

# Confounding of Location and Dispersion Effects in Unreplicated Fractional Factorials

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When studying both location and dispersion effects in unreplicated fractional factorial designs, a "standard" procedure is to identify location effects using ordinary least squares analysis, fit a model, and then identify dispersion effects by analyzing the residuals. In this paper, we show that if the model in the above procedure does not include all active location effects, then null dispersion effects may be mistakenly identified as active. We derive an exact relationship between location and dispersion effects, and we show that without information in addition to the unreplicated fractional factorial (such as replication) we can not determine whether a dispersion effect or two location effects are active.

## Introduction

SCREENING experiments are often used in industry to identify factors having an important impact on a response or responses. The intent is to reduce a relatively large list of potential factors to a manageable few. The  $2^{k-p}$  fractional factorials are screening designs where each factor is studied at 2 levels, but only a fraction of all factor-level combinations are run. Traditionally, the primary use of these designs has been in detecting and modeling location effects (changes in the mean response). An assumption of constant variance is usually made. Many techniques have been designed to address this problem. Examples of various approaches are presented by Daniel (1959, 1978), Box and Meyer (1986a), Juan and Peña (1992), Lenth (1989), and Ye, Hamada, and Wu (2001). See Hamada and Balakrishnan (1998) for an overview and comparison of different methods.

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Several papers have described techniques for studying dispersion effects in replicated experiments where  $r \geq 2$  observations are obtained at each design setting. See Davidian and Carroll (1987) and Nair and Pregibon (1988) for examples. For unreplicated fractional factorials however, there is no estimate of variation available at each design setting, making the study of dispersion effects more difficult. In their pioneering work, Box and Meyer (1986b) developed an informal method for identifying dispersion effects in unreplicated experiments by studying the logarithm of the ratio of residual variances. They noted the importance of first identifying location effects before studying dispersion effects. Montgomery (1990) extended this method by plotting these statistics on a normal probability plot in order to discriminate small dispersion effects from large effects. Wang (1989) developed a test statistic that has an approximate  $\chi^2$  distribution for a large sample size. Ferrer and Romero (1993a, b) used the residuals (or an appropriate transformation of the residuals) as a response to study dispersion. More recently, Bergman and Hynén (1997) developed an exact dispersion test using a statistic having an  $F$  distribution.

In this paper, we study the relationships derived in Box and Meyer (1986b) in more detail and derive

an *explicit* relationship between location and dispersion effects. This relationship explains why failure to remove location effects before studying dispersion can produce spurious dispersion effects. Furthermore, we show that without information in addition to the unreplicated fractional factorial we can not determine whether a dispersion effect or two location effects are active. Finally, suggestions about additional information that can help to resolve this confounding are made.

### An Example

Montgomery (1990) analyzed data from an injection molding experiment where the response to be optimized was shrinkage. The factors studied are mold temperature (A), screw speed (B), holding time (C), gate size (D), cycle time (E), moisture content (F), and holding pressure (G). The design is a  $2^{7-3}_{IV}$  fractional factorial, meaning it is a resolution IV,  $1/2^3$  fraction of a  $2^7$  design (See Box, Hunter, and

Hunter (1978)). The generators of this design are  $E=ABC$ ,  $F=BCD$ , and  $G=ACD$ . The data are shown in Table 1.

The least squares regression coefficients are obtained from fitting a saturated model. Figure 1 is a normal probability plot of the estimated regression coefficients ( $\hat{\beta}_j$ s). In  $2^{k-p}$  experiments, "effects" are calculated as the average difference in the response at the +1 and -1 levels of the column. Here,  $effect_j = 2\beta_j$ . Montgomery uses a normal probability plot of the estimated effects and determines that columns 1, 2, and 5 (A, B, and AB) produce active location effects. He fits this location model, which we denote M1.

$$(M1) \quad \hat{y} = 27.3125 + 6.9375A + 17.8125B + 5.9375AB$$

The estimated residuals under M1 are (-2.50, -0.50, -0.25, 2.00, -4.50, 4.50, -6.25, 2.00, -0.50, 1.50, 1.75, 2.00, 7.50, -5.50, 4.75, -6.00).

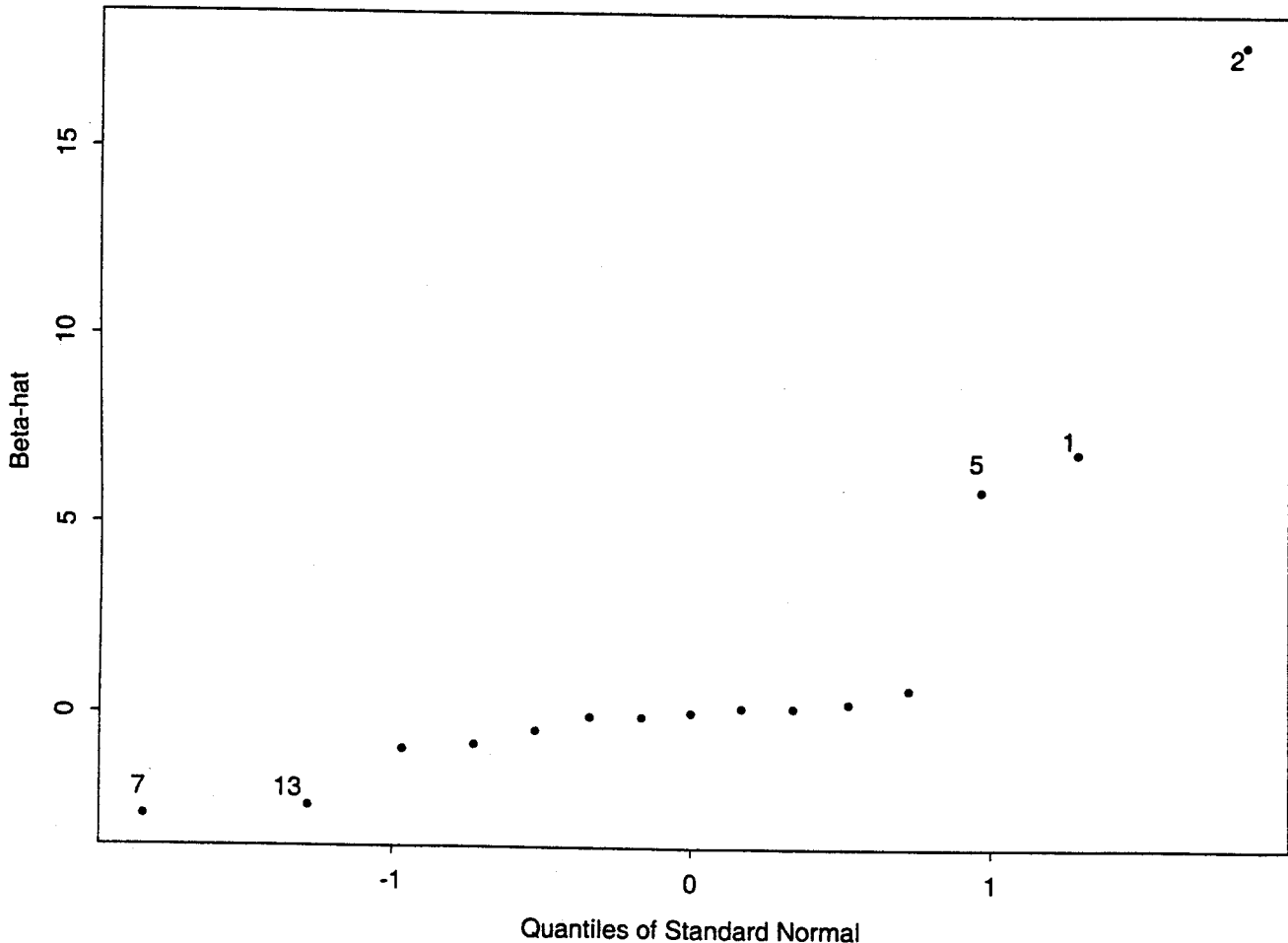


FIGURE 1. Injection Molding Experiment: Normal Plot of Location Coefficients

TABLE 1. Design Matrix and Response for Injection Molding Experiment

i \ j →	A B C D										E		G		F		y <sub>M</sub>
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	6
2	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	10
3	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	32
4	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	60
5	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	4
6	1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	15
7	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	26
8	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	60
9	1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	8
10	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	12
11	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	34
12	1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	60
13	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	16
14	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	5
15	1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	37
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	52

As a measure of the dispersion effect magnitude for column *j*, Montgomery calculates the statistic  $F_j^* = \ln(s_{j+}^2/s_{j-}^2)$ , which is the natural logarithm of the ratio of the sample variances of the residuals at the +1 and -1 levels of column *j*. Note that Box and Meyer (1986b) point out this statistic is *approximately* normally distributed with mean 0. Montgomery compares these statistics to an appropriate normal quantile to determine significance. He also uses a normal plot of these statistics. Using either the normal quantile or the probability plot, it is evident that column 3 (C) has a dispersion effect with

$$F_{3|M1}^* = \ln \frac{s_{3+|M1}^2}{s_{3-|M1}^2} = \ln \frac{32.44}{2.66} = 2.50.$$

Thus, Montgomery (1990) concludes that factors A (mold temperature) and B (screw speed) impact the mean shrinkage of the mold, and that factor C (holding time) impacts the variation in shrinkage. By studying the interaction between mold temperature and screw speed, it is apparent that the low screw speed is better for reducing mean shrinkage and that the setting of mold temperature is not crucial at this speed. To reduce the variation in shrinkage, holding time should be set at its low level.

This logical procedure has become a standard practice. However, the identification of dispersion effects is quite sensitive to the location model that is

fit. To illustrate, note that another reasonable interpretation of Figure 1 is that columns 7 and 13 have active location effects in addition to columns 1, 2, and 5. This is even more apparent if a half-normal plot is used as shown in Figure 2 (see Daniel, 1959).

We denote this model with five location effects (columns 1, 2, 5, 7, and 13) as M2.

$$(M2) \quad \hat{y} = 27.3125 + 6.9375A + 17.8125B + 5.9375AB - 2.6875CG - 2.4375G.$$

Due to the confounding associated with this design, column 13 represents the factor G main effect and interactions of three or more factors. The AD, CG, and EF interaction effects appear in column 7, and the interaction of columns 7 and 13 appears in column 3. So while the model may be represented many ways, we use the labels CG and G for columns 7 and 13, respectively, to emphasize that their interaction is in column 3, i.e. C.

The residuals from model M2 are (-2.250, -0.750, 0.000, 1.750, 0.625, -0.625, -1.125, -3.125, -0.750, 1.750, 1.500, 2.250, 2.375, -0.375, -0.375, -0.875). From this model we have the  $F_j^*$  statistic for column 3,

$$F_{3|M2}^* = \ln \frac{s_{3+|M2}^2}{s_{3-|M2}^2} = \ln \frac{2.42}{2.58} = -0.06.$$

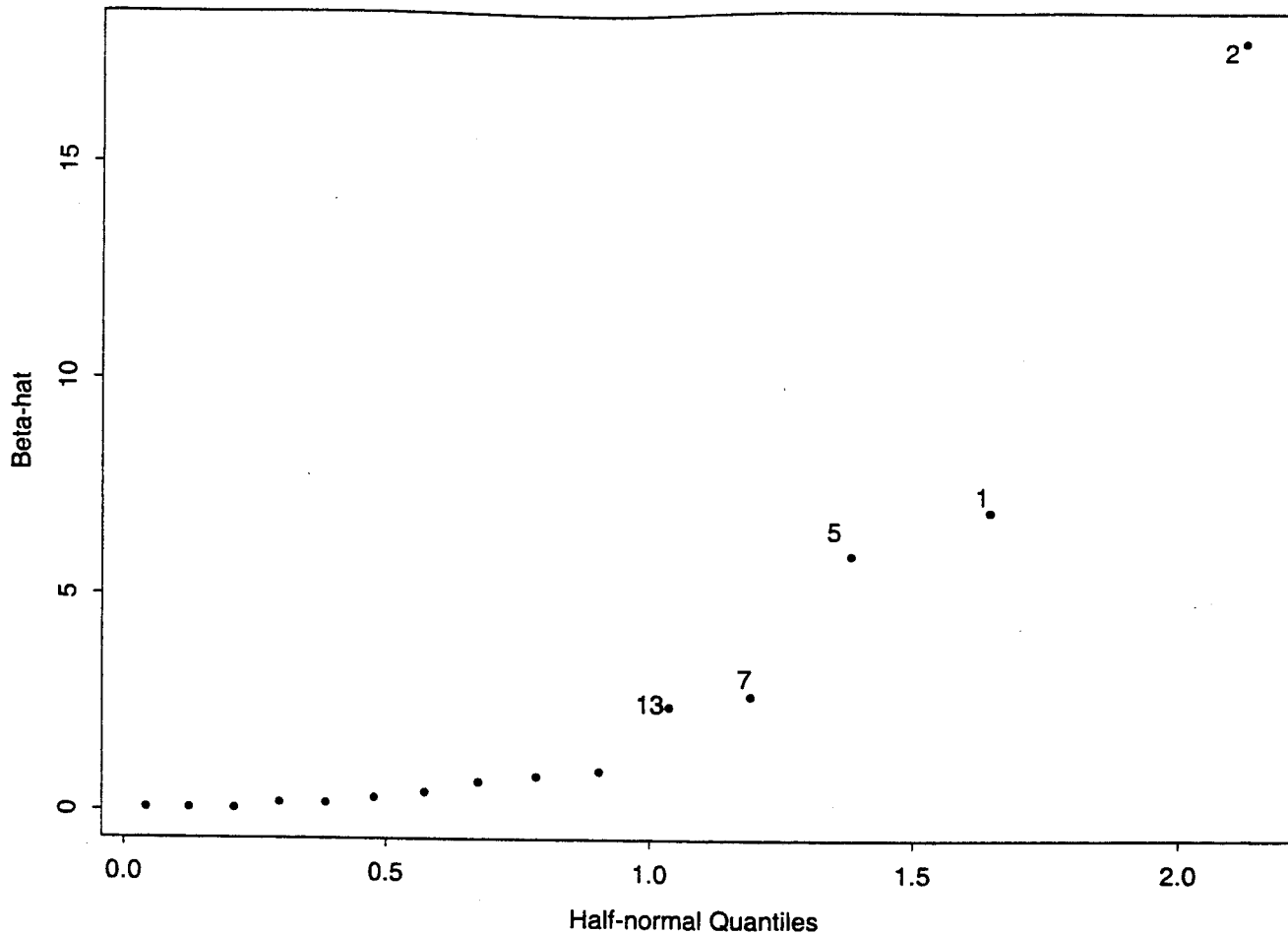


FIGURE 2. Injection Molding Experiment: Half-Normal Plot of Location Coefficients

Here, it is apparent that there is no dispersion effect associated with column 3 (factor C), since the sample variance of residuals is quite similar at the  $-1$  and  $+1$  levels of column 3.

So we have two feasible models for mold shrinkage, M1 and M2. M1 shows two factors important for determining the location (mean) of the response, and it also includes another factor that is important for controlling the variation in the response. M2 includes four factors that affect the mean response and no dispersion factors. Which model is more appropriate? Is one model better than the other? Some additional information may be helpful.

The experiment actually included four center runs (25, 29, 24, 27) in addition to the fractional factorial. From these center runs, we have an estimate of the variance of the response,  $\sigma^2$ , of  $\hat{\sigma}^2 = 4.92$ . M1 produces  $\hat{\sigma}_{M1}^2 = 20.73$  and M2 produces  $\hat{\sigma}_{M2}^2 = 3.81$ , so we see the M2 estimate is in much better agreement

with the center point estimate. As mentioned by one referee, if M1 is the "true" model, there is no common variance to be estimated so  $\hat{\sigma}_{M1}^2$  is meaningless. However, we would expect the variance of all residuals to equal the average of the variances at the  $+1$  and  $-1$  levels of a column if all location effects have been removed.

Therefore, a reasonable conclusion based on model M2 is that there are four important factors: mold temperature, screw speed, holding time and holding pressure (G). If this experiment is truly a screening experiment, then fitting M1 may have eliminated a potentially important factor, holding pressure.

So we have two distinctly different possibilities:

- Failing to include a pair of location effects created a spurious dispersion effect, or
- Failing to account for a dispersion effect created two location effects.

These spurious dispersion effects are not uncommon. We show that the exclusion of a pair of active location effects will create an apparent (spurious) dispersion effect in the interaction of these two columns. Box and Meyer (1986) and Bergman and Hynén (1997) both note a relationship between location and dispersion effects. We derive the exact relationship. In the next section, we provide a theoretical explanation showing that failure to include two location effects in a model before calculating residuals can produce a spurious dispersion effect.

### Spurious Dispersion Effects

Assume some method is used to identify  $m$  active location effects in an unreplicated fractional factorial design. A model is fit and residuals are estimated, but we assume that there are two active location effects that are excluded from this model. Let the excluded active location effects be in columns  $\mathbf{x}_j$  and  $\mathbf{x}_{j'}$ , and let  $\mathbf{x}_d$  be the column associated with the interaction of  $\mathbf{x}_j$  and  $\mathbf{x}_{j'}$ . Then  $x_{ij}x_{ij'} = x_{id}$ . Let  $\hat{\beta}_j$  and  $\hat{\beta}_{j'}$  be the usual least squares estimators of  $\beta_j$  and  $\beta_{j'}$ , the regression coefficients associated with  $\mathbf{x}_j$  and  $\mathbf{x}_{j'}$  respectively. We show that failure to include  $\beta_j$  and  $\beta_{j'}$  in the regression model will create a difference in the expected value of the sample variances at the +1 and -1 levels of  $\mathbf{x}_d$ .

Define the following sets of rows using the conventional  $P$  for 'plus' and  $M$  for 'minus':

$$M = \{i : x_{id} = -1\}, \quad P = \{i : x_{id} = +1\}.$$

A dispersion effect occurs when the variance of the response, independent of the location effects, or equivalently, the variance of the residuals from a known location model, is higher at one level of a column than the other. We can compare sample variances of the residuals at the plus and minus levels of a column to determine if it has a dispersion effect. Let

$$s_{d+}^2 = \frac{2}{n-2} \sum_{i \in P} (e_i - \bar{e}_p)^2$$

and

$$s_{d-}^2 = \frac{2}{n-2} \sum_{i \in M} (e_i - \bar{e}_m)^2,$$

where

$$\bar{e}_m = \frac{2}{n} \sum_{i \in M} e_i \quad \text{and} \quad \bar{e}_p = \frac{2}{n} \sum_{i \in P} e_i.$$

It is shown in the Appendix that, under the assumption that the errors are independently distributed such that  $E[\epsilon_i] = 0$  and  $\text{Var}[\epsilon_i] = \sigma^2$ , the expected sample variance of the residuals when  $x_{id} = -1$  ( $i \in M$ ) is

$$E[s_{d-}^2] = E \left[ \frac{2}{n-2} \sum_{i \in M} (e_i - \bar{e}_m)^2 \right] = \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} (\beta_j - \beta_{j'})^2. \quad (1)$$

When  $x_{id} = +1$  ( $i \in P$ ),

$$E[s_{d+}^2] = E \left[ \frac{2}{n-2} \sum_{i \in P} (e_i - \bar{e}_p)^2 \right] = \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} (\beta_j + \beta_{j'})^2. \quad (2)$$

From Equations (1) and (2) we have

$$E[s_{d+}^2] - E[s_{d-}^2] = \frac{4n}{n-2} \beta_j \beta_{j'}. \quad (3)$$

Consider the following three scenarios involving  $\beta_j$  and  $\beta_{j'}$ :

- If  $\beta_j = \beta_{j'} = 0$ , then these two location effects are not active and  $E[s_{d-}^2] = E[s_{d+}^2] = (n-1-m/n-2)\sigma^2$  and  $E[s_{d+}^2] - E[s_{d-}^2] = 0$ . Thus, any difference is just random error so there will be no spurious dispersion effect.
- If only one of the coefficients is nonzero, then Equation (3) is still zero as mentioned in Bergman and Hynén (1997), although both are biased upwards as estimates of  $\sigma^2$ .
- If  $\beta_j$  and  $\beta_{j'} \neq 0$ , the residuals will have different expected variance at the -1 and +1 levels of  $\mathbf{x}_d$ . Thus, excluding two location effects from a model and then studying residuals can create a spurious dispersion effect.

Returning to the injection molding example, if we assume columns 7 and 13 produce active location effects but were left out of the model, we then have

$$\begin{aligned} E[s_{3+|M1}^2] - E[s_{3-|M1}^2] &= \frac{4n}{n-2} \hat{\beta}_7 \hat{\beta}_{13} \\ &= \frac{(4)(16)}{14} (-2.688)(-2.438) \\ &= 29.95. \end{aligned}$$

Recalling that  $s_{3-|M1}^2 = 2.66$  and  $s_{3+|M1}^2 = 32.44$ , we have

$$s_{3+|M1}^2 - s_{3-|M1}^2 = 29.79.$$

So the observed difference in sample variances is almost the same as that caused by not including  $\beta_7$  and  $\beta_{13}$  in the model. This seems to indicate that the dispersion effect detected by fitting model M1 is spurious. Of course, as previously mentioned, we can not determine this with certainty. In the next section, we show an example of both an apparently spurious and an apparently real dispersion effect.

### A Second Example

In Box and Meyer (1986b), an unreplicated  $2_{III}^{9-5}$  fractional factorial experiment is analyzed. The data from this welding experiment performed by the National Railway of Japan are originally analyzed in Taguchi and Wu (1980) and are shown in Table 2. The response is observed tensile strength of a weld. We have labeled the columns in the same manner as Box and Meyer.

The normal probability plot of the 15 regression coefficients (excluding the overall mean) in Figure 3

clearly indicates active location effects in columns 14 and 15 (factors B and C) as found by Box and Meyer.

They pointed out that comparing the sample variances of the response at the high and low levels of each column falsely indicates the presence of a dispersion effect in column 1 (factor D). Note that column 1 is the interaction of the two active location effects (14 and 15). Denote the null model BM1 as

$$(BM1) \quad \hat{y} = \bar{y}.$$

Calculating the variance of the residuals from this model (or equivalently the variance of the observed responses) at each level of column 1 results in  $s_{1-|BM1}^2 = 0.5342$  and  $s_{1+|BM1}^2 = 8.1421$ . Therefore,

$$F_{1|BM1}^* = \ell_n \frac{s_{1+|BM1}^2}{s_{1-|BM1}^2} = \ell_n \frac{8.1421}{0.5342} = 2.72$$

and

$$s_{1+|BM1}^2 - s_{1-|BM1}^2 = 7.6079.$$

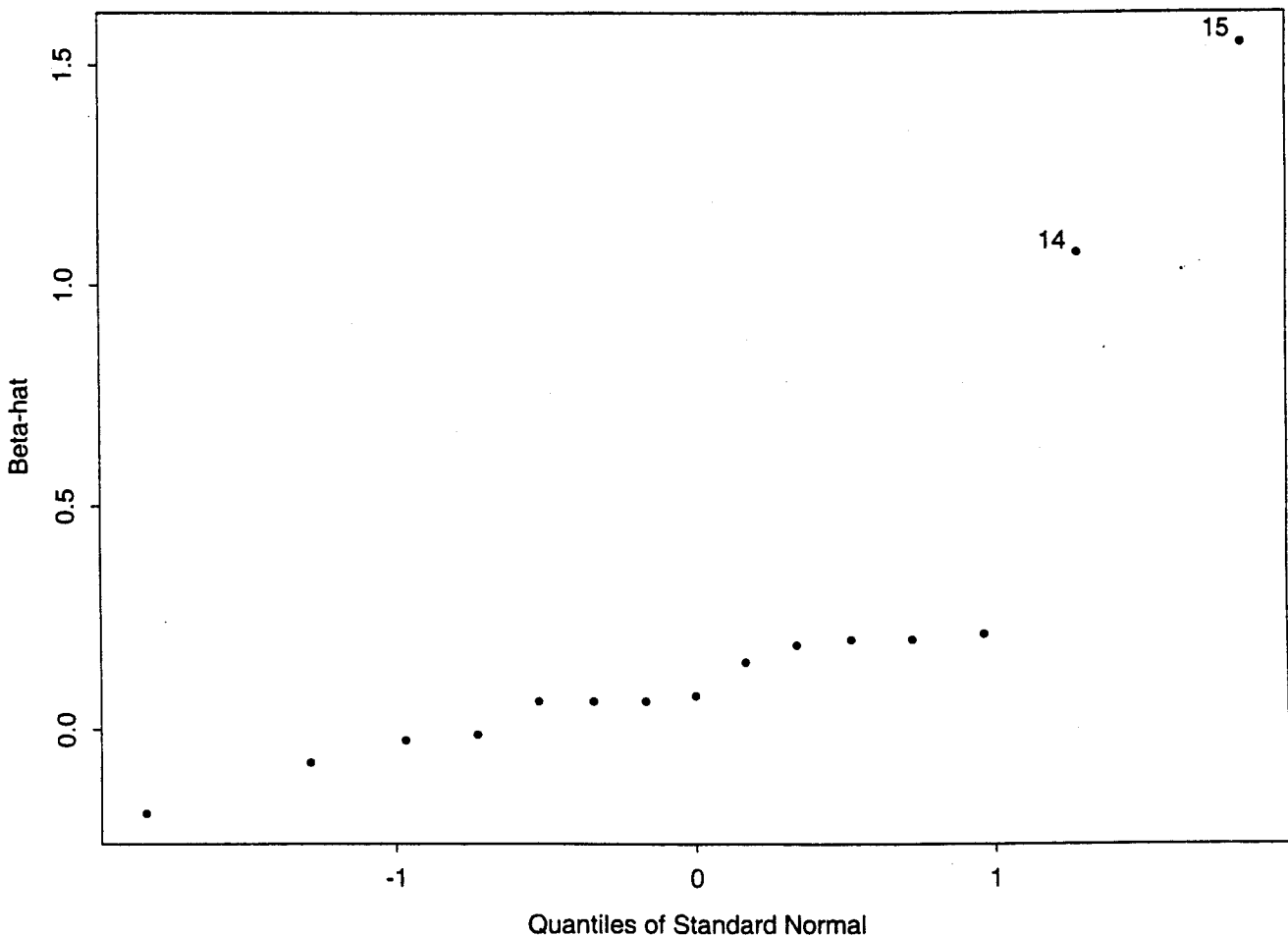


FIGURE 3. Welding Experiment: Normal Plot of Location Coefficients

TABLE 2. Design Matrix and Response for Welding Experiment

i \ j →	0	D	H	G	A	-F	-E							J	B	-C	y <sub>w</sub>
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15		
1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	43.7
2	1	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	40.2
3	1	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	42.4
4	1	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	44.7
5	1	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	42.4
6	1	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	45.9
7	1	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	42.2
8	1	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	40.6
9	1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	42.4
10	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	45.5
11	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	43.6
12	1	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	40.6
13	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	44.0
14	1	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	40.2
15	1	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	42.5
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	46.5

Using Equation (3) with the estimated coefficients we obtain

$$E [s_{1+|BM1}^2] - E [s_{1-|BM1}^2] = \frac{4n}{n-2} \hat{\beta}_{14} \hat{\beta}_{15} = \frac{(4)(16)}{14} (1.075)(1.55) = 7.6171.$$

Again, the difference between these two estimates of the difference between variances is quite small. This provides substantial evidence that there is not a dispersion effect in column 1. It seems this spurious dispersion effect is caused by not removing the location effects in columns 14 and 15.

Box and Meyer then fit a model with the active location effects in columns 14 and 15. We denote this model BM2 as

$$(BM2) \quad \hat{y} = 42.9625 + 1.075B - 1.550C.$$

In their analysis of residual variance, they find a dispersion effect in column 15, not in column 1. The sample variances of the residuals from model BM2 are  $s_{15-|BM2}^2 = 0.0284$  and  $s_{15+|BM2}^2 = 0.5241$ . Therefore,

$$F_{15|BM2}^* = \ell n \frac{s_{15+|BM2}^2}{s_{15-|BM2}^2} = \ell n \frac{0.5241}{0.0284} = 2.92$$

and

$$s_{15+|BM2}^2 - s_{15-|BM2}^2 = 0.4957.$$

There are seven pairs of columns that have their interaction appear in column 15. These pairs are 1:14, 2:13, 3:12, 4:11, 5:10, 6:9, and 7:8. Of these pairs, 6:9 have the largest coefficients. Assuming  $\beta_6$  and  $\beta_9 \neq 0$  and using Equation (6) on the residuals, we have

$$E [s_{15+|BM2}^2] - E [s_{15-|BM2}^2] = \frac{4n}{n-2} \hat{\beta}_6 \hat{\beta}_9 = \frac{(4)(16)}{14} (.2)(.2125) = 0.1943.$$

Of course this is not a formal test. Without a standard error for  $\hat{\beta}_6 \hat{\beta}_9$ , we can not determine statistical significance. However, the observed difference in sample variances of the residuals is much larger than can be attributed to excluded possibly-active location effects. This seems to imply that column 15 has an active dispersion effect. We thus conclude that the presence of this dispersion effect violates the homogeneity of variance assumption. This can cause some problems when studying location effects. Fortunately, the variance of each estimate is the same, because each coefficient is a linear combination (+ and -) of the same responses.

So we have seen that studying dispersion effects in the presence of location effects can be misleading. A pair of active location effects that are not included in the location model can cause a spurious dispersion effect in their interaction column. Techniques that model the variance of the response without first removing location effects are likely to identify factors as dispersion-active when they actually do not produce a dispersion effect. One last look at the injection molding experiment reinforces this concept.

Figure 4 is a normal probability plot of the  $F_j^*$  values calculated from the observations, i.e. without fitting any model first. Note that columns 1, 2, and 5 all appear to have dispersion effects. All three of these columns have highly significant location effects that have not been removed in this case. Also, each of these columns is the interaction of the other two. As shown earlier, these dispersion effects disappear when we study residuals from a model including the

active location effects. Thus, it is evident that location effects should be studied and removed from the data, at least in a preliminary manner, before studying dispersion effects.

### Discussion and Future Research

We have shown that there is a specific relationship between dispersion and location effects and that our suggested analysis may help avoid incorrectly identifying inactive dispersion effects as active. We recommend, as do Box and Meyer (1986), that location effects be identified and residuals from this location model be used to identify dispersion effects. If a dispersion effect is detected, then Equation (3) can be used with estimated location coefficients to estimate how much of the observed dispersion effect is due to the excluded location effects. As summarized below, this method can provide guidance, but it will not result in definitive conclusions.

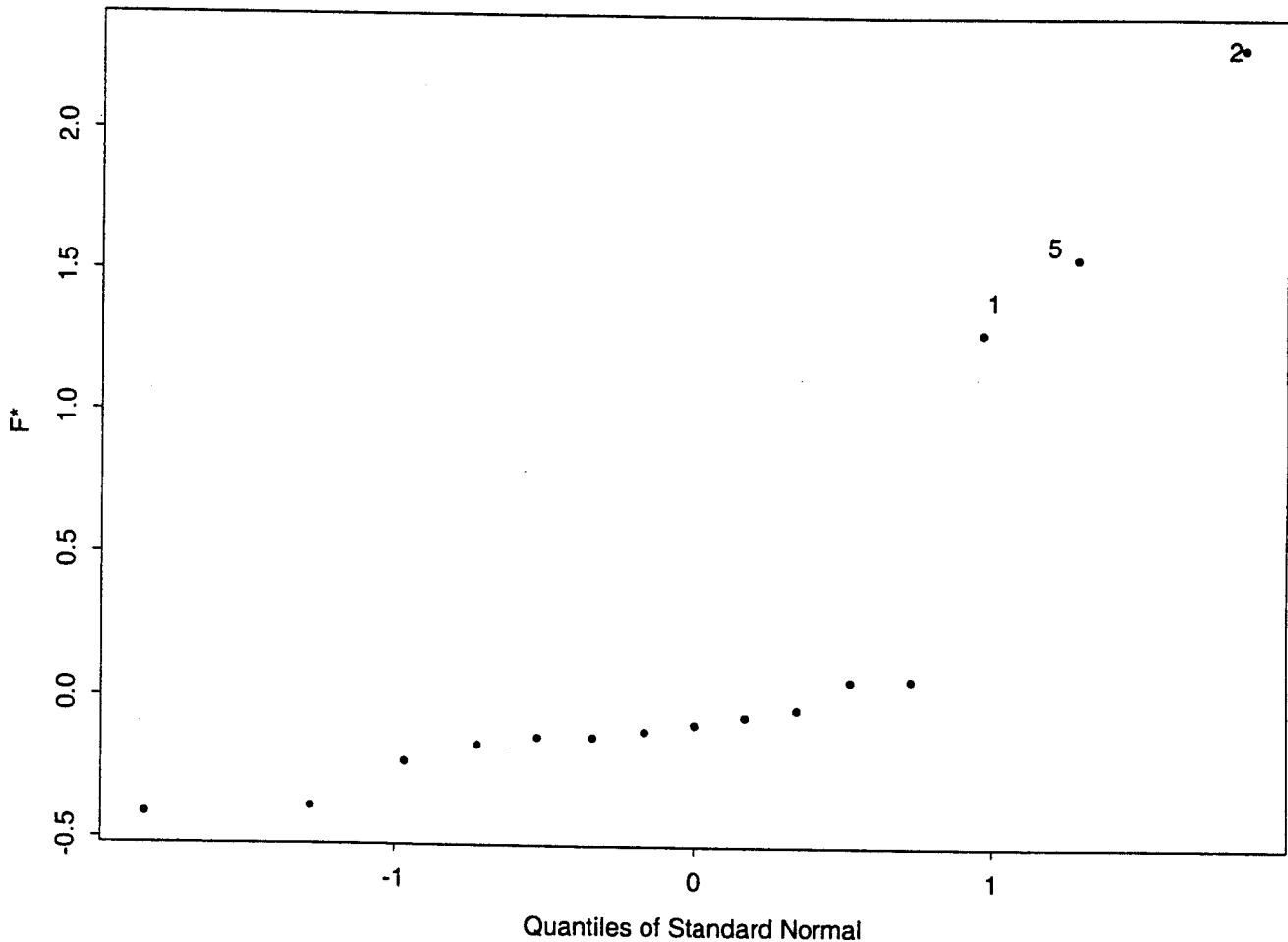


FIGURE 4. Injection Molding Experiment: Normal Plot of  $F^*$  Based on Observations



- In the injection molding example, there are two possibilities: 1) there are two location effects excluded from the model; or 2) there is one dispersion effect. We find that  $s_{3+|M2}^2 - s_{3-|M2}^2 = 29.79$  and  $(4n/n - 2)\widehat{\beta}_7\widehat{\beta}_{13} = 29.95$ . As these two quantities are almost equal, we conclude that we actually have two location effects, not a dispersion effect. Additionally, a separate estimate of variance is available from center runs, allowing us to confirm this conclusion.
- In the second (welding) example, it is shown that there is an active dispersion effect. Here we have  $s_{15+|BM2}^2 - s_{15-|BM2}^2 = .4957$  and  $(4n/n - 2)\widehat{\beta}_6\widehat{\beta}_9 = .1943$ . As these two quantities differ substantially, it appears that column 15 does produce a dispersion effect.
- However, in some cases, we can not distinguish which effects (one dispersion or two location) are active. This occurs if the values of  $s_{d+}^2 - s_{d-}^2$  and  $(4n/n - 2)\widehat{\beta}_j\widehat{\beta}_{j'}$  are neither "close" nor "far apart." The dispersion effect and the two location effects are confounded. Critical values based on distribution theory will be helpful in removing this confounding. Theoretical work in this direction is currently in progress by the authors. In the interim, we leave the decision to the knowledgeable team leading the experiment.

We have described a method to help distinguish whether a single dispersion effect or two location effects are active. Box and Meyer develop an approach for detecting dispersion effects based on residuals that does not have a formal method of testing. Bergman and Hynén develop a formal test for dispersion effects using linear combinations of all the estimated location effects. They also suggest an approach of analyzing two 1/2 fractions of the experiment, using the dispersion effect as a branching column to create the half fractions. This is an adaptation of a more general approach suggested by Goldfeld and Quandt (1965). These methods and others, however, do not address how to separate location effects from dispersion effects as our method does.

We have provided an exact confounding relationship between location and dispersion effects in unreplicated fractional factorials. Without additional information this confounding can not be removed. If possible, an experiment should be replicated in order to study dispersion effects. If complete replication is not feasible due to resource constraints, then the addition of center runs can provide some guidance to

help separate location effects from dispersion effects. Alternatively, a follow-up experiment can be run fixing all factors except the suspected dispersion effect. If  $k$  runs are made at each level of this factor, then a simple  $F$  test with  $k - 1$  and  $k - 1$  degrees of freedom can be used to test for the dispersion effect.

Some preliminary work done by the authors shows that a dispersion effect produces a correlation among pairs of location effects. Analysis of this correlation structure may provide additional help in removing the confounding between location and dispersion effects. We are continuing research in this area.

### Appendix

In an  $n = 2^{k-p}$  design, there are  $n/2$  pairs of columns  $(x_j, x_{j'})$  such that  $x_{ij}x_{ij'} = x_{id}$  for  $i = 1, \dots, n$ . We refer to these pairs as alias pairs. Define the following:

$$A = \{j : \beta_j \text{ is in the fitted model, } j \neq 0\},$$

$$A^c = \{j : \beta_j \text{ is not in the fitted model, } j \neq 0\},$$

$$e_i = y_i - \widehat{y}_i = y_i - \bar{y} - \sum_{j \in A} x_{ij}\widehat{\beta}_j,$$

$$\bar{e}_p = (2/n) \sum_{i \in P} e_i,$$

$$\bar{e}_m = (2/n) \sum_{i \in M} e_i.$$

Then

$$\begin{aligned} \bar{e}_p &= (2/n) \sum_{i \in P} (y_i - \bar{y} - \sum_{j \in A} x_{ij}\widehat{\beta}_j) \\ &= \widehat{\beta}_d - \sum_{i \in P} \sum_{j \in A} x_{ij}\widehat{\beta}_j \\ &= \widehat{\beta}_d - (2/n) \sum_{j \in A} \widehat{\beta}_j \sum_{i \in P} x_{ij}. \end{aligned}$$

But,  $\sum_{i \in P} x_{ij} = 0$  for  $j \neq d$  and  $(n/2)$  for  $j = d$ . Therefore,  $\beta_d$  being in the model implies  $\bar{e}_p = 0$ . If  $\beta_d$  is not in the model, then  $\bar{e}_p = \widehat{\beta}_d$ . Using a similar argument for  $\bar{e}_m$ , we have

$$\bar{e}_p = \widehat{\beta}_d I_{(\beta_d \in A^c)} \quad \bar{e}_m = -\widehat{\beta}_d I_{(\beta_d \in A^c)}.$$

If  $\beta_d$  is in the model, then  $\bar{e}_p = 0$  and

$$\begin{aligned} s_{d+}^2 &= \frac{2}{n-2} \sum_{i \in P} e_i^2 \\ &= \frac{2}{n-2} \sum_{i \in P} \left( y_i - \bar{y} - \sum_{j \in A} x_{ij}\widehat{\beta}_j \right)^2 \end{aligned}$$

$$= \frac{2}{n-2} \sum_{i \in P} \left( \sum_{j \in A^c} x_{ij} \hat{\beta}_j \right)^2 \tag{A1}$$

Suppose there are  $g$  pairs of effects,  $(\hat{\beta}_j^{(f)}, \hat{\beta}_{j'}^{(f)})$  with  $f = 1, \dots, g$ , such that  $x_j$  and  $x_{j'}$  are aliases and both  $\beta_j^{(f)}$  and  $\beta_{j'}^{(f)}$  are in not the model. Furthermore, suppose there are  $t$  effects,  $\hat{\beta}_q$ , such that  $\beta_q$  is not in the model but its alias is. Then Equation (A1) can be written as

$$s_{d+}^2 = \frac{2}{n-2} \sum_{i \in P} \left( \sum_{f=1}^g (x_{ij} \hat{\beta}_j^{(f)} + x_{ij'} \hat{\beta}_{j'}^{(f)}) + \sum_{q=1}^t x_{iq} \hat{\beta}_q \right)^2$$

Using  $x_{ij} = x_{ij'}$  for  $i \in P$  we have

$$s_{d+}^2 = \frac{2}{n-2} \sum_{i \in P} \left( \sum_{f=1}^g x_{ij} (\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)}) + \sum_{q=1}^t x_{iq} \hat{\beta}_q \right)^2$$

$$= \frac{2}{n-2} \sum_{i \in P} \left[ \left( \sum_{f=1}^g x_{ij} (\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)}) \right)^2 + \text{crossprods} + \left( \sum_{q=1}^t x_{iq} \hat{\beta}_q \right)^2 \right]$$

Now

$$\sum_{i \in P} x_{ij} x_{iu} = 0 \quad \text{for } j \neq j' \neq u \tag{A2}$$

so the crossproducts drop out when summing over  $i \in P$ . Similarly, when summing over  $i \in P$ , Equation (A2) tells us that all crossproducts within each summation also drop out so we have

$$s_{d+}^2 = \frac{2}{n-2} \sum_{i \in P} \left( \sum_{f=1}^g (\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)})^2 + \sum_{q=1}^t \hat{\beta}_q^2 \right)$$

$$= \frac{n}{n-2} \left( \sum_{f=1}^g (\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)})^2 + \sum_{q=1}^t \hat{\beta}_q^2 \right) \tag{A3}$$

If  $\beta_j$  is not in the model, then

$$s_{d+}^2 = \frac{2}{n-2} \sum_{i \in P} \left( y_i - \bar{y} - \sum_{j \in A} x_{ij} \hat{\beta}_j - \hat{\beta}_d \right)^2$$

$$= \frac{2}{n-2} \sum_{i \in P} \left( \sum_{j \in A^c-d} x_{ij} \hat{\beta}_j + x_{id} \hat{\beta}_d - \hat{\beta}_d \right)^2$$

$$= \frac{2}{n-2} \sum_{i \in P} \left( \sum_{j \in A^c-d} x_{ij} \hat{\beta}_j \right)^2$$

The proof follows as before where it is understood that  $q \neq d$ . Similarly,

$$s_{d-}^2 = \frac{n}{n-2} \left( \sum_{f=1}^g (\hat{\beta}_j^{(f)} - \hat{\beta}_{j'}^{(f)})^2 + \sum_{q=1}^t \hat{\beta}_q^2 \right) \tag{A4}$$

Subtracting Equation (A4) from Equation (A3) results in

$$s_{d+}^2 - s_{d-}^2 = \frac{4n}{n-2} \sum_{f=1}^g \hat{\beta}_j^{(f)} \hat{\beta}_{j'}^{(f)}$$

Under the equal variance hypothesis,  $\hat{\beta}_j \sim N(\beta_j, \sigma^2/n)$  independently. Thus, if of the coefficients considered to be inactive exactly one pair  $(\beta_j, \beta_{j'})$  is actually active, then

$$E [s_{d+}^2 - s_{d-}^2] = \frac{4n}{n-2} \beta_j \beta_{j'}$$

and

$$E [s_{d+}^2 - s_{d-}^2] = \frac{4n}{n-2} \hat{\beta}_j \hat{\beta}_{j'}$$

Also,

$$E [s_{d+}^2] = \frac{n}{n-2} \left( E \left[ \sum_{f=1}^g (\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)})^2 \right] + E \left[ \sum_{q=1}^t \hat{\beta}_q^2 \right] \right)$$

$$= \frac{2g+t}{n-2} \sigma^2 + \frac{n}{n-2} \left[ \sum_{f=1}^g (E [\hat{\beta}_j^{(f)} + \hat{\beta}_{j'}^{(f)}])^2 + \sum_{q=1}^t (E [\hat{\beta}_q])^2 \right]$$

$$= \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} \left( \sum_{f=1}^g (\beta_j^{(f)} + \beta_{j'}^{(f)})^2 + \sum_{q=1}^t (\beta_q)^2 \right) \tag{A5}$$

Using a similar argument,

$$E [s_{d-}^2] = \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} \left( \sum_{f=1}^g (\beta_j^{(f)} - \beta_{j'}^{(f)})^2 + \sum_{q=1}^t (\beta_q)^2 \right) \tag{A6}$$

Again, if exactly one pair  $(\beta_j, \beta_{j'})$  is actually active, then Equations (A5) and (A6) become

$$E [s_{d+}^2] = \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} (\beta_j + \beta_{j'})^2$$

$$E [s_{d-}^2] = \frac{n-1-m}{n-2} \sigma^2 + \frac{n}{n-2} (\beta_j - \beta_{j'})^2.$$

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