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Cumulative probability control charts for geometric and exponential process characteristics

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A statistical process control chart called the cumulative probability control chart (CPC-chart) is proposed. The CPC-chart is motivated from two existing statistical control charts, the cumulative count control chart (CCC-chart) and the cumulative quantity control chart (CQC-chart). The CCC- and CQC-charts are effective in monitoring production processes when the defect rate is low and the traditional *p*- and *c*-charts do not perform well. In a CPC-chart, the cumulative probability of the geometric or exponential random variable is plotted against the sample number, and hence the actual cumulative probability is indicated on the chart. Apart from maintaining all the favourable features of the CCC- and CQC-charts, the CPC-chart is more flexible and it can resolve a technical plotting inconvenience of the CCC- and CQC-charts.

1. Introduction

The traditional statistical process control charts, the *p*-chart (with control limits $\bar{p} \pm 3\sqrt{\bar{p}(1-\bar{p})/n}$) and the *c*-chart (with control limits $\bar{c} \pm 3\sqrt{\bar{c}}$) are widely used for monitoring industrial processes where defect occurrences in a sample follow binomial and Poisson distributions, respectively. Despite their wide-spread use, it is known that these charts have some pitfalls and perform poorly when the defect rate of the process is low (e.g. Xie and Goh 1992, Xie *et al.* 1997). This can be summarized as follows.

- The ± 3-sigma control limits of these charts are defined based on approximating the binomial and Poisson distributions by the normal distribution, but this approximation fails when the defect rate of the process is low. Even when the normal approximation holds, it is only accurate in the central part of the distribution, and the accuracy at the tails in the ± 3-sigma regions is usually poor.
- When the defect rate of the process is low, usually the lower control limits of the *p*- and *c*-charts are negative. Negative control limits are meaningless, because the number of defects in a sample cannot be negative. Without a positive lower control limit, it is not possible to detect a downward shift of the defect rate of the process, that is, an improvement.

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- When the defect rate of the process is low, the upper control limit of the p-chart is usually $< 1/n$, and that of the c-chart is usually <1. In this situation, every single defect becomes a signal of out of control (because the number of defects takes only non-negative integer values), which is clearly an overreaction to natural variations in the process.
- The in- or out-of-control decision based on the p- or c-chart depends heavily on the sample size, regardless of the magnitude of the defect rate. Examples in Chan (2000, appendix) and Chan *et al.* (2000: pp. 403–404) show that for the same data set, for a certain value of the sample size, the occurrence of every defect is an alarm of out of control, while for a slightly large sample size, none of the defects indicates that the process is out of control.

Similarly, the np-chart with control limits $n\bar{p} \pm 3\sqrt{n\bar{p}(1-\bar{p})}$ and u-chart with control limits $\bar{u} \pm 3\sqrt{\bar{u}/l}$ also have these disadvantages. To overcome these disadvantages, Calvin (1983), Goh (1987) and Chan *et al.* (1997) proposed the cumulative count control chart (CCC-chart) based on the geometric distribution, and Chan *et al.* (2000) proposed the cumulative quantity control chart (CQC-chart) based on the exponential distribution.

In most existing control charts, the variate is plotted against the sample number. For the p-chart, the variate is the proportion of defective items, while the c-chart plots the number of defects. For the CCC- and CQC-charts, the variate is the cumulative length of observation to observe one defect. The objective of this paper is to enhance the usability the CCC- and CQC-charts by introducing a cumulative probability control chart (CPC-chart) in which the cumulative probability is plotted against the sample number. This probability chart is constructed using the cumulative distribution function of the geometric or exponential random variable, and is more flexible and provides a better visual picture for decision making than the CCC- and CQC-charts. In section 2, the CCC- and CQC-charts are briefly reviewed. In section 3, the cumulative probability chart is introduced. In section 4, technical aspects of implementation of the CPC-chart are discussed; section 5 is devoted to average run length analysis. In section 6, examples are given and comparisons are made among the three charts. This paper is concluded in section 7.

2. The CCC- and CQC-charts

Suppose that the fraction of defective items produced by a binomial process has a constant value $p_0 (> 0)$, and let n be the number of items inspected until a defective item is observed. It is known (Walpole *et al.* 1998) that n follows a geometric distribution with expected value $E[n] = 1/p_0$ and cumulative distribution function:

$$F(n) = 1 - (1 - p_0)^n \quad (n = 1, 2, \ldots). \tag{1}$$

To construct a double-limit CCC-chart (Calvin 1983, Goh 1987), the probability of false alarm, α, is fixed first, and $F(n)$ in (2.1) is set equal to $\alpha/2$, $1-\alpha/2$ and $1/2$, which gives respectively the lower control limit (LCL) $n_L = \ln(1-\alpha/2)/\ln(1-p_0)$, the upper control limit (UCL) $n_U = \ln(\alpha/2)/\ln(1-p_0)$ and the centerline (CL) $n_C = \ln(1/2)/\ln(1-p_0)$. In a CCC-chart, the number of items inspected, n, is plotted on the chart when inspection of a sample is completed, or when a defective item is observed. When a defective item is observed, n is reset to 0. A plotted point corresponding to a defective item appearing below the LCL is an alarm that the fraction of defective items p of the process may have shifted upward (that is, the process may have deteriorated), and a plotted point above the UCL is an indication that p may have shifted downward (that is, the process may have improved). For a single-limit CCC-chart, $F(n)$ in (1) is set equal to α, giving only the LCL $n_L = \ln(1-\alpha)/\ln(1-p_0)$, and occurrence of a defect below the LCL indicates that p may have shifted upward.

For the case when a Poisson process has a constant rate of occurrence of defects $\lambda_0 (> 0)$ defect(s) per unit quantity of product produced, if Q is the quantity of product inspected until one defect is observed, it is known (Walpole *et al.* 1998)) that Q follows an exponential distribution with expected value $E[Q] = 1/\lambda_0$ and cumulative distribution function:

$$F(Q) = 1 - e^{-\lambda_0 Q} \quad (Q > 0). \tag{2}$$

To construct a double-limit CQC-chart (Chan *et al.* 2000), the probability of false alarm, α, is fixed first, and $F(Q)$ in (2.2) is set equal to $\alpha/2$, $1-\alpha/2$ and $1/2$, which gives the following LCL, UCL and CL, respectively: $Q_L = -\ln(1-\alpha/2)/\lambda_0$, $Q_U = -\ln(\alpha/2)/\lambda_0$, $Q_C = -\ln(1/2)/\lambda_0$. To construct a single-limit CQC-chart, $F(Q)$ in (2) is set equal to α, giving only the LCL $Q_L = -\ln(1-\alpha)/\lambda_0$. In a CQC-chart, the number of units inspected, Q, is plotted on the chart when inspection of a sample is completed, or when a defect is observed. When a defect is observed, Q is reset to 0. Occurrence of a defect below the LCL is an alarm that the rate of occurrence of defects λ of the process may have shifted upward (that is, the process may have deteriorated), and a plotted point above the UCL is an indication that λ may have shifted downward (that is, the process may have improved). A single-limit CQC-chart is administered in a similar way, except there is no UCL.

The CCC-chart is an alternative to the traditional p-chart and np-chart, and the CQC-chart is an alternative to the traditional c-chart and u-chart. In the CCC- and CQC-charts, the plotting of n and Q against the sample number gives a direct and intuitive interpretation of the state of the process. However, as the process continues, n or Q will become large and the chart will eventually exceed the boundary of the plot. To avoid this awkward situation, $\log n$ or $\log Q$ (with any convenient base, such as e or 10) instead of n and Q may be plotted. The use of logarithmic scale is discussed in Xie *et al.* (1995). However, plotting $\log n$ or $\log Q$ distorts the original shape of the charts for n and Q, and makes it difficult for practitioners to comprehend the intuitive meaning of the points plotted. The charts for $\log n$ and $\log Q$ rise very sharply when the charts are started, but flatten very quickly and hence has a decreasing accuracy as n and Q increase.

To overcome this technical difficulty of the CCC- and CQC-charts, a new control chart, namely the cumulative probability control chart (CPC-chart), is proposed. In a CPC-chart, the cumulative probability is plotted against the sample number. The sensitivity of a CPC-chart in a region of the control limits can be increased by using a larger scale. The CPC-chart is standardized, as its vertical axis is scaled to $[0, 1]$ for any variate. This makes possible to compare several quality characteristics simultaneously by plotting their corresponding CPC-charts on the same graph paper (or computer screen).

3. CPC-chart

The CPC-chart for a variate y is a control chart in which the observed cumulative probability of y is plotted against the sample number.

For a binomial process with fraction of defective items p_0, if n is the number of items inspected until one defective item is observed, the cumulative distribution function $F(n)$ of n is given by (1). In the CPC-chart for this process, $y = n$ and the observed values of $F(y) = F(n) = 1 - (1 - p_0)^n$ is plotted against the sample number. This CPC-chart may be denoted the CPC(C)-chart, as it is the counter-part of the CCC-chart.

For a Poisson process with rate of occurrence of defects λ_0, if Q is the quantity of product inspected until one defect is observed, the cumulative distribution function $F(Q)$ of Q is given by (2). In the CPC-chart for this process, $y = Q$, and the observed values of $F(y) = F(Q) = 1 - e^{-\lambda_0 Q}$ is plotted against the sample number. This CPC-chart may be denoted the CPC(Q)-chart, as it is the counter-part of the CQC-chart.

Given the probability of false alarm α, the LCL, UCL and CL of a double-limit CPC-chart are defined as follows:

$$\mathrm{LCL} = \alpha/2 \tag{3}$$

$$\mathrm{UCL} = 1 - \alpha/2 \tag{4}$$

$$\mathrm{CL} = 1/2. \tag{5}$$

If unequal probabilities of false alarm are set for detecting upward and downward shifts of the process defect rate, the $\alpha/2$'s in (3) and (4) may be replaced by α_L and α_U, respectively, where $\alpha_L + \alpha_U = \alpha$. For a single-limit CPC-chart with probability of false alarm set at α, the LCL is defined by

$$\mathrm{LCL} = \alpha. \tag{6}$$

Since the LCL, UCL and CL of a CPC-chart defined by (3–6) correspond to those of the equivalent CCC- or CQC-chart, decisions on the process made from the CPC-chart will be identical to those made from the CCC- or CQC-chart. On a CPC-chart, the actual cumulative probability is indicated on the vertical axis, while on a CCC- or CQC-chart, the cumulative count n or cumulative quantity Q is indicated on the vertical axis.

For any random variable Y, the plotted variate in a CPC-chart is the cumulative distribution function $F(y)$ of Y, which has a uniform distribution on the interval $[0, 1]$, no matter what distribution Y has. Thus, all CPC-charts are standardized, with 0 as the lower boundary and 1 as the upper boundary on the vertical axis. This makes possible to use the CPC-chart to compare the performance of different quality characteristics simultaneously. The vertical scale on a CPC-chart is linear, which makes possible to apply sensitizing rules (Western Electric Company 1956, Balkin and Lin 1997, Montgomery 2001) to improve the effectiveness of decision-making.

4. Implementation issues of the CPC-chart

The rules for plotting a CPC(C)-chart for a geometric random variable Y or a CPC(Q)-chart for an exponential random variable Y are as follows:

- The cumulative probability $F(y)$ is set to 0 when the chart is started.
- A circled dot '$\odot$' is plotted when inspection of a sample is completed and no defect is observed in this sample.
- When a defect is observed, a cross '$\times$' is plotted on the chart, and $F(y)$ is reset to 0 immediately.

The sample size can be determined according to operational convenience. Since a circled dot '$\odot$' is plotted every time when inspection of a sample is completed, the process is constantly monitored. The decision rules when a CPC-chart is used are as follows.

- A cross '$\times$' appearing below the LCL is an alarm that indicates that the defect rate of the process has increased, that is, the process is out of control. In this case, investigation should be carried out to identify the assignable cause(s) and corrective action should be taken to bring the process defect rate back to its previous level. If no assignable causes can be identified, the alarm is considered a false alarm.
- No conclusion will be made when a circled dot '$\odot$' appears below the LCL.
- The process is regarded as in control when a circled dot '$\odot$' or a cross '$\times$' appears on or above the LCL.
- A circled dot '$\odot$' appearing above the UCL is a signal that indicates the process defect rate may have decreased, that is, the process may have improved. In this case, investigation should be carried out to ensure that there is no mistake in data recording, and to identify and retain the assignable causes that produce such a decrease in the process defect rate. When the process has become stable at a lower defect rate, a new control chart will be constructed with this lower defect rate. If no assignable causes can be identified, the signal of improvement may be considered a false signal.

In the plotting of a CPC-chart, a circled dot '$\odot$' may also be plotted immediately when the cumulative probability $F(y)$ reaches the LCL even if inspection of a sample has not yet been completed (and no defect is observed), so that a decision of the process being in control can be made immediately, even before completion of inspection of the sample. Likewise, a circled dot '$\odot$' may be plotted immediately when the cumulative probability reaches the UCL even if inspection of a sample has not yet been completed (and no defect is observed), so that investigation can be carried out immediately to find out whether the process has actually improved or not.

If the probability of false alarm α is chosen to be small, say 2×0.00135 as in traditional statistical process control charts, and if the vertical axis of a CPC-chart has a uniform scale, the LCL will be close to the lower boundary 0 and the UCL will be close to the upper boundary 1, which makes it difficult to see whether a point is outside the control limits or not. However, since the points that are near the control limits are more crucial for decision making than the points that are far away from the control limits, a magnified linear scale can be used in a vicinity of the control limits in a CPC-chart for easier visual perception. On the other hand, since the part of the vertical axis in a vicinity of the CL is obviously in the acceptance region, a smaller linear scale can be used in this region. An advantage of using a linear scale in the central part of the chart is that sensitizing rules similar to the Western Electric Rules (Western Electric Company 1956, Nelson 1984, 1985, Montgomery 2001: Section 4–3.6)) can be applied to the crosses '$\times$' on the chart.

Non-linear scales such as that on a traditional normal probability paper can also be considered. However, for any distribution of Y, $F(y)$ (when regarded as a random variable) has a uniform distribution on the interval $[0, 1]$. Based on this fact, it is more justifiable to use linear scales rather than nonlinear scales on a CPC-chart.

5. Average run length analysis

The average run length (ARL) for a type of signal showing up on a control chart is defined as the expected number of points plotted on the chart in order to observe such a signal. If the probability for a point on the control chart to indicate a signal is P (which depends on the values of the process parameters), it can be proved mathematically that:

$$\mathrm{ARL} = \frac{1}{P} \tag{7}$$

(Wadsworth *et al.* 1986), Section 7–9.2)). If a control chart has a LCL and an UCL, there will be a separate ARL for each of these control limits, and there will also be an overall ARL for both the LCL and UCL.

If the length of inspection required to plot a point on the control chart is a random variable X with expected value $E[X]$, it is not difficult to understand intuitively that the average length of inspection (ALI) required to observe a signal is $E[X]/P = E[X] \times \mathrm{ARL}$. This result is stated as Proposition A in the appendix, and a rigorous mathematical proof is provided there.

The ARL's and the ALI's of the CCC- and CQC-charts are given in the following Propositions 5.1 and 5.2.

Proposition 5.1: Suppose a CCC-chart is constructed such that the probabilities of false alarms for LCL and UCL are α_L and α_U when the fraction of defective items is p_0. The ARLs and the ALIs for signals to show up on the chart below the LCL, above the UCL, and either below the LCL or above the UCL, when the fraction of defective items is p, are given by:

$$\mathrm{ARL}_L = \frac{1}{1-(1-\alpha_L)^v}, \qquad \mathrm{ALI}_L = \frac{1}{p}\,\mathrm{ARL}_L$$

$$\mathrm{ARL}_U = \frac{1}{\alpha_U^v}, \qquad \mathrm{ALI}_U = \frac{1}{p}\,\mathrm{ARL}_U,$$

$$\mathrm{ARL}_{L\cup U} = \frac{1}{1-(1-\alpha_L)^v + \alpha_U^v}, \qquad \mathrm{ALI}_{L\cup U} = \frac{1}{p}\,\mathrm{ARL}_{L\cup U}$$

respectively, where $v = \ln(1-p)/\ln(1-p_0)$.

Proposition 5.2: Suppose a CQC-chart is constructed such that the probability of false alarms for LCL and UCL are α_L and α_U when the rate of occurrence of defects is λ_0. The ARLs and ALIs for signals to show up on the chart below the LCL, above the UCL, and either below the LCL or above the UCL, when the rate of occurrence of defects is λ, are given by

$$\mathrm{ARL}_L = \frac{1}{1-(1-\alpha_L)^r}, \qquad \mathrm{ALI}_L = \frac{1}{\lambda}\,\mathrm{ARL}_L$$

$$\mathrm{ARL}_U = \frac{1}{\alpha_U^r}, \qquad \mathrm{ALI}_U = \frac{1}{\lambda}\,\mathrm{ARL}_U$$

$$\mathrm{ARL}_{L\cup U} = \frac{1}{1-(1-\alpha_L)^r + \alpha_U^r}, \qquad \mathrm{ALI}_{L\cup U} = \frac{1}{\lambda\,\mathrm{ARL}_{L\cup U}},$$

respectively, where $r = \lambda/\lambda_0$.

Proposition 5.1. follows from (7) and Proposition A in the appendix by substituting $n_L = \ln(1-\alpha_L)/\ln(1-p_0)$ into $1/P = 1/F(n_L) = 1/(1-(1-p)^{n_L})$, substituting $n_U = \log(\alpha_U)/\log(1-p_0)$ into $1/P = 1/(1-F(n_U)) = 1/(1-p)^{n_U}$, substituting such n_L and n_U into $1/P = 1/(F(n_L)+1-F(n_U)) = 1/(1-(1-p)^{n_L}+(1-p)^{n_U})$ (cf. Section 2), and using the fact that the expected number of items inspected in order to observe a defective item is $E[n] = 1/p$. Proposition 5.2 can be proved in a similar way.

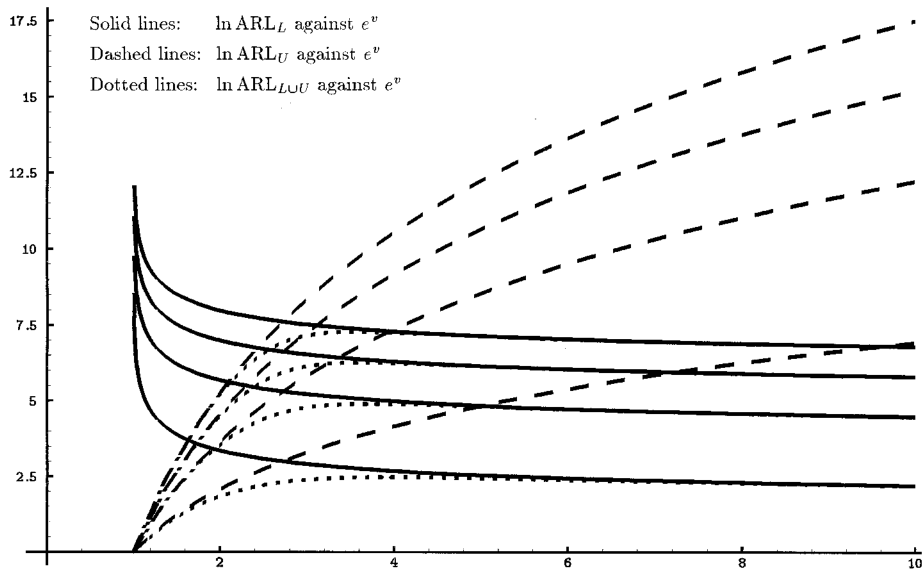

Figure 1a. Average run length plots for the CPC(C)-chart.

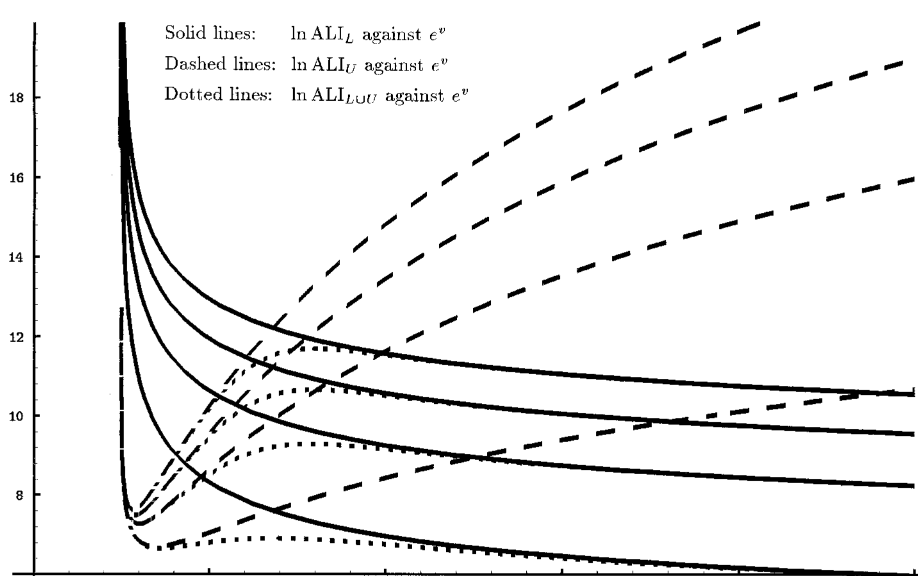

Figure 1b. Average length inspected plots for the CPC(C)-chart.

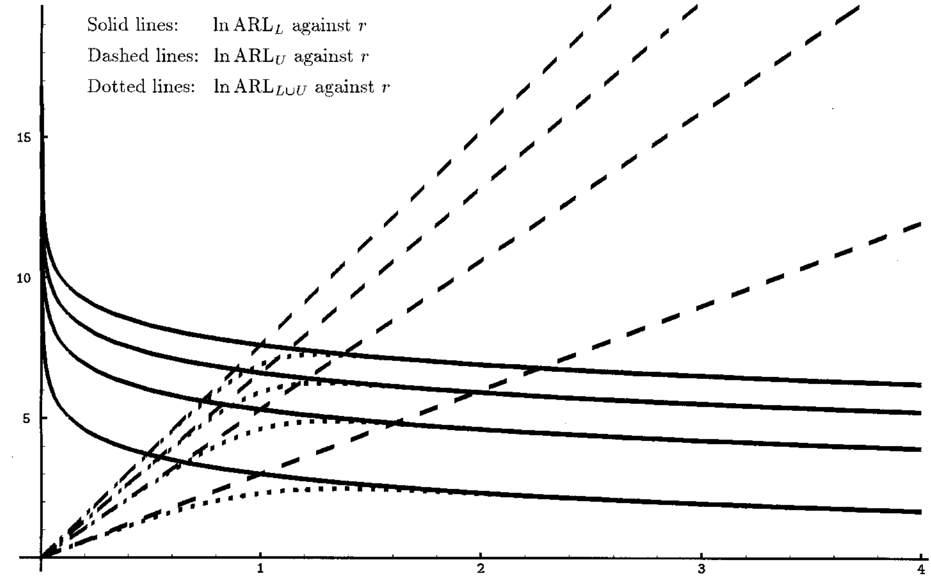

Figure 2a. Average run length plots for the CPC(Q)-chart.

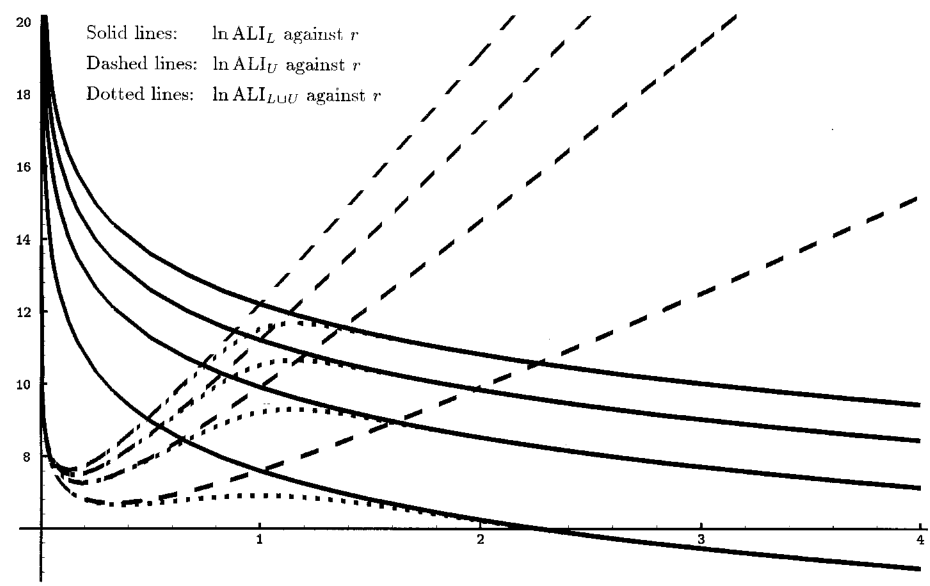

Figure 2b. Average length inspected plots for the CPC(Q)-chart.

Since decisions resulted from the CPC(C)-chart are the same as those resulting from the CCC-chart, the CPC(C)-chart and the CCC-chart have identical ARLs and ALIs. Likewise, the CPC(Q)-chart and the CQC-chart have identical ARLs and ALIs.

The behaviour of the ARLs and ALIs of these charts will be investigated graphically. In figure 1a, the solid lines down from the top are the graphs of ln ARL_L

against e^v for the CPC(C)- or CCC-chart at $\alpha_L = 0.0005, 0.00135, 0.005, 0.05$, the dashed lines down from the top are the graphs of $\ln \text{ARL}_U$ against e^v for the CPC(C)- or CCC-chart at $\alpha_U = 0.0005, 0.00135, 0.005, 0.05$, and the dotted lines down from the top are the graphs of $\ln \text{ARL}_{L\cup U}$ against e^v for the CPC(C)- or CCC-chart at $\alpha_L = \alpha_U = 0.0005, 0.00135, 0.005, 0.05$, where $v = \ln(1-p)/\ln(1-p_0)$. Plotting the graphs against e^v instead of v exaggerates the variations of the graphs at small values of v. Figure 1b shows the corresponding graphs of $\ln \text{ALI}_L$, $\ln \text{ALI}_U$ and $\ln \text{ALI}_{L\cup U}$ when $p_0 = 0.01$. Figure 1a shows that the graphs of $\ln \text{ARL}_U$'s are steeper than those of $\ln \text{ARL}_L$'s in a vicinity of $e^v = 2.718\ldots$ (or $v = 1$), which means that in a vicinity of p_0, the CPC(C)- or CCC-chart is more sensitive in detecting process improvement than process deterioration. When e^v is close to 1 (or v is close to 0), the graphs of $\ln \text{ARL}_L$s are extremely steep, which means that false signals indicating process deterioration rarely show up on the CPC(C)- or CCC-chart when p is close to 0. In figure 1b, the graphs of ln ALIs have the same pattern as those of ln ARLs in figure 1a except in a small neighbourhood of $e^v = 1$ (or $v = 0$). When e^v approaches 1 (or v approaches 0), both the dashed lines and dotted lines in figure 1b approaches infinity, which means that when p approaches 0 , although both ARL_U and $\text{ARL}_{L\cup U}$ are small (as shown in figure 1a), ALI_U and $\text{ALI}_{L\cup U}$ are large owing to the fact that a large number of items have to be inspected in order to observe a defective one.

Figure 2a shows graphs of $\ln \text{ARL}_L$, $\ln \text{ARL}_U$ and $\ln \text{ARL}_{L\cup U}$ against r for the CPC(Q)- or CQC-chart at $\alpha_L = 0.0005$, 0.00135, 0.005, 0.05 and $\alpha_U = 0.0005$, $0.00135, 0.005, 0.05$, where $r = \lambda/\lambda_0$. Figure 2b are those of $\ln \text{ALI}_L$, $\ln \text{ALI}_U$ and $\ln \text{ALI}_{L\cup U}$ against r for the CPC(Q)- or CQC-chart at $\alpha_L = 0.0005$, 0.00135, 0.005, 0.05, $\alpha_U = 0.0005$, 0.00135, 0.005, 0.05, and $r_0 = 0.01$. These graphs can be interpreted similarly to those of the CPC(C)- or CCC-chart.

6. Examples

6.1. *Example 1*

In the manufacture of integrated circuit chips, the die is connected to the lead-frame by gold wires using a high-speed bonding machine. In the plant of a multi-national manufacturer, this is a high-quality process with fraction of defective joints $p_0 = 0.0001$ (one defective joint per 10 000 joints) when the process in normal condition. A special type of chip with 100 bondings in each chip is being produced. Each chip is inspected automatically after the bonding process, and the number of defective bondings on the chip is recorded. In applying the CPC(C)-chart to monitor the process, a chip is considered as a sample, so that the size of each sample is 100 (bondings). A CPC(C)-chart is constructed with the probability of false alarm set at $\alpha = 2 \times 0.00135 = 0.0027$ at $p_0 = 0.0001$. From (1), the cumulative distribution function of the CPC(C)-chart is $F(n) = 1 - 0.9999^n$, and the LCL, UCL and CL are $\alpha/2 = 0.00135$, $1 - \alpha/2 = 0.99865$ and 1/2, respectively. On the other hand, it follows from Section 2 that the LCL, UCL and CL of the CCC-chart are

$$n_L = \ln(1 - 0.0027/2)/\ln(1 - 0.0001) = 13.5084,$$

$$n_U = \ln(0.0027/2)/\ln(1 - 0.0001) = 66073.2$$

and

$$n_C = \ln(1/2)/\ln(1 - 0.0001) = 6931.5,$$

respectively.

Sample number	Cumulative items inspected n	$\log_{10} n$	$F(n)$	Defect observed	Indication
1	14 $\geq$LCL	1.146	$0.001399 \geq \alpha/2$	yes	i.c.
1	*14 $\geq$LCL	1.146	$0.001399 \geq \alpha/2$	no	i.c.
1	72	1.857	0.007175	no	i.c.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	972	2.988	0.092630	no	i.c.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
661	66072	4.820	0.998650	no	i.c.
662	66172 > UCL	4.821	$0.998663 > 1 - \alpha/2$	no	im.
663	66245 $\geq$ LCL	4.821	$0.998673 \geq \alpha/2$	yes	i.c.
663	*14 $\geq$ LCL	1.146	$0.001399 \geq \alpha/2$	no	i.c.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1. Inspection results for a binomial process.

Some observed data are shown in table 1, where ‘i.c.’ and ‘im.’ stand for ‘in-control’ and ‘improved’, respectively. The number immediately to the right of an asterisk ‘*’ is the number of items inspected since the last resetting of n to 0 (which happens when a defect is observed but the process is still deemed to be in control). The CPC(C)-chart is shown in figure 3a, and the CCC-chart with $\log_{10}n$ as the vertical axis is shown in figure 3b. In figure 3a, the range $[0, 1]$ of the vertical axis is divided into segments with different scales, with the segments [0.00130, 0.00140], [0.10, 0.90] and [0.99860, 0.99870] covering the LCL, CL and UCL, respectively. In figure 3b, the LCL, CL and UCL are $\log_{10}13.5084 = 1.13061$, $\log_{10}6931.5 = 3.84083$ and $\log_{10}66073.2 = 4.82003$, respectively.

On the CPC(C)-chart, the cross ‘$\times$’ at sample number 1 is clearly shown to be above the LCL ($=0.00135$), which does not indicate that the process is out-of-control. On the corresponding CCC-chart, however, this point is very close to the LCL which makes it much harder to see that it is above the LCL. When inspection of sample number 662 was completed, the cumulative number of items inspected was $66172 >$ UCL, and the circled dot ‘$\odot$’ corresponding to $F(n) = 0.998663$ on the CPC(C)-chart is clearly shown to be beyond the UCL, which is an indication that the process may have improved. On the CCC-chart, this point visually overlaps the UCL, and it is very difficult to see that it is beyond the UCL.

6.2. *Example 2*

The manufacture of high-quality optical cable is a stringent process, but imperfections on the product still occur occasionally. The defect rate of the process, however, is very low, and in a certain plant it is maintained at four flaws per 10,000 metres of cable produced which is a standard established by the manufacturer. Regarding the occurrence of flaws as a Poisson process, the defect rate of the process is $\lambda_0 = 0.0004$ (flaw/metre). To monitor the process, samples of size 50 (metres) are inspected continuously, and the probability of false alarm is set at $\alpha = 0.05$. For the CPC(Q)-chart, from (2) the cumulative distribution function is given by $F(Q) = 1 - e^{-0.0004Q}$, and the CLC, UCL and CL are $\alpha/2 = 0.025$, 1/2 and $1 - \alpha/2 = 0.975$, respectively. On the other hand, for the CQC-chart, it follows from Section 2 that the LCL, UCL and CL are $Q_L = -\ln(1 - 0.05/2)/0.0004 = 63.2945$,

$Q_U = \ln(0.05/2)/0.0004 = 9222.1986$, $Q_C = \ln(1/2)/0.0004 = 1732.8680$, respectively.

Some observed data are shown in table 2, where 'o.c.', 'n.i.', 'i.c.' and 'im.' stand for 'out of control', 'no indication', 'in control' and 'improved', respectively. The number immediately to the right of an asterisk is the quantity of product inspected since the last resetting of Q to 0 (which happens when a defect is observed but the process is still deemed to be in control). The CPC(Q)-chart is shown in figure 4a, and the CQC-chart with $\log_{10}Q$ as the vertical axis is shown in figure 4b. In figure 4a, the range [0, 1] of the vertical axis is divided into segments with different scales, with the segments [0.020, 0.030], [0.10, 0.90] and [0.970, 0.980] covering the LCL, CL and

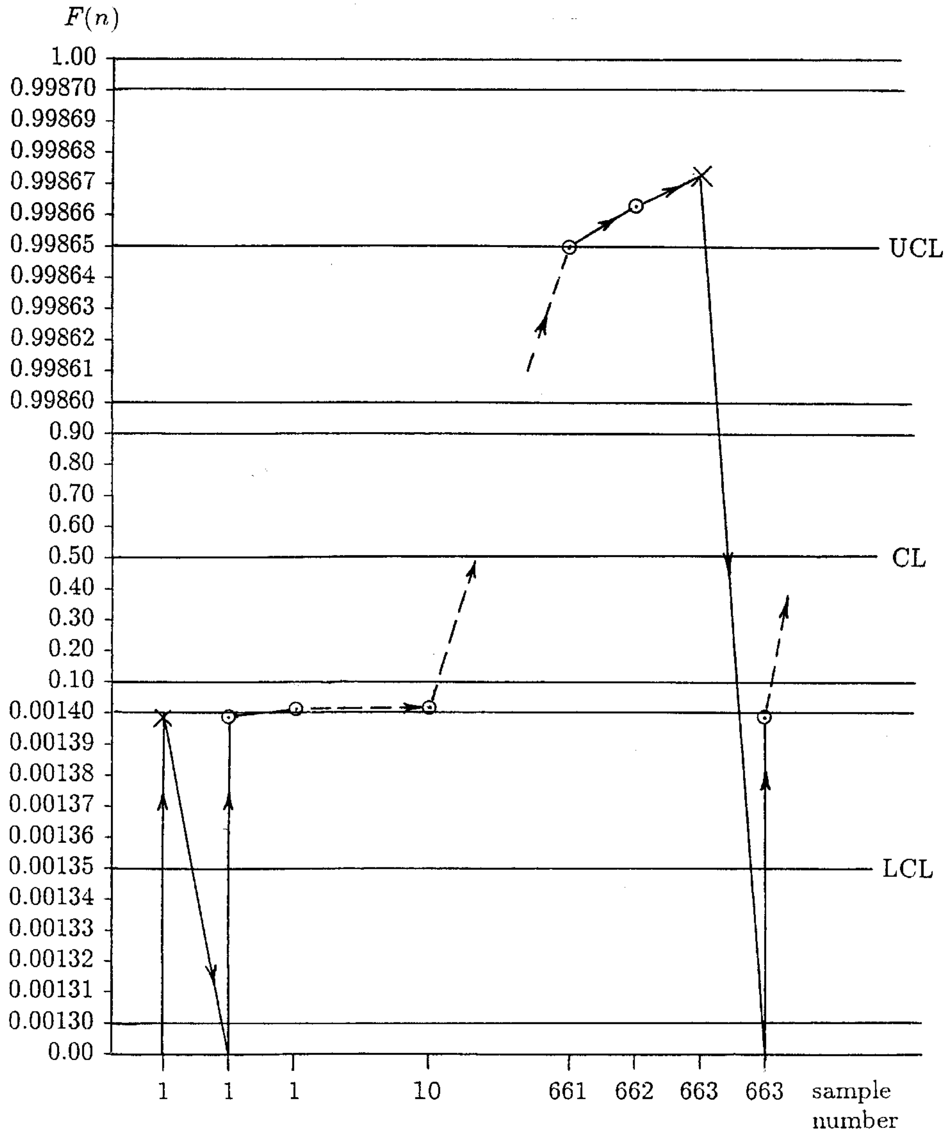

Figure 3a. CPC(C)-chart for a binomial process for the data in table 1.

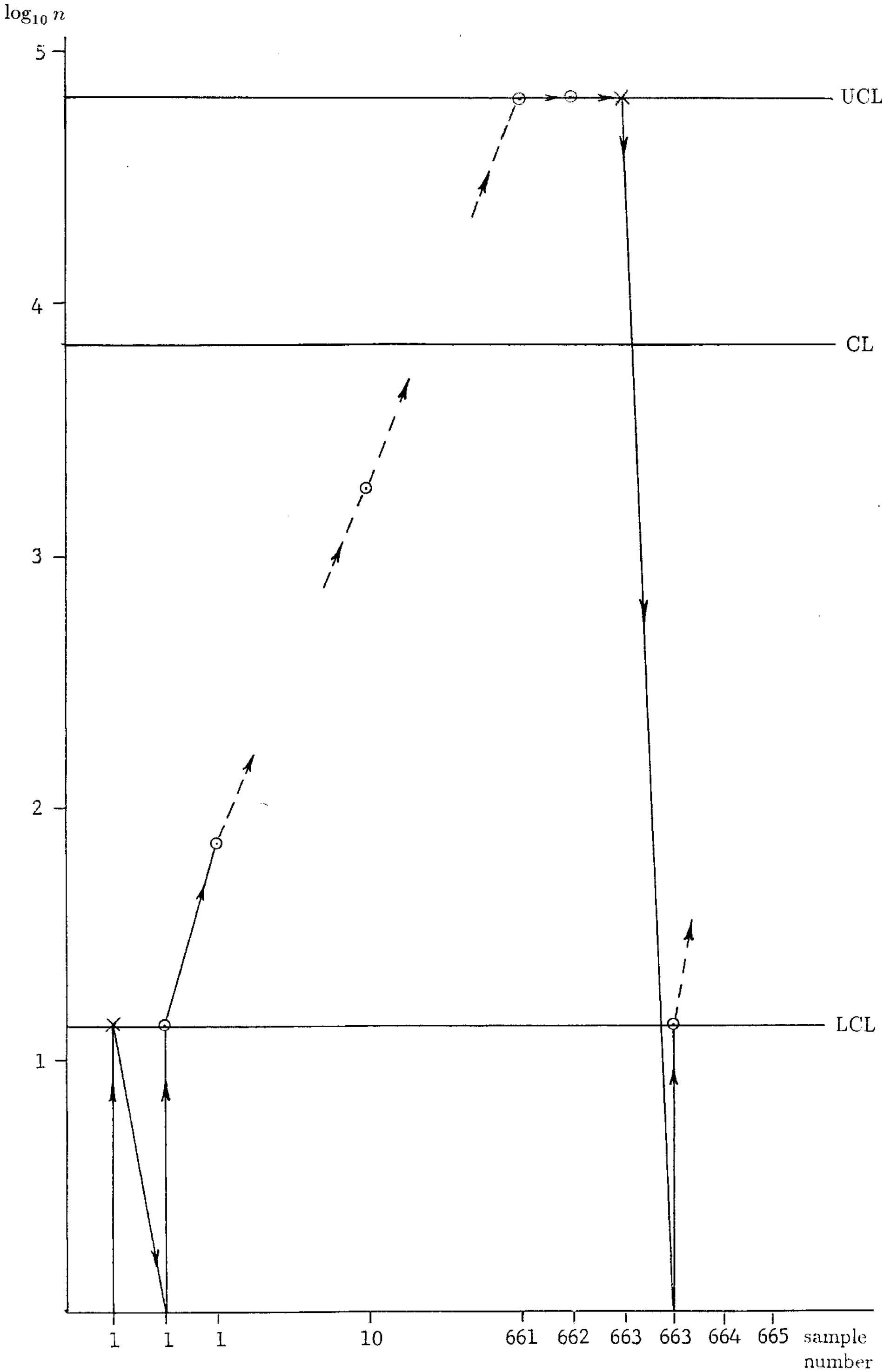

Figure 3b. CCC-chart for a binomial process for the data in table 1.

Sample number	Cumulative items inspected Q	$\log_{10} Q$	$F(Q)$	Defect observed	Indication
1	$47.5 < \text{LCL}$	1.6767	$0.01882 < \alpha/2$	yes	o.c.
1	$50 < \text{LCL}$	1.6990	$0.01980 < \alpha/2$	no	n.i.
2	$63.2945 \geq \text{LCL}$	1.8014	$0.025 \geq \alpha/2$	no	i.c.
2	100	2.0000	0.03921	no	i.c.
⋮	⋮	⋮	⋮	⋮	⋮
100	$467.8 \geq \text{LCL}$	2.6701	$0.1707 \geq \alpha/2$	yes	i.c.
100	$*32.2 < \text{LCL}$	1.5079	$0.01280 < \alpha/2$	no	n.i.
101	$63.2945 \geq \text{LCL}$	1.8014	$0.025 \geq \alpha/2$	no	i.c
101	82.2	1.9149	0.03235	no	i.c.
⋮	⋮	⋮	⋮	⋮	⋮
283	9182.2	3.9630	0.9746	no	i.c.
284	$9232.2 > \text{UCL}$	3.9653	$0.9751 > 1 - \alpha/2$	no	im.
285	9282.2	3.9677	0.9756	no	im.
⋮	⋮	⋮	⋮	⋮	⋮

Table 2. Inspection results for a Poisson process.

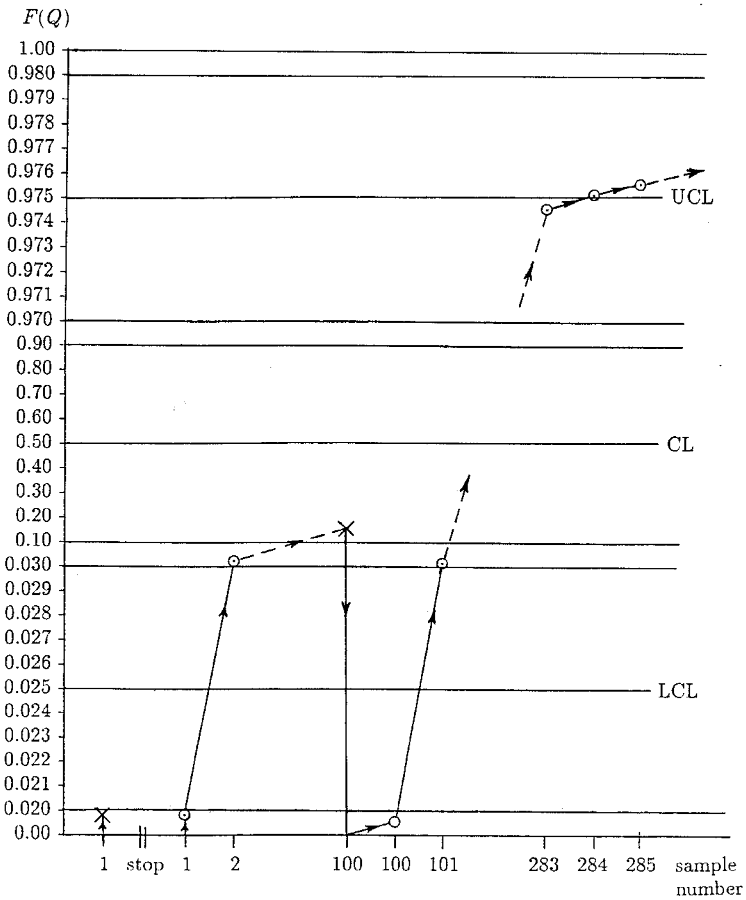

Figure 4a. CPC(Q)-chart for a Poisson process for the data in table 2.

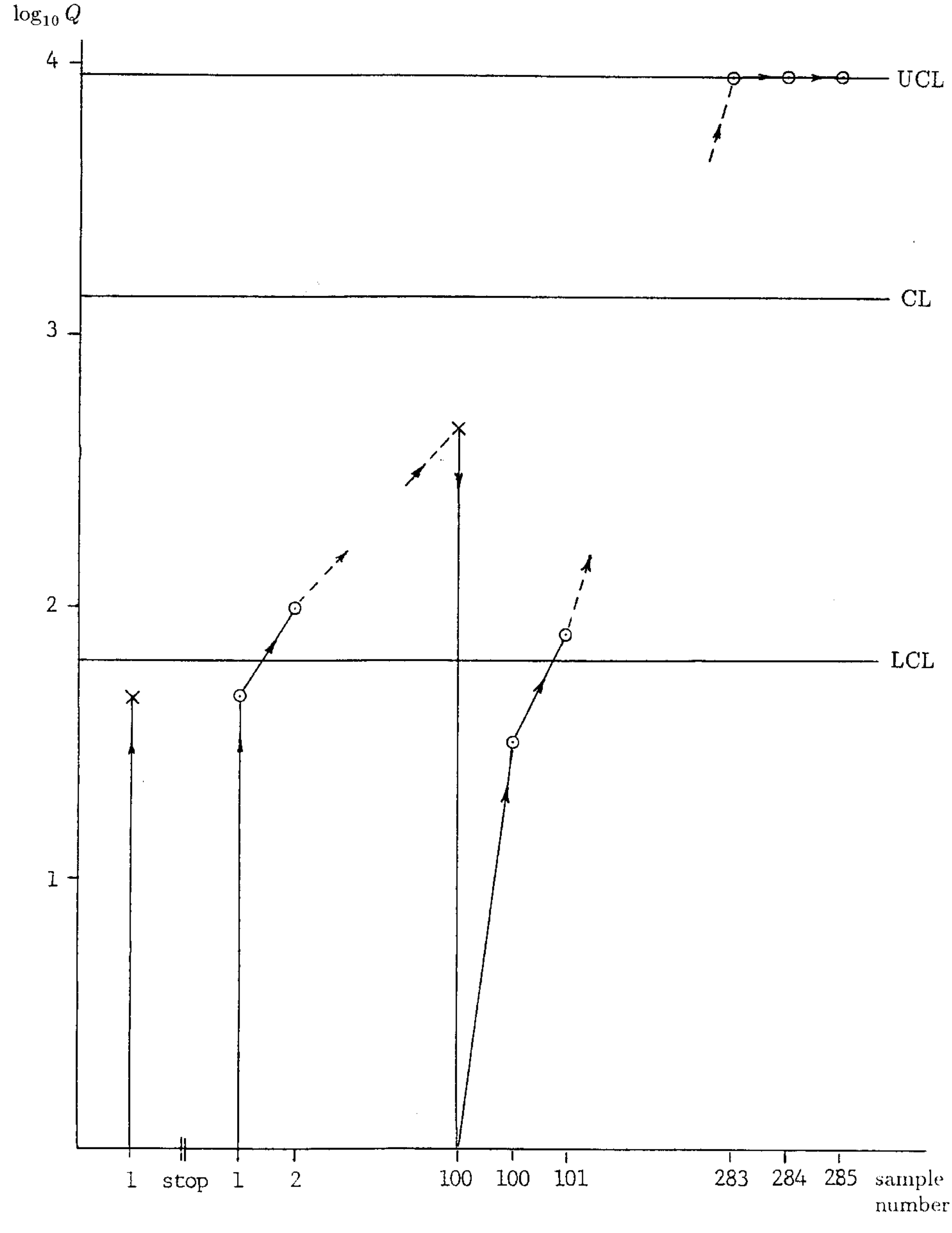

Figure 4b. CQC-chart for a Poisson process for the data in table 2.

UCL, respectively. In figure 4b, the LCL, CL and UCL are $\log_{10} 63.2945 = 1.80137$, $\log_{10} 1732.8680 = 3.23877$ and $\log_{10} 9222.1986 = 3.96493$, respectively.

On the CQC-chart, the region of in-control (between the LCL and UCL) covers only about the upper half of the chart, and therefore the region between 0 and the LCL is not used efficiently. (This phenomenon will be even more prominent for smaller values λ_0.)

The defect found in the left-most sample number 1 in figures 4a and b was an indication of out-of-control, and assignable causes were found during investigation.

The process was then rectified. After that, the process was started again and the first sample was numbered as 1. Indication of improvement appeared at sample number 284, and investigation would be carried out to find out whether this was a false signal or the process had actually improved. This indication of improvement is much more clearly shown by the circled dot on the CPC(Q)-chart at sample number 285 (which corresponds to $F(Q) = 0.9756$) than that on the CQC-chart.

7. Conclusion

Processes with low defect rates cannot be satisfactorily monitored using the traditional *p*-chart, *np*-chart, *c*-chart and *u*-chart. The cumulative count control chart (CCC-chart) and the cumulative quantity control chart (CQC-chart) are possible alternatives that can be used to monitor such processes. When the CCC- or CQC-chart is used, in many cases $\log n$ or $\log Q$ will be plotted instead of n or Q, since the cumulative count n or cumulative quantity Q increases linearly as the sample number increases. Plotting of $\log n$ and $\log Q$, however, produces charts that are over-sensitive for small n or Q and extremely insensitive for large n and Q. To overcome this disadvantage, using an idea in Yeh and Lin (1997) in the studies of two-dimensional charts, the cumulative probability control chart (CPC-chart) is proposed here. In a CPC-chart, the cumulative probability is plotted, instead of the cumulative count or cumulative quantity. Decisions resulted from the CPC-chart is the same as those resulting from the CCC- or CQC-chart, because if the CPC-chart indicates 'in control' or 'out-of-control', so will the CCC- or CQC-chart, and vice versa. To increase the sensitivity of the CPC-chart near the decision lines (the lower and upper control limits), the vertical axis of a CPC-chart may be divided into several segments, in such a way that the segments containing the decision lines have a larger linear scale.

Using linear scales in the CPC-chart, sensitizing rules can be applied to increase the effectiveness of the chart. Commonly used sensitizing rules for the Shewhart control chart are defined based on dividing the area between the LCL and UCL into Zones A–C. Zone C is the region between the ± 1 sigma limits on either side of the centre line, Zone B is the regions between 1 and 2 sigmas on either side of the centre line, and Zone A is the regions between 2 and 3 sigmas on either side of the centre line. Some sensitizing rules are shown in table 3. To develop sensitizing rules for the CPC-chart, Zones A–C can be defined in terms of probabilities instead of the ± 1, ± 2 and ± 3 sigma lines.

1. Point falls outside the ± 3 sigma limits.
2. Eight points in a row in Zone C or beyond on the same side of the center line.
3. Six points in a row increasing or decreasing.
4. Fourteen points in a row alternating up and down.
5. Two out of three points in a row in Zone A or beyond on the same side of the center line.
6. Four out of five points in a row in Zone B or beyond on the same side of the center line.
7. Fifteen points in a row in Zone C.
8. Eight points in a row not in Zone C.

Table 3. Sensitizing rules.

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Appendix A

Proposition A: Suppose that the length of inspection in order to observe a defect in the process is a random variable X with expected value $E[X]$. Suppose also that a point is plotted on the control chart when a defect is observed in the process. If P is the probability for a point on the control chart to indicate a type of signal, the average length of inspection in order to observe such a signal on the control chart is $\mathrm{ALI} = E[X]/P$.

Proof of Proposition A: Let x denote an observed value of X. Divide the range of variation of X into two sets A and B, in such a way that $x \in A$ gives a signal on the control chart, and $x \in B$ does not give a signal on the control chart. Let the distribution function of X be $F(x)$. Hence:

$$\int_{x\in A} \mathrm{d}F(x) = P, \quad \int_{x\in B} \mathrm{d}F(x) = 1 - P,$$

$$\int_{x\in A} x\,\mathrm{d}F(x) + \int_{x\in B} x\,\mathrm{d}F(x) = \int_{x\in A\cup B} x\,\mathrm{d}F(x) = E[X].$$

The expected length of observation until the first defect occurs, where this defect gives a signal on the control chart, is

$$L_0 = \int_{x\in A} x\,\mathrm{d}F(x).$$

Let $n \geq 1$ be an integer. Suppose that n defects have occurred and none of these defects gives a signal on the control chart, followed by the $(n+1)^{\text{th}}$ defect which gives a signal on the control chart. Let the observed values of X corresponding to the occurrence of these defects be $x_1, \ldots, x_n \in B$ and $x_{n+1} \in A$. For each $n \geq 1$, the expected value of the total length of observation until the signal occurs on the control chart is:

$$\begin{aligned}
L_n &= \int_{x_1\in B} \cdots \int_{x_n\in B} \int_{x_{n+1}\in A} (x_1 + \cdots + x_n + x_{n+1})\,\mathrm{d}F(x_1)\cdots \mathrm{d}F(x_n)\,\mathrm{d}F(x_{n+1}) \\
&= \sum_{j=1}^{n} \left(\int_{x_1\in B} \cdots \int_{x_n\in B} \int_{x_{n+1}\in A} x_j\,\mathrm{d}F(x_1)\cdots \mathrm{d}F(x_n)\,\mathrm{d}F(x_{n+1}) \right) \\
&\quad + \int_{x_1\in B} \cdots \int_{x_n\in B} \int_{x_{n+1}\in A} x_{n+1}\,\mathrm{d}F(x_1)\cdots \mathrm{d}F(x_n)\,\mathrm{d}F(x_{n+1}) \\
&= \sum_{j=1}^{n} \left(\int_{x_j\in B} x_j\,\mathrm{d}F(x_j) \right) \left(\prod_{\ell\neq j}^{n} \int_{x_\ell\in B} \mathrm{d}F(x_\ell) \right) \int_{x_{n+1}\in A} \mathrm{d}F(x_{n+1}) \\
&\quad + \left(\prod_{j=1}^{n} \int_{x_j\in B} \mathrm{d}F(x_j) \right) \int_{x_{n+1}\in A} x_{n+1}\,\mathrm{d}F(x_{n+1})
\end{aligned}$$

$$= \sum_{j=1}^{n}\left(\int_{x\in B} x\,\mathrm{d}F(x)\right)\left(\int_{x\in B}\mathrm{d}F(x)\right)^{n-1}\int_{x\in A}\mathrm{d}F(x)$$

$$+\left(\int_{x\in B}\mathrm{d}F(x)\right)^{n}\int_{x\in A} x\,\mathrm{d}F(x)$$

$$= \int_{x\in B} x\,\mathrm{d}F(x)n(1-P)^{n-1}P + \int_{x\in A} x\,\mathrm{d}F(x)\ (1-P)^{n}.$$

Therefore

$$\mathrm{ARL} = L_0 + \sum_{n=1}^{\infty} L_n$$

$$= \sum_{n=0}^{\infty}\left(\int_{x\in B} x\,\mathrm{d}F(x)n(1-P)^{n-1}P + \int_{x\in A} x\,\mathrm{d}F(x)(1-P)^{n}\right)$$

$$= \int_{x\in B} x\,\mathrm{d}F(x)P\sum_{n=0}^{\infty} n(1-P)^{n-1} + \int_{x\in A} x\,\mathrm{d}F(x)\sum_{n=0}^{\infty}(1-P)^{n}$$

$$= \int_{x\in B} x\,\mathrm{d}F(x)/P + \int_{x\in A} x\,\mathrm{d}F(x)/P = E[X]/P.$$

This proves Proposition A. □

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