Testing Multiple Dispersion Effects in Unreplicated Fractional Factorial Designs

Richard N. McGrath

Department of Applied Statistics
and Operations Research
Bowling Green State University
Bowling Green, OH 43403
(*rnmcgra@cba.bgsu.edu*)

Dennis K. J. Lin

Department of Management Science
and Information Systems
The Pennsylvania State University
University Park, PA 16802
(*dkl5@psu.edu*)

In unreplicated 2^{k-p} designs, the assumption of constant variance is commonly made. When the variance of the response differs between the two levels of a column in the effect matrix, that column produces a dispersion effect. In this article we show that two active dispersion effects may create a spurious dispersion effect in their interaction column. Most existing methods for dispersion-effect testing in unreplicated fractional factorial designs are subject to these spurious effects. We propose a method of dispersion-effect testing based on geometric means of residual sample variances. We show through examples from the literature and simulations that the proposed test has many desirable properties that are lacking in other tests.

KEY WORDS: Experimental design; Geometric mean; Multiple effects; Robust design; Screening experiments; Variance.

Traditionally, the primary use of fractional factorial designs has been in detecting the factors that produce location effects (changes in the mean response). An assumption of constant variance (no dispersion effects) is usually made. Many techniques have been designed to attack this problem. Examples of studies of location effects in unreplicated fractional factorial designs include those of Daniel (1959, 1976), Box and Meyer (1986a), Lenth (1989), Juan and Peña (1992), and Loughin and Noble (1997). See Hamada and Balakrishnan (1998) for an overview and comparison of different methods.

Taguchi (e.g., see Taguchi and Wu 1980) emphasized the importance of detecting dispersion effects. If a factor produces a dispersion effect in the response, the factor level can be adjusted to reduce the variation in a manufactured product. Thus, identifying and studying dispersion effects can result in a product or process that is robust to environmental variations (noise). Techniques have been developed for studying dispersion effects in replicated experiments with $r \geq 2$ observations at each design setting. See Davidian and Carroll (1987) and Nair and Pregibon (1988) for examples.

For unreplicated fractional factorials, however, no estimate of variation is available at each design setting, making the study of dispersion effects more challenging. If the design is a fractional factorial, the confounding greatly increases the complexity. In their pioneering work, Box and Meyer (1986b) developed an informal method for identifying dispersion effects in unreplicated experiments by studying the logarithm of the ratio of residual variances. They noted, as did Pan (1999) and McGrath and Lin (2001), the importance of first identifying location effects before studying dispersion effects. Montgomery (1990) extended this method by plotting these statistics on a normal probability plot to distinguish between small and large dispersion effects. Wang (1989) developed a test statistic that has an approximate χ^2 distribution for a large sample size. Ferrer and Romero (1993a,b) used the residuals (or an appropriate transformation of the residuals) as a response to study dispersion. More recently, Bergman and Hynén (1997) developed an exact dispersion test using a statistic having an F distribution, while McGrath and Lin (1999) developed a nonparametric version of this test. We show in Section 3 that these tests behave as expected only when a lone dispersion effect exists. Wolfinger and Tobias (1998) used a mixed-model approach that is applicable for unreplicated full factorials but not highly fractionated designs. In this article, we discuss a new test that is applicable even when there are multiple dispersion effects in unreplicated 2^{k-p} experiments.

The article is organized as follows. In Section 1 we discuss the test of Bergman and Hynén (1997), an appropriate dispersion-effect testing method when only a single dispersion effect is present. In Section 2 we show that two dispersion effects create a spurious dispersion effect in their interaction column and develop a new test that is insensitive to this problem. Examples from the literature are used to compare these methods. In Section 3 various simulations show the superiority of the proposed method. Finally, Section 4 provides some practical considerations and recommendations.

1. AN EXISTING METHOD

Suppose an $n = 2^{k-p}$ fractional factorial design is run. The design matrix, $\mathbf{X} = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{n-1})$ represents k factors and interactions between these factors depending on the degree

© 2001 American Statistical Association and
the American Society for Quality

of fractionation, $\mathbf{x}_0 = (1, \ldots, 1)'$, and $(\mathbf{x}_j = x_{1j}, x_{2j}, \ldots, x_{nj})'$ with $x_{ij} = \pm 1$, $j = 1, \ldots, n-1$. We assume that the observations are independently normally distributed with

$$Y_i = \sum_{j=0}^{n-1} x_{ij}\beta_j + \epsilon_i, \quad \epsilon_i \sim \mathrm{N}(0, \sigma_i^2), \;\; \sigma_i^2 = \sigma^2 \prod_{j=1}^{n-1} \Delta_j^{x_{ij}/2}, \quad (1)$$

with β_j, Δ_j, and σ^2 unknown parameters. This model reparameterization was used by Cook and Weisberg (1983), among others; β_j is a measure of the location (additive) effect for column j, and Δ_j is a measure of the dispersion (multiplicative) effect for j. The absence of dispersion effects—that is, $\Delta_j = 1, j = 1, \ldots, n-1$—corresponds to the usual model, assuming homogeneity of variance.

Bergman and Hynén (1997) developed a dispersion-effect test statistic, D^{BH}, that has an F distribution. Because our proposed approach has some similarities, we briefly describe their procedure for comparison purposes. Defining e_i as the residual from the fitted model in row i, it is straightforward to show that when testing the null hypothesis $\Delta_j = 1$, D_j^{BH} is the ratio of the sample variances of residuals from $P_j = \{e_i : x_{ij} = +1\}$ and $M_j = \{e_i : x_{ij} = -1\}$—that is, s_{j+}^2 / s_{j-}^2. However, a specific model must be fit for this statistic to have an F distribution. It must include exactly the overall mean, all active (nonzero) location effects, the location effect of the column to be tested for dispersion, and the interaction of the dispersion-effect column with the other terms in the model. Accordingly, several different location models need to be fit to test all of the columns.

Example 1. As an example of calculating D^{BH}, Bergman and Hynén (1997) studied data originally analyzed by Davies (1956). The effect of five factors on the quality of a dyestuff was studied in an unreplicated 2_{V}^{5-1} design. The five factors were temperature (A), starting material (B), reduction pressure (C), oven drying pressure (D), and vacuum leak (E). Table 1 shows the design matrix and responses where the y_1 column contains the responses. All authors found that D has a large impact on location—that is, the mean dyestuff quality—and appears to be the only location effect. This finding also agrees with the results using the procedure of Lenth (1989) with $\alpha = .05$.

Using their D^{BH}, Bergman and Hynén also found a dispersion effect due to E and mild dispersion effects due to D and DE (see col. 2 of Table 2). To calculate D_D^{BH}, the location model is simply $Y_i = \beta_0 + x_{iD}\beta_D + \epsilon_i$. (We will use the notation D^{BH} when referring to the general test statistic and D_j^{BH} when referring to the statistic calculated for column j.) Using the residuals from this model, $D_D^{\mathrm{BH}} = s_{D+}^2 / s_{D-}^2 = 447.64/100.05 = 4.474$. With two terms in the model, there are $(16-2)/2 = 7$ df in each variance estimate. Comparing D_D^{BH} to an $F_{7,7}$ distribution gives a p value of .009. As previously mentioned, to calculate D_E^{BH} the model must be adapted to

$$Y_i = \beta_0 + x_{iD}\beta_D + x_{iE}\beta_E + x_{iDE}\beta_{DE} + \epsilon_i. \quad (2)$$

Residuals from this same model can also be used to calculate D_{DE}^{BH}. With four terms in this model, there are $(16-4)/2 = 6$ df in each variance estimate, so the D^{BH} can be compared to an $F_{6,6}$. The same procedure can be used with the appropriate model to test for dispersion in each column.

In the next section we show that when there are multiple dispersion effects, as there may be in this example, this test can be misleading because spurious dispersion effects may be created. In Section 3 we show that D^{BH} tests for null dispersion effects have inflated significance levels in the presence of a single active dispersion effect.

2. THE PROPOSED METHOD

Define an interaction triple $(\mathbf{x}_j, \mathbf{x}_{j'}, \mathbf{x}_{j''})$ such that $x_{ij}x_{ij'} = x_{ij''}$ for $i = 1, \ldots, n$ and the actual magnitudes of dispersion effects in these columns as Δ_j, $\Delta_{j'}$, and $\Delta_{j''}$. In Section 2.1 we derive a dispersion-effect test in which the location model consists exactly of an overall mean and an interaction triple. From this case, we derive the general test statistic in Section 2.2. Throughout this article, we assume that the observations are normally distributed in accordance with (1). Unless otherwise specified, we will assume that all active location effects have

Table 1. Experimental Designs and Responses

$i \backslash j$	0	A 1	B 2	C 3	D 4	AB 5	AC 6	AD 7	BC 8	BD 9	CD 10	DE 11	CE 12	BE 13	AE 14	E 15	y_1	y_2
1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	−1	−1	1	201.5	13
2	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	178.0	54
3	1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	1	−1	1	−1	183.5	44
4	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	1	1	176.0	49
5	1	−1	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	1	−1	188.5	13
6	1	1	−1	1	−1	−1	1	−1	−1	1	−1	−1	1	−1	1	1	178.5	14
7	1	−1	1	1	−1	−1	−1	1	1	−1	−1	−1	1	1	−1	1	174.5	18
8	1	1	1	1	−1	1	1	−1	1	−1	−1	1	−1	−1	−1	−1	196.5	85
9	1	−1	−1	−1	1	1	1	−1	1	−1	−1	−1	1	1	1	−1	255.5	41
10	1	1	−1	−1	1	−1	−1	1	1	−1	−1	1	−1	−1	1	1	240.5	73
11	1	−1	1	−1	1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	208.5	79
12	1	1	1	−1	1	1	−1	1	−1	1	−1	−1	1	−1	−1	−1	244.0	17
13	1	−1	−1	1	1	1	−1	−1	−1	−1	1	1	1	−1	−1	1	274.0	82
14	1	1	−1	1	1	−1	1	1	−1	−1	1	−1	−1	1	−1	−1	257.5	58
15	1	−1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	1	−1	256.0	10
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	274.5	29

Table 2. Dispersion-Effect Statistics and p Values for Example 1

Effect	D^{BH} (p value)	F^{ML}	p value	$F_{c,c}$ approx. p value
D	4.47 (.066)	1.97	.463	.464
E	11.51 (.009)	8.19	.033	.033
DE	5.29 (.062)	3.14	.222	.224

been identified and are included in the model. In Section 3 we discuss the impact of unidentified location effects.

2.1 Interaction Triple Dispersion-Effect Testing

Suppose two columns of an interaction triple have dispersion effects ($\Delta \neq 1$). Then a dispersion effect is induced in their interaction column (the third member of the interaction triple). This can be seen as follows. Any interaction triple creates four unique sets of rows with the variance of each row (assuming that columns not shown do not produce dispersion effects) to be given hereafter:

j	j'	j''	Variance/σ^2
−1	−1	+1	$(\Delta_{j''}/\Delta_j\Delta_{j'})^{1/2}$
+1	−1	−1	$(\Delta_j/\Delta_{j'}\Delta_{j''})^{1/2}$
−1	+1	−1	$(\Delta_{j'}/\Delta_j\Delta_{j''})^{1/2}$
+1	+1	+1	$(\Delta_j\Delta_{j'}\Delta_{j''})^{1/2}$.

Suppose a location model is fit that includes exactly the overall mean and the location-effect estimates of an interaction triple, as in (2). Defining C_q, $q = 1, \ldots, 4$, as the four sets of residuals corresponding to the unique rows, D_j^{BH} can be written as

$$D_j^{\mathrm{BH}} = \frac{s_{j+}^2}{s_{j-}^2} = \frac{\sum_{e_i \in P_j} e_i^2}{\sum_{e_i \in M_j} e_i^2} = \frac{\sum_{q:C_q \subset P_j} s_q^2}{\sum_{q:C_q \subset M_j} s_q^2}, \qquad (3)$$

where s_q^2 is the sample variance of the residuals contained in C_q. Thus, we can view D^{BH} as the ratio of the sum (or arithmetic mean) of two sample variances and the sum (or arithmetic mean) of two others. Calculating this ratio using the variances of the observations and denoting it R_j^{BH}, we have

$$R_j^{\mathrm{BH}} = \frac{(\Delta_j/\Delta_{j'}\Delta_{j''})^{1/2} + (\Delta_j\Delta_{j'}\Delta_{j''})^{1/2}}{(\Delta_{j''}/\Delta_j\Delta_{j'})^{1/2} + (\Delta_{j'}/\Delta_j\Delta_{j''})^{1/2}} = \left(\frac{1+\Delta_{j'}\Delta_{j''}}{\Delta_{j'}+\Delta_{j''}}\right)\Delta_j.$$

Additional pairs of active dispersion effects having their interaction in column j would create additional multipliers of the same form. This product is the dispersion effect that D^{BH} estimates. Other tests such as those of Box and Meyer (1986b), Wang (1989), and McGrath and Lin (1999) also estimate this biased effect. In general, letting Δ_j^I be the dispersion effect induced by the other two dispersion effects,

$$\Delta_j^I = (1+\Delta_{j'}\Delta_{j''})/(\Delta_{j'}+\Delta_{j''}). \qquad (4)$$

So Δ_j^I is a multiplicative factor that increases (or decreases) the underlying dispersion effect of column j, possibly creating a spurious dispersion effect or dampening a true effect. Note that $\Delta_j^I = 1$ iff $\Delta_{j'}$ or $\Delta_{j''}$ or both $= 1$.

We propose a geometric mean approach to avoid the spurious dispersion effect. Viewing the far right side of (3) as a ratio of arithmetic means, we instead calculate a ratio of geometric means resulting in the test statistic

$$F_j^{\mathrm{ML}} = \frac{\left(\prod_{q:C_q\subset P_j} s_q^2\right)^{1/2}}{\left(\prod_{q:C_q\subset M_j} s_q^2\right)^{1/2}} = \left(\frac{\prod_{q:C_q\subset P_j} s_q^2}{\prod_{q:C_q\subset M_j} s_q^2}\right)^{1/2}. \qquad (5)$$

Again, we will use the notation F^{ML} when referring to the general test statistic and F_j^{ML} when referring to the statistic calculated for column j. Calculating this ratio using the variances of the observations and denoting it R_j^{ML}, we have

$$R_j^{\mathrm{ML}} = \left(\frac{(\Delta_j/\Delta_{j'}\Delta_{j''})^{1/2}(\Delta_j\Delta_{j'}\Delta_{j''})^{1/2}}{(\Delta_{j''}/\Delta_j\Delta_{j'})^{1/2}(\Delta_{j'}/\Delta_j\Delta_{j''})^{1/2}}\right)^{1/2} = \Delta_j.$$

Thus, by studying products of variances instead of sums, no spurious dispersion effect is created.

F^{ML}, as defined in (5), can be viewed as the square root of the product of two independently and identically $F_{d,d}$-distributed random variables, where d is the degrees of freedom associated with each s_q^2. As pointed out by one referee, we have assumed that all active location effects are included in the interaction triple. We also assume that no two columns with active dispersion effects are excluded from the interaction triple while their interaction column is included. In Section 3 we show that this assumption is less restrictive than that of D^{BH}.

The distribution of F^{ML} can be easily simulated by generating two large samples of independent $F_{d,d}$ random variables, forming pairs, and calculating the geometric mean of each pair. In addition, we show in the Appendix that

$$E(F^{\mathrm{ML}}) = \left(\Gamma[(d+1)/2]\Gamma[(d-1)/2]\Gamma^{-2}[d/2]\right)^2, \qquad (6)$$

where $\Gamma[\cdot]$ is the gamma function. The distribution of F^{ML}, however, can be approximated by an $F_{c,c}$ distribution. It is well known that, if $F \sim F_{c,c}$, then $E(F) = c/(c-2)$, $c > 2$. Setting $E(F^{\mathrm{ML}}) = c/(c-2)$ and solving for c gives

$$c = \frac{2\{\Gamma[(d+1)/2]\Gamma[(d-1)/2]\}^2}{\{\Gamma[(d+1)/2]\Gamma[(d-1)/2]\}^2 - \Gamma^4[d/2]}. \qquad (7)$$

Example 1 Revisited. Returning to Example 1, we calculate F^{ML} for D, E, and DE. Fitting Model (2) we can calculate D_E^{BH} and D_{DE}^{BH}, as well as the residuals. Note that the model containing these three columns' location effects can also be used to calculate D_D^{BH}. The residuals fall into the sets $C_1 = \{e_1, e_4, e_6, e_7\}$, $C_2 = \{e_2, e_3, e_5, e_8\}$, $C_3 = \{e_9, e_{12}, e_{14}, e_{15}\}$, and $C_4 = \{e_{10}, e_{11}, e_{13}, e_{16}\}$. Calculating the sample variance of the residuals in each set yields $s_1^2 = 161.06$, $s_2^2 = 61.73$, $s_3^2 = 38.75$, and $s_4^2 = 995.73$. With $d = 3$, we find $E(F^{\mathrm{ML}}) = 1.62411$ and $c = 5.21989$ from (6) and (7). Thus, we can approximate the distribution of F_j^{ML} with an $F_{5.22,5.22}$ distribution. (Although noninteger degrees-of-freedom critical values are not commonly found in published tables, statistical packages can easily calculate p values or provide critical values in these cases.) Applying (5) we have the results shown in the last three columns of Table 2. (The simulated distribution was based on 200,000 pairs of independent $F_{5.22,5.22}$ random variables, so the standard error of the reported p values is $\approx$.0005 for p value = .05.) Thus it appears that E has a dispersion effect but that D and DE do not. We will further discuss the conflicting results of D^{BH} and F^{ML} in this example in Section 4.

2.2 The General Test

The use of F^{ML} is not limited to the case in which the fitted model consists solely of an interaction triple as in Example 1. It may be used for any unsaturated model from an unreplicated 2^{k-p} experiment such that the interaction of every pair of terms in the model is also in the model. In other words, the fitted location model must include exactly the following terms:

1. The overall mean (β_0)
2. All active location effects and their interactions
3. The location effects of the columns to be tested for dispersion and their interactions
4. The interactions of all of the preceding terms

Model (2) is of this form since it includes exactly the overall mean and an interaction triple. The next smallest model would include this model, one additional term, and the interaction of the additional term with each term in the interaction triple. With such a model, seven columns can be tested for dispersion effects. For example, suppose the researcher wanted to test factor C for dispersion. Then (2) would be adapted to

$$Y_i = \beta_0 + x_{iD}\beta_D + x_{iE}\beta_E + x_{iDE}\beta_{DE} + x_{iC}\beta_C + x_{iCD}\beta_{CD} + x_{iCE}\beta_{CE} + x_{iCDE}\beta_{CDE} + \epsilon_i.$$

The preceding conditions imply that the maximum number of dispersion effects to be tested by F^{ML} is $(n-2)/2$ in an unreplicated design. Although this may seem restrictive, it is sensible. If the adapted location model includes $(n-2)/2$ terms (plus the overall mean), then only $(n-2)/2$ df remain for estimating dispersion effects plus the overall variance.

In general, the covariance matrix of the residuals from the appropriate model meeting the preceding conditions can be calculated to determine what sets of residuals are correlated with each other, forming the C_q's. Alternatively, each C_q consists of the residuals from rows of the effect matrix with identical entries for the adapted model. Thus we see that a replicated design of a fraction of the original design is formed. The residuals from the adapted model fall into m mutually exclusive sets $C_q, q = 1, \ldots, m$, each with n/m residuals such that the residuals are correlated within set and uncorrelated between sets. For each q and each j such that $\hat{\beta}_j$ is in the fitted model, either $C_q \subset P_j$ or $C_q \subset M_j$. Define s_q^2 as $\sum_{e_i \in C_q} e_i^2/(n/m-1)$—that is, the sample variance of the residuals. If the sample variance of each set has d df, then we have the following general formulas:

$$F_j^{\text{ML}} = \left(\prod_{q:C_q \subset P_j} s_q^2 \prod_{q:C_q \subset M_j} s_q^{-2} \right)^{2/m} \quad (8)$$

and

$$E\left(F_j^{\text{ML}}\right) = \left(\Gamma[d/2+2/m]\Gamma[d/2-2/m]\Gamma^{-2}[d/2]\right)^{m/2}. \quad (9)$$

Here, F^{ML} may be viewed as the $2/m$th power of the product of $m/2$ iid $F_{d,d}$ random variables based on the same assumptions used in the interaction triple case. Its distribution can be simulated in a straightforward manner or can be approximated by an $F_{c,c}$ distribution with

$$c = \frac{2\{\Gamma[d/2+2/m]\Gamma[d/2-2/m]\}^{m/2}}{\{\Gamma[d/2+2/m]\Gamma[d/2-2/m]\}^{m/2} - \Gamma^m[d/2]}. \quad (10)$$

Simulations have shown that the approximation works quite well. The possible combinations of m and d for unreplicated designs of $n = 16$ runs are $(m, d) = (4, 3), (8, 1)$ and for $n = 32$, $(m, d) = (4, 7), (8, 3), (16, 1)$. (Note: Whenever $m = 2$, $F^{\text{ML}} = D^{\text{BH}} \sim F_{d,d}$ because there is only a single dispersion effect being tested.) The approximation seems least accurate for $d = 1$, specifically $(m, d) = (8, 1)$, a case we discuss in the following example and in the simulations of Section 3.

Example 2. Consider data originally analyzed by Anderson and McLean (1974). The experiment was a 2_{V}^{5-1} design to study the impact of five factors on an index of "goodness" of asphalt concrete. The response is shown in the y_2 column of Table 1. Anderson and McLean used this example to illustrate the analysis of a 1/2 fraction and did not intend to discuss dispersion effects. They correctly indicated that the main effects and two-factor interactions consume all 15 df leaving no error term. They also stated that a previous estimate of 200 for the error mean square was available. Using this value and performing analysis of variance, the F tests are based on 1 and ∞ df. If $\alpha = .05$ is used, four effects are found active: *AD, AE, BD*, and *DE*. Column 2 of Table 3 shows the D^{BH} statistics and associated p values using this four-location-effect model as a base. We see that *AB* and *E* have mildly significant dispersion effects with p values of .0567 and .0424, respectively.

Note that the four location effects form an interaction triple (*AD, AE, DE*) and another term, *BD*. If we add the interactions between *BD* and the interaction triple terms, we get the model

$$Y_i = \beta_0 + x_{iC}\beta_C + x_{iAB}\beta_{AB} + x_{iAD}\beta_{AD} + x_{iBD}\beta_{BD} + x_{iAE}\beta_{AE} + x_{iBE}\beta_{BE} + x_{iDE}\beta_{DE} + \epsilon_i. \quad (11)$$

As we see in this example, the adapted location model may not be a hierarchical model. The main effects A, B, D, and E do not appear in the model, yet interactions involving these factors do appear. One must note, however, that this is not the fitted location model. It is an adapted model used solely for the purpose of dispersion-effect testing.

The residuals are correlated in pairs resulting in the sets $C_1 = \{e_1, e_{12}\}$, $C_2 = \{e_2, e_{11}\}$, $C_3 = \{e_3, e_{10}\}$, $C_4 = \{e_4, e_9\}$, $C_5 = \{e_5, e_{16}\}$, $C_6 = \{e_6, e_{15}\}$, $C_7 = \{e_7, e_{14}\}$, and $C_8 = \{e_8, e_{13}\}$, so each s_q^2 has $d = 1$ df. We may test the columns

Table 3. Dispersion-Effect Statistics and p Values for Example 2

Effect	D^{BH} *(p value)*	F^{ML}	*p value*	$F_{c,c}$ *approx. p value*
A	0.14 (0.141)			
B	1.16 (0.908)			
C	1.22 (0.876)	0.58	.708	.682
D	1.83 (0.631)			
AB	0.11 (0.057)	0.12	.159	.134
AC	0.47 (0.552)			
AD	3.01 (0.251)	5.56	.259	.223
BC	0.94 (0.963)			
BD	0.36 (0.350)	0.48	.622	.588
CD	0.24 (0.275)			
DE	1.20 (0.848)	2.61	.522	.483
CE	0.31 (0.359)			
BE	2.89 (0.329)	9.59	.144	.120
AE	0.87 (0.879)	1.11	.944	.937
E	17.37 (0.042)			

associated with each of the location terms for dispersion by calculating F_j^{ML}. Here we are performing seven tests using the test statistic of the form in (8) with $d = 1$ and $m = 8$. The degrees of freedom for the approximate F test are $c = 8/3$ from (10). The last three columns of Table 3 show the F^{ML} values for these columns as well p values.

F^{ML} can only be calculated for the $(n-2)/2 = 7$ columns associated with the terms in the location model given in (11). Due to the required form of location model, this is the maximum number of dispersion effects that can be tested using this approach for an unreplicated 16-run experiment. Note that D_{AB}^{BH} is mildly significant, whereas F_{AB}^{ML} is not significant. Moreover, F_E^{ML} cannot be calculated to compare with the mildly significant D_E^{BH}. We further discuss this example in Section 4 after studying the two tests' behaviors through simulations in the next section.

3. COMPARISON BY SIMULATION

To compare F^{ML} and D^{BH}, we use the common family of $n = 2^{k-p} = 16$ designs. In each situation studied, 10,000 sets of 16 responses were randomly generated based on a normality assumption. Assuming that all active location effects are included in the model, we can assume without loss of generality that all $\beta_j = 0$. To fit a model of the same form as (11), any seven location effects that meet the criterion of each pair's interaction being another term in the model may be used. We fit the model

$$Y_i = \beta_0 + x_{iA}\beta_A + x_{iB}\beta_B + x_{iC}\beta_C + x_{iAB}\beta_{AB} + x_{iAC}\beta_{AC} + x_{iBC}\beta_{BC} + x_{iABC}\beta_{ABC} + \epsilon_i, \quad (12)$$

which results in $(m, d) = (8, 1)$. [Note that for this study, the column labels (*A, B, C, AB, AC, BC, ABC*) are irrelevant other than for defining the relationships among columns.] Here, $D^{\mathrm{BH}} \sim F_{4,4}$. F^{ML} was compared to both a simulated reference distribution based on 200,000 simulations and an $F_{8/3,8/3}$. All tests were performed with a nominal significance level of $\alpha = .05$. Section 3.1 studies the performance of both tests when there are no unidentified location effects but perhaps multiple dispersion effects. Section 3.2 studies the impact of unidentified location effects on both tests.

3.1 Multiple Dispersion-Effect Simulations

Table 4 shows the results of the simulations with the possibility of multiple dispersion effects, assuming that all location effects are identified. In this table, $\overline{D}^{\mathrm{BH}}$ and $\overline{F}^{\mathrm{ML}}$ refer to the arithmetic means of the generated statistics. Because the p values are based on 10,000 simulations, the standard error of reported p values is $\approx .0022$ for p value $= .05$. The same standard errors and definitions apply to Table 5, Section 3.2.

No Dispersion Effects. For the first comparison, 10,000 sets of 16 standard normal variates are generated and stored in a $(16 \times 10{,}000)$ matrix. This is the null case with no dispersion effects. The first block of rows of Table 4 shows that both D^{BH} and F^{ML} using the simulated distribution behave as expected and yield roughly the specified significance level of .05 for all effects. The $\overline{D}^{\mathrm{BH}}$ and $\overline{F}^{\mathrm{ML}}$ values are close to their expected values of 2 and 4, respectively. The $F_{c,c}$ approximation yields an inflated significance level of about .06. So, although Table 4 shows the $F_{c,c}$ significance levels, we will use the simulated reference distribution for comparison in the rest of the table.

One Dispersion Effect. Initially, a dispersion effect was created by multiplying all observations from the preceding matrix by 5 in the rows where $A = +1$ resulting in $\Delta_A = 25$. The results are shown in the second block of Table 4. In this case, $\overline{D}_A^{\mathrm{BH}} = 51.500 = (25)(2.060)$ and $\overline{F}_A^{\mathrm{ML}} = (25)(4.377)$, or Δ_A times their values under the null case as they should be. D^{BH} has greater power than F^{ML} for detecting this effect. However, the $\overline{D}^{\mathrm{BH}}$ statistics for the null effects have risen compared to the null case discussed previously leading to inflated significance levels of about .13 or .14. This is because D^{BH} does not have an F distribution for these columns because both the numerator and the denominator are sums of χ^2 random variables that are not identically distributed. So a single dispersion effect increases the probability of falsely detecting another dispersion effect when using D^{BH}. Thus, the added power for the active effect comes at the expense of the significance level of the others. On the other hand, the null F^{ML} statistics are identical to the preceding null case, indicating that F^{ML} is totally insensitive to another dispersion effect.

Next, the original observations (null case) were adapted to have a single dispersion effect of $\Delta_D = 25$. Note that β_D is not in Model (12). The third block of rows of Table 4 shows this has almost no impact on D^{BH} and absolutely no impact on F^{ML}. Thus we see that F^{ML} is not affected by a another single dispersion effect regardless of whether the location effect of that column is included in the model or not. D^{BH} is only practically affected when another dispersion effect exists in a column that has its location effect in the fitted model.

Two Dispersion Effects. In the next case, we create effects of $\Delta_A = 25$ and $\Delta_C = 9$. Again the $\overline{F}^{\mathrm{ML}}$ values are identical to those of the single effect of $\Delta_A = 25$ with the exception of $\overline{F}_C^{\mathrm{ML}}$, which is greater by a factor of 9 as it should be. However, the $\overline{D}^{\mathrm{BH}}$ values are inflated even more than before. So again, although D^{BH} has greater power than F^{ML} for the active effects, it cannot hold the specified significance level for the null effects. This is especially true for AC (significance level = .325), which is showing the induced (spurious) effect due to the interaction of A and C.

Next we created dispersion effects of $\Delta_D = 25$ and $\Delta_{BD} = 9$. The two dispersion columns, D and BD, are not in the location model, but their interaction column B is. As the fifth block of rows in Table 4 shows, all significance levels for D^{BH} are greater than .08, with D_B^{BH} having a very large significance level of about .326. On the other hand, the only F^{ML} statistic that is affected by the other two dispersion effects is B, with a significance level of about .125. Thus, F^{ML} is more robust to multiple dispersion effects than D^{BH} even when these effects occur in columns not included in the adapted location model.

Several Dispersion Effects. To study the impact several small dispersion effects have on the other dispersion effects, we created dispersion effects for every column that is not included in (12) using $\Delta = 1.5^2$. Additionally, the large dispersion effect of $\Delta_A = 5^2$ was created. The results are shown in the last block of Table 4. The D^{BH} significance levels are all

Table 4. Multiple-Dispersion-Effect Simulation Results (nominal significance level $\alpha = .05$)

Active effects	Effect	$\bar{D}^{BH}$	Sig. level (power)	$\bar{F}^{ML}$	Sig. level (power)	$F_{c,c}$ approx. sig. level (power)
None (Null)	AC	1.964	.051	3.693	.050	.062
	B	1.876	.045	3.833	.049	.058
	BC	1.971	.048	3.620	.046	.067
	C	1.942	.050	3.647	.049	.061
	ABC	1.941	.049	3.945	.050	.063
	AB	1.994	.051	3.685	.051	.063
	A	2.060	.050	4.377	.050	.063
$\Delta_A = 5^2$	AC	3.424	.139	3.693	.050	.062
	B	3.421	.136	3.833	.049	.058
	BC	3.233	.139	3.620	.046	.067
	C	3.359	.138	3.647	.049	.061
	ABC	3.542	.141	3.945	.050	.063
	AB	3.289	.138	3.685	.051	.063
	A	51.500	.818	109.428	.528	.573
$\Delta_D = 5^2$	AC	1.968	.051	3.693	.050	.062
	B	1.888	.046	3.833	.049	.058
	BC	1.985	.047	3.620	.046	.067
	C	1.953	.049	3.647	.049	.061
	ABC	1.956	.050	3.945	.050	.063
	AB	2.001	.051	3.685	.051	.063
	A	2.077	.052	4.377	.045	.063
$\Delta_A = 5^2$	AC	14.256	.365	3.693	.050	.062
$\Delta_C = 3^2$	B	4.808	.203	3.833	.049	.058
	BC	4.525	.197	3.620	.046	.067
	C	30.236	.483	32.829	.264	.300
	ABC	5.122	.200	3.945	.050	.063
	AB	4.645	.205	3.685	.051	.063
	A	70.295	.764	109.428	.528	.573
$\Delta_D = 5^2$	AC	2.388	.085	3.693	.050	.062
$\Delta_{BD} = 3^2$	B	11.269	.326	15.331	.125	.147
	BC	2.445	.087	3.620	.046	.067
	C	2.402	.090	3.647	.049	.061
	ABC	2.423	.086	3.945	.050	.063
	AB	2.509	.089	3.685	.051	.063
	A	2.521	.086	4.377	.050	.063
$\Delta_D = \Delta_{AD} = \Delta_{BD} =$	AC	21.067	.351	5.989	.058	.071
$\Delta_{CD} = \Delta_{ABD} = \Delta_{ACD} =$	B	19.226	.347	6.216	.056	.069
$\Delta_{BCD} = \Delta_{ABCD} = 1.5^2$	BC	19.100	.359	5.871	.054	.066
	C	20.311	.353	5.914	.055	.069
$\Delta_A = 5^2$	ABC	21.124	.356	6.398	.058	.069
	AB	19.900	.351	5.975	.058	.071
	A	180.342	.935	177.463	.660	.701

tremendously high ($\approx .35$) for the null effects while the F^{ML} significance levels are only slightly inflated ($\approx .057$).

3.2 Unidentified Location Effects

Finally, the impact of unidentified location effects was studied through simulations. The general impact of unidentified location effects on dispersion-effect testing was studied by Pan (1999) and McGrath and Lin (2001). In this section, we study the specific impact on F^{ML} and D^{BH}. For this study, again 10,000 experiments with 16 runs were simulated using a standard normal distribution for the responses. Residuals from the fitted model (12) were used for all cases. The results are shown in Table 5 with the first block of rows showing the results for a null model. (This is the same case studied in the first block of rows of Table 4.)

One Unidentified Location Effect. In the second block of rows in this table, an unidentified location effect $\beta_D = 1$ was added. Comparing the results from the first two blocks of rows in Table 5, one notices that all significance levels have decreased for both tests, as expected, since this unidentified effect inflates the sample variances in both the numerator and the denominator of both tests. Although firm conclusions cannot be drawn from this small study, it does appear that the significance level of F^{ML} is affected less than that of D^{BH}.

Two Unidentified Location Effects. The third block of rows considers two unidentified location effects, $\beta_D = \beta_{BD} = 1$, whose interaction is in column B. McGrath and Lin (2001) showed that two unidentified location effects create a difference in the expected value of the sample variance of residuals at the high and low levels of their interaction column. Our simulations verify this result: Both tests show that the

Table 5. Unidentified-Location-Effect Simulation Results (nominal significance level $\alpha = .05$)

Active effects	Effect	$\overline{D}^{BH}$	Sig. level (power)	$\overline{F}^{ML}$	Sig. level (power)	$F_{c,c}$ approx. sig. level (power)
None (Null)	AC	1.964	.051	3.693	.050	.062
	B	1.876	.045	3.833	.049	.058
	BC	1.971	.048	3.620	.046	.067
	C	1.942	.050	3.647	.049	.061
	ABC	1.941	.049	3.945	.050	.063
	AB	1.994	.051	3.685	.051	.063
	A	2.060	.050	4.377	.050	.063
$\beta_D = 1$	AC	1.906	.043	3.505	.045	.057
	B	1.888	.038	4.044	.046	.059
	BC	1.884	.044	3.687	.045	.056
	C	2.011	.043	3.945	.047	.058
	ABC	1.905	.041	4.198	.045	.058
	AB	1.858	.040	3.470	.046	.056
	A	1.811	.042	3.797	.046	.059
$\beta_D = 1$	AC	1.756	.034	3.543	.042	.053
$\beta_{BD} = 1$	B	6.159	.141	16.650	.125	.149
	BC	1.758	.033	3.776	.037	.048
	C	1.758	.032	3.304	.042	.052
	ABC	1.689	.033	3.317	.038	.048
	AB	1.683	.030	3.518	.041	.051
	A	1.709	.033	3.341	.043	.052
$\Delta_A = 5^2$	AC	3.424	.139	3.693	.050	.062
	B	3.421	.136	3.833	.049	.058
	BC	3.233	.139	3.620	.046	.067
	C	3.359	.138	3.647	.049	.061
	ABC	3.542	.141	3.945	.050	.063
	AB	3.289	.138	3.685	.051	.063
	A	51.500	.818	109.428	.528	.573
$\Delta_A = 5^2$	AC	3.052	.125	3.864	.051	.061
$\beta_B = 1$	B	3.047	.128	4.340	.052	.060
	BC	3.198	.123	3.913	.051	.062
	C	3.035	.129	3.996	.047	.059
	ABC	2.972	.122	4.449	.049	.058
	AB	3.270	.127	4.069	.049	.064
	A	30.921	.688	63.945	.400	.443
$\Delta_A = 5^2$	AC	2.914	.118	3.746	.046	.057
$\beta_D = 1$	B	4.155	.122	8.761	.069	.082
$\beta_{BD} = 1$	BC	2.836	.115	4.132	.044	.053
	C	2.953	.116	3.563	.048	.059
	ABC	2.747	.113	3.573	.044	.054
	AB	2.488	.116	1.821	.057	.071
	A	22.435	.590	43.713	.333	.375

significance level is inflated for Δ_B, although somewhat less so for the F_B^{ML} test. For the other null dispersion effects, the significance level is dampened even more than with a single unidentified location effect. Again, it appears that the F^{ML} significance level is affected less by these unidentified effects.

One Dispersion Effect With Zero, One, or Two Unidentified Location Effects. The fourth block of rows shows results when only a single dispersion effect of $\Delta_A = 25$ exists with no unidentified location effects. (This is the same case studied in the second block of rows in Table 4.) This block may be compared to blocks 5 and 6, which include this dispersion effect plus the unidentified location effects of $\beta_D = 1$ and $\beta_D = \beta_{BD} = 1$, respectively. Again, a single unidentified location effect (block 5) dampens the significance level for all effects (including the active dispersion effect) for both tests. However, F^{ML} roughly holds the nominal value of .05 for the null effects, whereas the significance level for $D^{\text{BH}} \approx .12$. With two unidentified location effects (block 6), the same pattern holds with $\overline{D}_B^{\text{BH}}$ and $\overline{F}_B^{\text{ML}}$ being larger than the other null effects as expected. These simulations show that both D^{BH} and F^{ML} are sensitive to unidentified location effects. However, F^{ML} appears to be less sensitive than D^{BH}.

All of the simulations discussed in Sections 3.1 and 3.2 were based on the multiplicative dispersion model given in (1). If an additive dispersion model were assumed—that is, $\sigma_i^2 = \sigma^2 + \sum_{j=1}^{n-1} x_{ij}\gamma_j$—then the conclusions might differ from those reported here. Of course, other dispersion models could also be entertained. We chose to study the multiplicative model because it appears to be the most common dispersion model, being used by Cook and Weisberg (1983), Nair and Pregibon (1988), Wang (1989), Ferrer and Romero (1993a,b), and Wolfinger and Tobias (1998) among others.

4. PRACTICAL CONSIDERATIONS AND RECOMMENDATIONS

If factors are included in an experiment, the experimenter must suspect that they may have an impact on the response in some manner. In an unreplicated 2^{k-p} design, the goal may not be to identify and estimate all active effects (indeed all may be active) but to find the largest (i.e., most important) effects. When testing for location effects in these designs, the location effect estimates are the same whether active effects are mistakenly assumed null or not. Ignoring active effects just reduces the power of location-effect tests.

However, when testing for dispersion effects, the presence of one or more dispersion effects may affect the *estimation* of other dispersion effects, not just the detection power. As shown in Section 3.1, D^{BH} is biased whenever there are multiple dispersion effects. If two active dispersion effects occur in columns that are not in the adapted location model, F^{ML} is biased for their interaction column but no others. Section 3 shows that unidentified location effects create bias for both tests, although F^{ML} seems less susceptible. Thus, when multiple dispersion effects exist, truly clean dispersion-effect estimates seem difficult to attain in unreplicated 2^{k-p} designs.

We may draw some conclusions about the two examples studied previously. In Example 1, D^{BH}_D and D^{BH}_{DE} had p values $\approx .06$ and may have been considered significant. However, F^{ML}_D and F^{ML}_{DE} had much higher p values, indicating little, if any, evidence of significance. As shown by the simulations, small dispersion effects may increase the other dispersion-effect estimates, thus increasing the probability of falsely detecting null effects when using D^{BH}. Therefore, we conclude that D and DE do not have significant dispersion effects or have effects of relatively small magnitude.

In Example 2, we fit several different models and calculated D^{BH} for each column. Based on these statistics, AB and E were mildly significant with p values $\approx .05$. Only seven columns can be tested for dispersion using F^{ML} because of the specific form of the location model here. F^{ML}_{AB} does not indicate a significant dispersion effect and, unfortunately, E cannot be tested. So we conclude that AB does not have a significant dispersion effect. Based on the F^{ML} results, none of the seven columns tested have significant dispersion effects. Of the remaining eight columns, only E seems to be significant using D^{BH}. So here it appears that E has a mildly significant effect and it seems reasonable to trust the D^{BH} results for E.

This last example shows that it is wise to use D^{BH} in conjunction with F^{ML}. The D^{BH} statistics can be used to tentatively identify dispersion effects, and F^{ML} can be used to study them simultaneously. Note, however, that it does not seem possible to independently test all columns for dispersion when residuals are used. (In fact, it is possible, with a large number of location effects that are not interactions of each other, that F^{ML} cannot be calculated.) So we see that there are choices in how to perform dispersion-effect testing:

1. If we are confident that at most one dispersion effect exists, then the D^{BH} test of Bergman and Hynén (1997) may be used. If the normality assumption is not plausible, the nonparametric SSDR test of McGrath and Lin (1999) may be used. However, it must be noted that the null effects have inflated significance levels and therefore have an increased probability of false detection.

2. If we suspect that there may be more than one dispersion effect, then we can use F^{ML} and test several dispersion columns independently, assuming that the untested columns do not produce dispersion effects. The columns that can be tested are determined, to some extent, by the location model.

Other tests seem to work only when there is a single dispersion effect present. Even then, the significance level of null effects is inflated by this lone effect. F^{ML}, as described in this article, allows independent testing of multiple dispersion effects (under normality) as long as all columns having active dispersion effects have their location effects included in the fitted model. Although this article has studied only unreplicated $n = 2^{k-p} = 16$ run designs, the extension to 2^{k-p} designs with $n > 16$ is immediate.

ACKNOWLEDGMENTS

We thank the editor (Karen Kafadar), an associate editor, and two referees for their insightful comments and suggestions that resulted in a much improved article. Some of this work was completed while Richard McGrath was a doctoral candidate in the Department of Statistics at The Pennsylvania State University. Dennis Lin was partially supported by the National Science Foundation via grant DMS-9704711 and National Science Council of ROC via contract NSC 87-2119-M-001-007.

APPENDIX: PROOF OF EQUATION (6)

Assume, without loss of generality, that $C_1 \subset P_j$, $C_2 \subset P_j$, $C_3 \subset M_j$, and $C_4 \subset M_j$. Now $W = ds_1^2/\sigma^2 \sim \chi^2_d$, $X = ds_2^2/\sigma^2 \sim \chi^2_d$, $Y = \sigma^2/ds_3^2 \sim \chi_d^{-2}$, and $Z = \sigma^2/ds_4^2 \sim \chi_d^{-2}$, independently, where χ_d^{-2} is the inverse χ^2_d distribution. Then F^{ML}_j can be written as

$$F^{ML}_j = \left((s_1^2 s_2^2)/(s_3^2 s_4^2)\right)^{1/2} = W^{1/2}Y^{1/2}X^{1/2}Z^{1/2}$$

and

$$\begin{aligned}
&E(W^{1/2}Y^{1/2})\\
&\quad = \int_0^\infty \int_0^\infty \{\Gamma[d/2]2^{(d/2)}\}^{-1} w^{1/2} w^{(d/2)-1}\exp(-w/2)\\
&\qquad \times \{\Gamma[d/2]2^{(d/2)}\}^{-1} y^{1/2} y^{-(d/2)-1}\exp(-1/2y)\,dw\,dy\\
&\quad = \Gamma[(d+1)/2]2^{(d+1)/2}\{\Gamma[d/2]2^{d/2}\}^{-1}\\
&\qquad \times \int_0^\infty \{\Gamma[(d+1)/2]2^{(d+1)/2}\}^{-1} w^{(d+1)/2-1}\exp(-w/2)\,dw\\
&\qquad \times \Gamma[(d-1)/2]2^{(d-1)/2}\{\Gamma[d/2]2^{d/2}\}^{-1}\\
&\qquad \times \int_0^\infty \{\Gamma[(d-1)/2]2^{(d-1)/2}\}^{-1} y^{-(d-1)/2-1}\exp(-1/2y)\,dy\\
&\quad = \Gamma[(d+1)/2]\Gamma[(d-1)/2]\Gamma^{-2}[d/2].
\end{aligned}$$

Similarly $E(X^{1/2}Z^{1/2}) = \Gamma[(d+1)/2]\Gamma[(d-1)/2]\Gamma^{-2}[d/2]$. Then, by the mutual independence of W, X, Y, Z, we have

$$\begin{aligned}
E(F^{ML}_j) &= E(W^{1/2}Y^{1/2}X^{1/2}Z^{1/2})\\
&= E(W^{1/2}Y^{1/2})E(X^{1/2}Z^{1/2})\\
&= (\Gamma[(d+1)/2]\Gamma[(d-1)/2]\Gamma^{-2}[d/2])^2.
\end{aligned}$$

The same procedure may be used to prove the general case given in (9).

[Received June 1999. Revised November 2000.]

REFERENCES

Anderson, V. L., and McLean, R. A. (1974), *Design of Experiments*, New York: Marcel Dekker, .

Bergman, B., and Hynén, A. (1997), "Dispersion Effects From Unreplicated Designs in the 2^{k-p} Series," *Technometrics*, 39, 191–198.

Box, G. E. P., and Meyer, R. D. (1986a), "An Analysis for Unreplicated Fractional Factorials," *Technometrics*, 28, 11–18.

——— (1986b), "Dispersion Effects From Fractional Designs," *Technometrics*, 28, 19–27.

Cook, R. D., and Weisberg, S. (1983), "Diagnostics for Heteroscedasticity in Regression," *Biometrika*, 70, 1–10.

Daniel, C. (1959), "Use of Half-Normal Plots in Interpreting Factorial Two-Level Experiments," *Technometrics*, 1, 311–341.

——— (1976), *Applications of Statistics to Industrial Experimentation*, New York: Wiley.

Davidian, M., and Carroll, R. J. (1987), "Variance Function Estimation," *Journal of the American Statistical Association*, 82, 1079–1091.

Davies, O. L. (ed.) (1956), *Design and Analysis of Industrial Experiments* (2nd ed.), London: Oliver and Boyd.

Ferrer, A. J., and Romero, R. (1993a), "Small Samples Estimation of Dispersion Effects From Unreplicated Data," *Communications in Statistics—Simulations*, 22, 975–995.

——— (1993b), "A Simple Method to Study Dispersion Effects From Non-necessarily Replicated Data in Industrial Contexts," *Quality Engineering*, 7, 747–755.

Hamada, M., and Balakrishnan, N. (1998), "Analyzing Unreplicated Factorial Experiments: A Review With Some New Proposals" (with discussion), *Statistica Sinica*, 8, 1–38.

Juan, J., and Peña, D. (1992), "A Simple Method to Identify Significant Effects in Unreplicated Two-Level Factorial Designs," *Communications in Statistics—Theory and Methods*, 21, 1383–1403.

Lenth, R. (1989), "Quick and Easy Analysis of Unreplicated Factorials," *Technometrics*, 31, 469–473.

Loughin, T. M., and Noble, W. (1997), "A Permutation Test for Effects in an Unreplicated Factorial Design," *Technometrics*, 39, 180–190.

McGrath, R. N., and Lin, D. K. J. (1999), "A Nonparametric Dispersion Test for Unreplicated Two-Level Fractional Factorial Designs," Technical Report 99–04, Pennsylvania State University, Dept. of Statistics.

——— (2001), "The Confounding Relationship of Location and Dispersion Effects in Unreplicated Fractional Factorials," *Journal of Quality Technology*, 33, 129–139.

Montgomery, D. C. (1990), "Using Fractional Factorial Designs for Robust Process Development," *Quality Engineering*, 3, 193–205.

Nair, V. N., and Pregibon, D. (1988), "Analyzing Dispersion Effects From Replicated Factorial Experiments," *Technometrics*, 30, 247–257.

Pan, G. (1999), "The Impact of Unidentified Location Effects on Dispersion-Effects Identification From Unreplicated Factorial Designs," *Technometrics*, 41, 313–326.

Taguchi, G., and Wu, Y. (1980), *An Introduction to Off-Line Quality Control*, Nagoya, Japan: Central Japan Quality Control Association.

Wang, P. C. (1989), "Tests for Dispersion Effects from Orthogonal Arrays," *Computational Statistics & Data Analysis*, 8, 109–117.

Wolfinger, R. D., and Tobias, R. D. (1998), "Joint Estimation of Location, Dispersion, and Random Effects in Robust Design," *Technometrics*, 40, 62–71.