l_j}$ $(1 \le i \le p_j)$ and $C_n^{l_j} = [c_1^{l_j}, c_2^{l_j}, \ldots, c_{p_j}^{l_j}]$ $(1 \le j \le q)$.

In this index, the numerator implies the lower bound of the sum of χ^2 values for any mixed-level supersaturated design given n and v and the denominator implies the sum of χ^2 values of the constructed design $\boldsymbol{C}_n$. So, the χ^2-efficiency takes a value from 0 to 1 and indicates the degree of attainment compared with an optimal design in terms of χ^2 dependency. For example, when χ^2-efficiency is equal to 1, the design is optimal in terms of χ^2 dependency.

Another class of design criteria is the maximum value of χ^2 dependency:

$$\max \chi^2_{l_i l_j} = \max\{\chi^2(\boldsymbol{c}_k^{l_i}, \boldsymbol{c}_m^{l_j}) \mid 1 \le k < m \le p_i, 1 \le m \le p_j\}. \tag{7}$$

The constructing problem of mixed-level supersaturated design consisting of $l_1, \dots, l_q$-levels is an enumeration of the columns from the sets $\mathscr{C}_n^{l_1}, \dots, \mathscr{C}_n^{l_q}$ while maintaining low dependency, *i.e.* maintaining a small value of $\text{ave}\,\chi^2_{l_i l_i}$, $\text{ave}\,\chi^2_{l_i l_j}$, χ^2 efficiency, $\max \chi^2_{l_i l_i}$ and $\max \chi^2_{l_i l_j}$.

3 A constructing method and its results

3.1 Algorithm

We consider here construction of a two- and three-level supersaturated design which includes at least one full-dimensional orthogonal base of two-level columns. Specifically, let $q = 2$, $l_1 = 2$, $l_2 = 3$, $\boldsymbol{C}_n = [\boldsymbol{C}_n^2, \boldsymbol{C}_n^3]$, $\boldsymbol{C}_n^2 = [\boldsymbol{c}_1^2, \dots, \boldsymbol{c}_{n-1}^2, \boldsymbol{c}_n^2, \dots, \boldsymbol{c}_{p_1}^2]$, $\boldsymbol{C}_n^3 = [\boldsymbol{c}_1^3, \dots, \boldsymbol{c}_{p_2}^3]$ and the first $n-1$ columns in $\boldsymbol{C}_n^2$ are mutually orthogonal. The proposed method consists of three major steps: a construction of initial columns, selection of three-level columns and selection of two-level columns. The maximum values of χ^2 dependency on constructed designs are assured in all steps.

(1) Construction of initial columns:
Construct initial two-level columns $\boldsymbol{c}_1^2, \dots, \boldsymbol{c}_{n-1}^2, \boldsymbol{c}_n^2, \dots, \boldsymbol{c}_{p_1^*}^2$ $(p_1^* > p_1)$ and initial three-level columns $\boldsymbol{c}_1^3, \dots, \boldsymbol{c}_{p_2^*}^3$ $(p_2^* > p_2)$, where the first $n-1$ columns, $\boldsymbol{c}_1^2, \dots, \boldsymbol{c}_{n-1}^2$ are mutually orthogonal such that $\chi^2(\boldsymbol{c}_i^2, \boldsymbol{c}_j^2) = 0$ $(1 \le i < j \le n-1)$. The initial two-level and three-level columns must be selected under a consideration of the maximum dependencies, *i.e.* $\max \chi^2_{22}$ and $\max \chi^2_{33}$ must be equal to threshold values. A natural choice for $\boldsymbol{c}_i^2$'s and $\boldsymbol{c}_i^3$'s are the design columns from Yamada and Lin (1997) and Yamada and Lin (1999) respectively because the maximum dependencies have been pre-controlled in these designs.

(2) Selection of three-level columns:
A three-level column $\boldsymbol{c}^3$ satisfying

$$\max\{\chi(\boldsymbol{c}_i^2, \boldsymbol{c}^3) \mid 1 \le i \le n-1\} \le \delta_{23}^2, \tag{8}$$

where δ_{23}^2 is a pre-specified threshold value and $\boldsymbol{c}_i^2$ is a two-level column in the orthogonal base. Let $\boldsymbol{c}_1^3, \dots, \boldsymbol{c}_{p_2}^3$ be all the selected columns which satisfy Equation (8). A three-level design portion is then formed by $C_n^3 = [c_1^3, \ldots, c_{p_2}^3]$. We determine the threshold value δ_{23}^2 under a consideration of the distribution of all possible χ^2 values. For example, when $n = 24$, it can be shown that $0 \le \chi^2(c^2, c^3) \le 16$, for any $c^2 \in \mathscr{C}_{24}^2$ and $c^3 \in \mathscr{C}_{24}^3$. In fact, it takes only nine χ^2 values: 0, 1, 3, 4, 7, 9, 12, 13, and 16.

(3) Selection of two-level columns:
A two-level column c^2 satisfying

$$\max\{\chi(c^2, c_j^3) \mid 1 \le j \le p_2\} \le \delta_{23}^2 \tag{9}$$

is selected for the two-level deign matrix C_n^2. Note that there is no need to re-examine $c_1^2, \ldots, c_{n-1}^2$ since it obviously satisfies Equation (9). Let $c_n^2, \ldots, c_{p_1}^2$ be all the selected columns which satisfy Equation (9). The two-level design portion is constructed by $C_n^2 = [c_1^2, \ldots, c_{p_1}^2]$, and finally, a mixed-level design consisting of $(p_1 + p_2)$ columns with n runs is constructed by $C_n = [C_n^2, C_n^3]$.

3.2 Some properties of the constructed designs

Next we consider some properties of designs constructed here. Since the first $n - 1$ columns are mutually orthogonal, we have

$$\max\{\chi^2(c_i^2, c_j^2) \mid 1 \le i < j \le n - 1\} = 0. \tag{10}$$

In the selection of initial designs, if we use supersaturated designs whose maximum dependency is assured, the dependency of the constructed design is also assured. Let δ_{22}^2 and δ_{33}^2 be the maximum χ^2 values of the initial two-level and three-level supersaturated design, respectively. Obviously, the maximum values are maintained for the constructed design such that

$$\max\{\chi^2(c_i^2, c_j^2) \mid 1 \le i \le n - 1, n \le j \le p_1\} = \delta_{22}^2 \tag{11}$$

$$\max\{\chi^2(c_i^2, c_j^2) \mid n \le i < j \le p_1\} = \delta_{22}^2 \tag{12}$$

$$\max\{\chi^2(c_i^3, c_j^3) \mid 1 \le i < j \le p_2\} = \delta_{33}^2. \tag{13}$$

Step (2) in the algorithm screens out three-level columns whose maximum dependency to the two-level columns is higher than the threshold value. In the same manner, Step (3) screens out two-level columns whose maximum dependency to two-level columns is greater than the threshold value. Thus, maximum dependency among selected columns are assured such that

$$\max\{\chi^2(c_i^2, c_j^3) \mid 1 \le i \le p_1, 1 \le j \le p_2\} = \delta_{23}^2. \tag{14}$$

As previously mentioned, a natural choice for the initial design is the combination of two-level designs from Yamada and Lin (1997) and three-level designs from Yamada and Lin (1999) because the dependencies among all design columns have been pre-controlled and an orthogonal design base is included.

Table 1. Some results of constructed designs

n	maximum of χ^2/χ_F^2			number of columns		
	$\max \chi_{22}^2/\chi_{22F}^2$	$\max \chi_{23}^2/\chi_{23F}^2$	$\max \chi_{33}^2/\chi_{33F}^2$	Two-level	Three-level	Total
24	2.67/24.00	4.00/16.00	18.75/48.00	86	8	94
	2.67/24.00	7.00/16.00	18.75/48.00	78	84	162
36	0.00/36.00	4.67/24.00	18.00/72.00	35	16	51
	0.00/36.00	8.00/24.00	18.00/72.00	35	134	150
48	5.33/48.00	8.00/32.00	37.50/96.00	112	86	198
	5.33/48.00	9.50/32.00	37.50/96.00	136	142	278

$\chi_{22F}^2, \chi_{23F}^2, \chi_{33F}^2$: χ^2 values for fully aliased

Specifically, the construction method by Yamada and Lin (1997) generates a two-level supersaturated design with n runs from a design with $n/2$ runs. When initial design with $n/2$ runs includes an orthogonal base, the constructed design with n runs also includes an orthogonal base. The three-level design construction by Yamada and Lin (1999) is similar to the two-level construction method, where this method generates three-level supersaturated design with n runs from a design with $n/3$ runs. Since the two construction methods assure the maximum value of the constructed designs, Equations (11) to (13) can be easily accomplished by using these two methods. A thorough example will be given in the next section.

Regarding determination of the threshold value δ_{23}^2, we first consider χ^2 values when two columns are fully aliased. The χ^2 value takes the maximum in all possible combinations when two columns are fully aliased. It can be shown that $\chi^2(c_i^2, c_j^2) = n$ for any two fully aliased two-level columns and $\chi^2(c_i^3, c_j^3) = 2n$ for any two fully aliased three-level columns. By an analogy from this fact, we define the fully aliased relation for a two-level column and a three-level column when the χ^2 value is maximized. Furthermore, it can be shown that $\max\{\chi^2(c^2, c^3) \mid c^2 \in \mathscr{C}_n^2, c^3 \in \mathscr{C}_n^3\} = \frac{2n}{3}$. Thus for $n = 24$, the maximum for δ_{22}^2, δ_{33}^2, and δ_{23}^2 are 24, 48 and 16 respectively. In the computation of mixed-level supersaturated design, we determine several levels of δ_{23}^2 under consideration of all possible χ^2 values.

Table 1 summarizes the result of the constructed designs in this paper. The values of design criteria $\max \chi_{22}^2, \max \chi_{23}^2, \max \chi_{33}^2$ are very small in comparison with the χ^2 values for fully aliased two-columns.

4 An example: $n = 24$

In this section, we show more details on the construction of mixed-level supersaturated designs with $n = 24$ runs as an example of construction.

(1) Construction of initial columns

The following eleven two-level mutually independent columns are utilized for constructing two-level initial columns.

$$[c_1^2, \ldots, c_{11}^2] = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} \tag{15}$$

Let $(\boldsymbol{c}_i \otimes \boldsymbol{c}_j)$ be the cross product between $\boldsymbol{c}_i$ and $\boldsymbol{c}_j$. This constructing method generates 55 columns, say $\boldsymbol{c}_{12}^2 = (\boldsymbol{c}_1^2 \otimes \boldsymbol{c}_2^2)$, $\boldsymbol{c}_{13}^2 = (\boldsymbol{c}_1^2 \otimes \boldsymbol{c}_3^2), \ldots, \boldsymbol{c}_{21}^2 = (\boldsymbol{c}_1^2 \otimes \boldsymbol{c}_{11}^2)$, $\boldsymbol{c}_{22}^2 = (\boldsymbol{c}_2^2 \otimes \boldsymbol{c}_3^2), \ldots, \boldsymbol{c}_{66}^2 = (\boldsymbol{c}_{10}^2 \otimes \boldsymbol{c}_{11}^2)$, and we have totally $55 + 11 = 66$ columns. Note that Wu (1993) shows the above constructing method of two-level supersaturated design with $n = 12$ runs.

Second, the $\boldsymbol{C}_{12}^2 = [\boldsymbol{c}_1^2, \ldots, \boldsymbol{c}_{66}^2]$ generates supersaturated design with $n = 24$ runs by

$$\boldsymbol{C}_n^2 = \begin{bmatrix} \mathbf{1}_{n/2} & \boldsymbol{C}_{n/2}^2 & \boldsymbol{C}_{n/2}^2 \\ -\mathbf{1}_{n/2} & \boldsymbol{C}_{n/2}^2 & -\boldsymbol{C}_{n/2}^2 \end{bmatrix}, \tag{16}$$

where $\mathbf{1}_{n/2}$ is the $n/2$ dimensional column consisting of $n/2$ 1's. We treat the 133 columns generated by Equation (16) as the initial two-level columns. This is denoted by $\boldsymbol{C}_n^2$.

Let $\phi^{ab}(\ )$ be an operator which transforms the elements from 1 to a and from 2 to b on the matrix in the brackets( ). Consider a matrix with n rows and p_2^* columns constructed by

$$[\boldsymbol{c}_1^3, \ldots, \boldsymbol{c}_{p_2^*}^3] = [\boldsymbol{C}_{n\cdot 1}^3, \boldsymbol{C}_{n\cdot 2}^3, \boldsymbol{C}_{n\cdot 3}^3, \boldsymbol{C}_{n\cdot 4}^3], \tag{17}$$

where

$$\boldsymbol{C}_{n\cdot 1}^3 = \begin{bmatrix} \phi^{12}(\boldsymbol{C}_{n/3}^2) \\ \phi^{23}(\boldsymbol{C}_{n/3}^2) \\ \phi^{31}(\boldsymbol{C}_{n/3}^2) \end{bmatrix}, \quad \boldsymbol{C}_{n\cdot 2}^3 = \begin{bmatrix} \phi^{12}(\boldsymbol{C}_{n/3}^2) \\ \phi^{13}(\boldsymbol{C}_{n/3}^2) \\ \phi^{23}(\boldsymbol{C}_{n/3}^2) \end{bmatrix}$$
$$\boldsymbol{C}_{n\cdot 3}^3 = \begin{bmatrix} \phi^{13}(\boldsymbol{C}_{n/3}^2) \\ \phi^{23}(\boldsymbol{C}_{n/3}^2) \\ \phi^{12}(\boldsymbol{C}_{n/3}^2) \end{bmatrix}, \quad \boldsymbol{C}_{n\cdot 4}^3 = \begin{bmatrix} \phi^{23}(\boldsymbol{C}_{n/3}^2) \\ \phi^{12}(\boldsymbol{C}_{n/3}^2) \\ \phi^{13}(\boldsymbol{C}_{n/3}^2) \end{bmatrix} \tag{18}$$

Table 2. Example of constructed designs with $n = 24$

δ_{23}^2	Selected columns														
4.0	Two-level columns (total number of columns: 86)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	16	17	18	19	20	21	22	23	24	25	26	27	31	32	37
	39	42	43	44	45	47	48	52	53	54	55	57	59	60	61
	62	65	67	68	71	72	73	74	76	77	78	79	80	84	85
	87	88	89	92	93	96	97	100	101	102	105	107	108	112	115
	116	118	120	121	124	125	126	127	128	129	132				
	Three-level columns (total number of columns: 8)														
	11	15	76	81	89	99	123	138							
7.0	Two-level columns (total number of columns: 78)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	16	17	18	19	20	21	22	23	24	25	29	31	32	33	36
	37	39	40	43	47	48	51	53	56	57	59	60	61	62	63
	64	67	69	72	73	75	79	80	83	85	86	88	91	95	96
	97	100	101	102	104	105	107	108	110	111	115	117	118	120	121
	126	127	130												
	Three-level columns (number of columns: 84)														
	5	6	8	9	10	11	12	13	14	15	16	17	18	19	37
	38	39	40	42	43	44	45	46	47	48	49	50	51	52	53
	54	73	74	76	77	78	79	80	81	82	83	84	85	86	87
	88	89	91	92	93	94	96	97	98	99	101	102	104	105	107
	113	114	115	116	117	118	120	121	122	123	124	125	126	127	128
	129	131	132	133	134	136	137	138	139						

and $C_{n/3}^2$ is an $n/3 \times p_2^*/4$ two-level design matrix whose maximum dependency is equal to the threshold value. The maximum dependency of the constructed design can be represented by

$$\max \chi_{33}^2 = \max\left\{ \left(n + 3\sqrt{3n\chi_o^2}\right)^2 / (8n), n/2 \right\}, \tag{19}$$

where χ_o^2 is the maximum χ^2 value of the two-level design matrix $C_{n/3}^2$ (Yamada and Lin (1999)). Let C_8^2 be a two-level supersaturated design consisting of 35 columns and $n = 8$ rows, which is shown in Yamada and Lin (1997, p. 207). In this design, the first seven columns are mutually independent. The 35 two-level columns generate 140 three-level columns by Equation (17), say $[c_1^3, \ldots, c_{140}^3]$. This is denoted by C_n^3. We next choose columns from $C = [C_n^2, C_n^3]$.

Table 2 shows the list of selected columns of mixed-level constructed designs with $n = 24$ rows from the initial two-level columns $[c_1^2, \ldots, c_{133}^2]$ and three-level columns $[c_1^2, \ldots, c_{140}^2]$. In Table 2, when $\max \chi_{23}^2 = 4.0$, 86 two-level columns and 8 three-level columns are selected. Table 3 shows the properties, such as the degree of saturation, $\max \chi_{22}^2, \max \chi_{33}^2$, and so on, for the constructed design. Furthermore, the construction method generates 78 two-level columns and 84 three-level columns for $\chi_{23}^2 = 7.0$ and the selected columns are also listed in Table 2.

Moreover, we construct mixed-level supersaturated design from two-level saturated design consisting of mutually independent columns, say $c_1^2, \ldots, c_{23}^2$.

Table 3. Evaluation of the constructed design with $n = 24$

# of columns c^2	c^3	Design criteria χ^2-eff.	ave χ^2_{22}	ave χ^2_{23}	ave χ^2_{33}	max χ^2_{22}/χ^2_{22F}	max χ^2_{23}/χ^2_{23F}	max χ^2_{33}/χ^2_{33F}
(i) initial: 133 two-level and 140 three-level columns								
86	8	0.92	0.88	1.74	5.46	2.67/24.00	4.00/16.00	18.75/48.00
78	84	0.83	0.88	1.97	5.39	2.67/24.00	7.00/16.00	18.75/48.00
105	125	0.86	0.89	1.95	5.28	2.67/24.00	9.00/16.00	18.75/48.00
122	136	0.87	0.91	1.96	5.27	2.67/24.00	12.00/16.00	18.75/48.00
128	137	0.87	0.90	1.99	5.27	2.67/24.00	13.00/16.00	18.75/48.00
133	140	0.87	0.90	1.99	5.27	2.67/24.00	16.00/16.00	18.75/48.00
(ii) initial: 23 two-level and 28 three-level columns								
23	1	0.54	0.00	2.09	-	0.00/24.00	4.00/16.00	12.00/48.00
23	12	0.72	0.00	2.09	3.82	0.00/24.00	7.00/16.00	12.00/48.00
23	21	0.75	0.00	2.09	4.20	0.00/24.00	9.00/16.00	12.00/48.00
23	25	0.79	0.00	2.09	4.02	0.00/24.00	12.00/16.00	12.00/48.00
23	26	0.80	0.00	2.09	4.02	0.00/24.00	13.00/16.00	12.00/48.00
23	28	0.81	0.00	2.09	4.00	0.00/24.00	16.00/16.00	12.00/48.00

$\chi^2_{22F}, \chi^2_{23F}, \chi^2_{33F}$: χ^2 values for fully aliased

On the three-level columns, the first seven columns shown in the design C^2_8 generate 28 three-level columns by Equation (17). The evaluations of the mixed-level designs constructed by the two and three-level initial columns are shown in Table 3. The extension to larger n is straightforward. Results for $n = 36$ and 48 are available through the first author (shu@ms.kagu.sut.ac.jp).

5 Concluding remarks

The constructing method used in this paper consists of three steps: the selection of initial columns, the selection of three-level columns and two-level columns. The method has an advantage to assure the maximum value of the χ^2 dependency. Thus, the resulting designs are apparently new and shown to be useful for the practitioners, where the essence of the constructing method can be regarded as an integration of our previous works with some modifications.

For the case where the number of factors is smaller than the number of columns in the full design, a sub-design can be used. For example, one referee considers the case when there are 30 two-level and 10 three-level factors. As a matter of fact, we can randomly select specified number of columns from the constructed designs, such as selection of 30 out of 78 two-level and 10 out 84 three-level columns. It is easily confirmed that this procedure guarantees the maximum χ^2 value of the resulting designs, while the average χ^2 value depends on the selected columns.

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