Dual Response Surface Optimization: A Fuzzy Modeling Approach

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In modern quality engineering, dual response surface methodology is a powerful tool. In this paper, we introduce a fuzzy modeling approach to optimize the dual response system. We demonstrate our approach in two examples and show the advantages of our method by comparing it with existing methods.

Introduction

RESPONSE surface methodology consists of a group of techniques used in the empirical study of the relationship between the response (y) and a number of input variables (x_i's). Consequently, the experimenter is able to find the optimal setting for the input variables that maximizes (or minimizes) the response. A quadratic (second-order) polynomial model, along with least squares fitting, is widely used to study such an empirical relationship. As a result, all observations are typically assumed to have equal variation.

However, evidence from real problems suggests that the equal variation assumption may not be practically valid. Indeed, when the variances for all observations are not equal, classical response surface methodology can be misleading. Recently the dual response surface approach, popularized by Vining and Myers (1990), has received a great deal of attention in response to its attempt to tackle such a nonequal variance problem (see, e.g., Del Castillo and Montgomery (1993); Lin and Tu (1995); Del Castillo (1996); Copeland and Nelson (1996)). Basically, the dual response surface approach builds two empirical models—one for the mean and one for the standard deviation—and then optimizes one of these responses subject to an appropriate constraint on the other's value.

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The dual response surface approach consists of roughly three stages: data collection (design of experiment), model building and optimization. In this paper we focus on optimization, assuming the data have been collected and suitable models have been fitted. Two questions must be addressed in the optimization stage: "what to optimize" (determining the objective function) and "how to optimize" (the optimization algorithm). Here we are particularly interested in "what to optimize" rather than "how to optimize". The latter issue can be solved by popular standard software packages such as EXCEL (Microsoft, 1993), the Generalized Reduced Gradient method (see, e.g., Del Castillo and Montgomery (1993)) or LANCELOT (see, e.g., Lin and Tu (1995)).

Consider the situation in which a response y depends on k variables, coded $x_1, x_2, \ldots, x_k$. The true response function is unknown so we shall approximate it over a limited experimental region by a polynomial representation. If a first-order model, $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$, suffers lack of fit arising from the existence of surface curvature, we might then wish to fit, by least squares, a quadratic response of the form

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum \sum_{i<j}^{k} \beta_{ij} x_i x_j + \varepsilon.$$

Again, such a model works well when the variance of the response is relatively small and stable (a constant value), but when the variance of y is not a constant, classical response surface methodology could be misleading.

Vining and Myers (1990) used the dual response approach, introduced by Myers and Carter (1973), and proposed an ingenious method to tackle such a problem. They modeled both the location effect (w_μ) and the dispersion effect (w_σ) as separate responses. Namely,

$$w_\mu = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j}\sum^{k} \beta_{ij} x_i x_j + \varepsilon_\mu,$$

$$w_\sigma = \gamma_0 + \sum_{i=1}^{k} \gamma_i x_i + \sum_{i=1}^{k} \gamma_{ii} x_i^2 + \sum_{i<j}\sum^{k} \gamma_{ij} x_i x_j + \varepsilon_\sigma.$$

They fit second-order models to both of the responses and then optimize the two fitted response surface models simultaneously. Specifically, they optimize one fitted response subject to an appropriate constraint on the value of the other fitted response using the Lagrangian multiplier approach. For example, for the case "Nominal the best" (NTB), the optimization scheme will

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Minimize}} && \hat{w}_\sigma \\ &\text{subject to} && \hat{w}_\mu = T \text{ (target)}. \end{aligned}$$

Del Castillo and Montgomery (1993) optimize the dual response system based on the same objective function, but use a more advanced computational algorithm to avoid the dimensionality problem (see also, Del Castillo (1996)). Lin and Tu (1995) point out that the optimization scheme based on Lagrangian multipliers can be misleading due to the unrealistic restriction of forcing the estimated mean ($\hat{w}_\mu$) to a specific value. Consequently, they propose a new objective function to be minimized, namely, the Mean Squared Error, MSE $= (\hat{w}_\mu - T)^2 + \hat{w}_\sigma^2$. Copeland and Nelson (1996) note that minimizing the MSE places no restriction on how far the resulting value of $\hat{w}_\mu$ might be from the target value T. They suggest the use of direct function minimization.

Proposed Optimization Scheme

We propose a novel mathematical programming formulation for the dual response problem based on fuzzy optimization methodology. Our approach considers both the deviation of $\hat{w}_\mu$ from T and the magnitude of $\hat{w}_\sigma$, simultaneously. We shall focus on the NTB case for the mean to illustrate the basic idea. It is assumed that the degree of satisfaction of the experimenter (or decision maker, DM) with respect to the mean is maximized when $\hat{w}_\mu$ equals T and decreases as $\hat{w}_\mu$ moves away from T. If $w_\mu^{\min}$ and $w_\mu^{\max}$ represent the lower and upper bounds of aspirations, respectively, then the DM does not accept a solution $\mathbf{x}$ for which $\hat{w}_\mu \le w_\mu^{\min}$ or $\hat{w}_\mu \ge w_\mu^{\max}$. Thus the satisfaction level with respect to the mean can be modeled by a function which decreases monotonically from 1, at $\hat{w}_\mu = T$, to 0, at $\hat{w}_\mu \le w_\mu^{\min}$ or $\hat{w}_\mu \ge w_\mu^{\max}$. In this paper, we will refer to such a function as a *membership function* (as in fuzzy set theory).

The membership function value of the mean response, denoted as $m(\hat{w}_\mu)$, is interpreted as the degree to which $\hat{w}_\mu$ satisfies the target on the mean, and is a value between 0 and 1. Assuming the degree of satisfaction changes linearly as a function of ($\hat{w}_\mu - T$), a membership function (also depicted in Figure 1(a)) can be expressed as

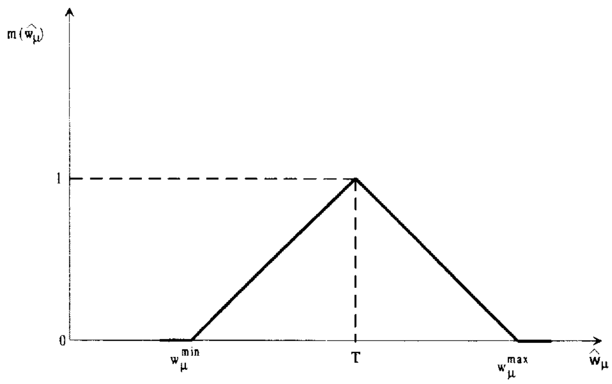
$$m(\hat{w}_\mu) = \begin{cases} 0 & \text{if } \hat{w}_\mu \le w_\mu^{\min} \\ & \text{or if } \hat{w}_\mu \ge w_\mu^{\max} \\ 1 - \frac{T - \hat{w}_\mu}{T - w_\mu^{\min}} & \text{if } w_\mu^{\min} < \hat{w}_\mu \le T \\ 1 - \frac{\hat{w}_\mu - T}{w_\mu^{\max} - T} & \text{if } T \le \hat{w}_\mu < w_\mu^{\max}. \end{cases} \quad (1)$$

The degree of satisfaction with respect to the estimated standard deviation, $\hat{w}_\sigma$, can be modeled in a similar way. The membership function value of the standard deviation, $m(\hat{w}_\sigma)$, would decrease monotonically from 1, at $w_\sigma^{\min}$, to 0, at $w_\sigma^{\max}$, where $w_\sigma^{\min}$ and $w_\sigma^{\max}$ represent the lower and upper bounds of aspirations with respect to the standard deviation. Figure 1(b) shows a linear membership function which is stated as

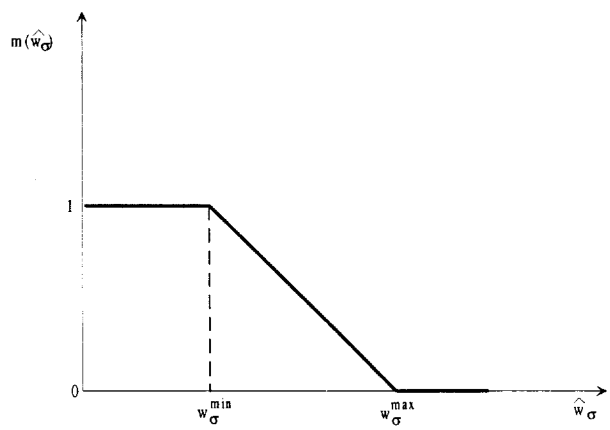
$$m(\hat{w}_\sigma) = \begin{cases} 1 & \text{if } \hat{w}_\sigma \le w_\sigma^{\min} \\ \frac{w_\sigma^{\max} - \hat{w}_\sigma}{w_\sigma^{\max} - w_\sigma^{\min}} & \text{if } w_\sigma^{\min} < \hat{w}_\mu < w_\sigma^{\max} \\ 0 & \text{if } \hat{w}_\sigma \ge w_\sigma^{\max}. \end{cases} \quad (2)$$

The value of $w_\sigma^{\min}$ is typically set equal to zero. If the DM, however, does not care about the variability up to a certain level, $w_\sigma^{\min}$ can be given a positive value (in Figure 1(b)). [Figure 1(c) shows an example of a nonlinear membership function. The determination of a membership function shape will be discussed later in this paper.]

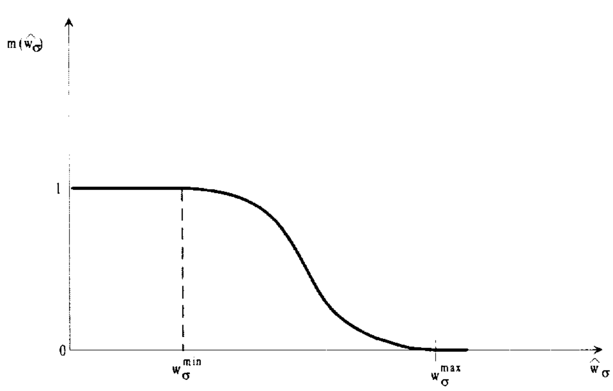
A dual response problem requires an overall optimization—that is, a simultaneous satisfaction with respect to both the mean and the standard deviation. If a "minimum" operator is employed for aggregating the two objectives, a dual response optimization



(a) Membership Function of $\hat{w}_\mu$: Linear Case.



(b) Membership Function of $\hat{w}_\sigma$: Linear Case.



(c) Membership Function of $\hat{w}_\sigma$: Nonlinear Case.

FIGURE 1. Example Membership Functions of $\hat{w}_\mu$ and $\hat{w}_\sigma$.

problem can be stated as

$$\begin{aligned} &\underset{\mathbf{x}}{\text{Maximize}}\ \lambda \\ &\text{subject to} \quad m(\hat{w}_\mu) \geq \lambda \qquad (3) \\ &\qquad\qquad\quad m(\hat{w}_\sigma) \geq \lambda \qquad (4) \\ &\qquad\qquad\quad \mathbf{x} \in \Omega \end{aligned}$$

where $\hat{w}_\mu$ and $\hat{w}_\sigma$ are functions of $\mathbf{x}$ (i.e., fitted response surfaces) and Ω defines the feasible region of $\mathbf{x}$. The above formulation aims to identify $\mathbf{x}^*$ which would maximize the minimum degree of satisfaction, λ, with respect to the mean and standard deviation within the feasible region, that is, maximize (minimum$[m(\hat{w}_\mu), m(\hat{w}_\sigma)]$) with respect to $\mathbf{x} \in \Omega$.

The optimization approach proposed has two main methodological advantages over the existing methods. First, the proposed approach achieves a better balance between bias and variance compared to the existing methods. As an example, in the approach proposed by Vining and Myers (1990), the requirement on the mean (or variance) is expressed in an equality constraint, which may be unrealistically constraining. The MSE criterion (Lin and Tu (1995)) minimizes the MSE value without regard to the relative magnitude of bias and variance. For instance, in the printing process example given in Lin and Tu (1995, p.37, Table 3, $\mathbf{x}'\mathbf{x} \leq 1$), the MSE criterion results in an optimal setting at which the bias is 29.81 ($= (494.54 - 500)^2$) and the variance is 1,992.73. The bias explains only 1.47% of the MSE, and thus the minimization of MSE was essentially driven by the minimization of variance, although a small amount of bias might have a significant impact on the design performance. The opposite scenario, where variance is dominated by bias, is also possible. The formulation proposed by Copeland and Nelson (1996) is deemed more realistic in this regard. However, it does not consider the effect of the bias up to a certain point. Specifically, all solutions within tolerance limits are not necessarily equally desirable (which is a fundamental concept from Taguchi). The proposed approach explicitly takes into account practically allowable ranges on the mean and standard deviation and then maximizes the satisfaction level which is defined within the specified range. Therefore, the optimal setting obtained from the proposed approach would be much more balanced in the sense that the contribution of both bias and variance is properly reflected in the optimization process. This point will be demonstrated in an example later in the paper.

Secondly, the proposed approach is flexible as it allows the DM to incorporate his or her preference into the model. For example, the membership function of $\hat{w}_\mu$ can be asymmetric, as when the DM's degree of satisfaction changes at a different rate when $\hat{w}_\mu$ moves away from T in the left or right direction. The DM's preference can be easily implemented by properly specifying $w_\mu^{\min}$ and $w_\mu^{\max}$. If an "undersize" deviation is more costly than an "oversize" deviation, the bounds should be set in such a way that $(T - w_\mu^{\min})$ is smaller than $(w_\mu^{\max} - T)$. As an extreme case, setting $w_\mu^{\min} = T = 0$ (assuming $\hat{w}_\mu$ is nonnegative) represents the "smaller-the-better", while setting $w_\mu^{\max} = T$ at a sufficiently large value represents the "larger-the-better" type situation.

Moreover, a nonlinear membership function (shown in Figure 1(c)) can model the DM's preference change very flexibly compared to the existing approaches, where the DM's degree of satisfaction changes linearly with bias and/or variance. Another possible advantage of the proposed approach is that the objective function value, λ, allows a good physical interpretation: λ is the overall degree of satisfaction ($0 \leq \lambda \leq 1$) based on the specified ranges of both $\hat{w}_\mu$ and $\hat{w}_\sigma$.

It is worth noting the similarities and differences between the membership function and the desirability function (see, e.g., Derringer and Suich (1980)). The membership function of a fuzzy set assigns each possible element a value representing its grade of membership in the set (see, e.g., Zadeh (1965)). Objects may belong in a fuzzy set to a greater or lesser degree as indicated by a larger or smaller membership value. The desirability function approach is probably one of the most frequently used multiresponse optimization techniques. The methodological basis for the desirability function approach is to transform the estimated response on a quality characteristic to a value, known as "desirability," for each response. The geometric mean of all individual desirability values is then used to represent the overall desirability—the larger, the better.

Although the two functions have been developed from different perspectives, they have some common characteristics: they are scale-free, they are between 0 and 1 (0 being the worst and 1 the best), and they are flexible in shape. Conceptually, a desirability function can be viewed as a special case of the membership function in the sense that the degree of desirability of an estimated response is essentially the grade of membership to a fuzzy set representing the "ideal" response. For a discussion on the use of fuzzy methods in statistical problems, see Laviolette, Seaman, Barrett and Woodall (1995).

Membership Function Assessment

As discussed in the previous section, the proposed approach requires that the membership function of the estimated mean and standard deviation be specified. A linear membership function, as in (1) and (2), is defined by fixing the upper and lower levels of acceptability (and a target value in the NTB case). If the marginal rate of change of membership values as a function of the response is not constant, a nonlinear membership function should be employed. Nonlinear shapes offer potential benefits in terms of realism and are chosen with varying perception of the DM.

When a nonlinear membership function is desired, the process of selecting an admissible functional form is difficult and time-consuming. However, it can be simplified by employing a general functional form which can generate a rich variety of shapes by adjustment of its parameters. In view of this, we suggest the use of an exponential function of the form

$$m(z) = \begin{cases} \frac{e^d - e^{d|z|}}{e^d - 1} & \text{if } d \neq 0 \\ 1 - |z| & \text{if } d = 0, \end{cases} \tag{5}$$

where d is a constant ($-\infty < d < \infty$), called the exponential constant, and z is a standardized parameter representing the distance of the response from its target in units of the maximum allowable deviation. For a symmetric case, the estimated mean, denoted as z_μ, is defined as

$$z_\mu = \frac{\hat{w}_\mu - T}{w_\mu^{\max} - T} = \frac{\hat{w}_\mu - T}{T - w_\mu^{\min}} \quad \text{for } w_\mu^{\min} \leq \hat{w}_\mu \leq w_\mu^{\max}. \tag{6}$$

Equation (6) can be easily modified for an asymmetric case. Similarly, the standard deviation, denoted as z_σ, is defined as

$$z_\sigma = \frac{\hat{w}_\sigma - w_\sigma^{\min}}{w_\sigma^{\max} - w_\sigma^{\min}} \quad \text{for } w_\sigma^{\min} \leq \hat{w}_\sigma \leq w_\sigma^{\max}. \tag{7}$$

Note that z_μ ranges between -1 and 1 while z_σ ranges between 0 and 1. In both cases the membership function value $m(z)$ achieves its maximum value of 1 when $z = 0$, that is, when $\hat{w}_\mu = T$ and $\hat{w}_\sigma = w_\sigma^{\min}$. The function $m(z)$ given in (5) has been proven to provide a reasonable and flexible representation of human perception (Kirkwood and Sarin (1980), Moskowitz and Kim (1993)) and is convenient to handle analytically.

The function $m(z)$ can represent many different shapes depending upon the exponential constant d; it is convex, linear, and concave when $d < 0$, $d = 0$, and $d > 0$, respectively. As d increases (from negative infinity to positive infinity), $m(z)$ becomes decreasingly convex and increasingly concave. If a convex-shaped membership function is used, the membership value changes more rapidly when z is close to 0 (and more slowly when z gets close to 1) than when using a linear or concave-shaped membership function. Therefore, using a convex membership function implies that the deviation of the response from its target value should be smaller (in units of z) than when using a linear or concave membership function to maintain the same degree of satisfaction. (See the points a, b, c at $m(z) = m_0$ in Figure 2.) Examples of the exponential membership function with several different d values are shown in Figure 2.

The membership function reflects the DM's belief and, by some, has been viewed analogous to a utility function in decision analysis (Zimmermann (1987)). A membership function can thus be measured using procedures similar to those used for assessing a utility function. For the exponential membership function case, $m(z)$ can be assessed by identifying just one point on the curve because it has only one unknown parameter d. At an arbitrary point z_0, $(0 < z_0 < 1)$, the DM assesses the degree of satisfaction, denoted as s, and then solves the equation $m(z_0) = s$ for d:

$$\frac{e^d - e^{dz_0}}{e^d - 1} = s, \quad 0 < z_0 < 1,\ 0 < s < 1. \tag{8}$$

There is no closed form solution to (8), hence it must be solved numerically. However, when $z_0 = 0.5$, d is obtained as

$$d = 2\ell\text{n}\left(\frac{s}{1-s}\right). \tag{9}$$

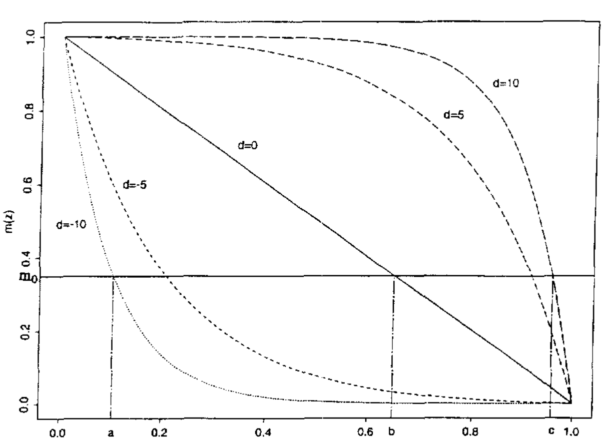


FIGURE 2. Example of Exponential Membership Functions. Note: The graph is shown only for $0 \le z \le 1$; $m(z_\mu)$ may not be symmetric for $-1 \le z \le 0$.

TABLE 1. The Printing Process Study Data

u	x_1	x_2	x_3	y_{u1}	y_{u2}	y_{u3}	$\bar{y}_u$	s_u
1	−1	−1	−1	34	10	28	24.0	12.49
2	0	−1	−1	115	116	130	120.3	8.39
3	1	−1	−1	192	186	263	213.7	42.80
4	−1	0	−1	82	88	88	86.0	3.46
5	0	0	−1	44	178	188	136.7	80.41
6	1	0	−1	322	350	350	340.7	16.17
7	−1	1	−1	141	110	86	112.3	27.57
8	0	1	−1	259	251	259	256.3	4.62
9	1	1	−1	290	280	245	271.7	23.63
10	−1	−1	0	81	81	81	81.0	0.00
11	0	−1	0	90	122	93	101.7	17.67
12	1	−1	0	319	376	376	357.0	32.91
13	−1	0	0	180	180	154	171.3	15.01
14	0	0	0	372	372	372	372.0	0.00
15	1	0	0	541	568	396	501.7	92.50
16	−1	1	0	288	192	312	264.0	63.50
17	0	1	0	432	336	513	427.0	88.61
18	1	1	0	713	725	754	730.7	21.08
19	−1	−1	1	364	99	199	220.7	133.80
20	0	−1	1	232	221	266	239.7	23.46
21	1	−1	1	408	415	443	422.0	18.52
22	−1	0	1	182	233	182	199.0	29.45
23	0	0	1	507	515	434	485.3	44.64
24	1	0	1	846	535	640	673.7	158.20
25	−1	1	1	236	126	168	176.7	55.51
26	0	1	1	660	440	403	501.0	138.90
27	1	1	1	878	991	1161	1010.0	142.50

The function $m(z)$ is convex, linear, and concave when $0 < s < 0.5$, $s = 0.5$, and $0.5 < s < 1$, respectively.

Example 1

As an example, we use a problem taken from Box and Draper (1987), which was also used in Vining and Myers (1990) and Lin and Tu (1995), referred to as VM and LT, respectively, to make a fair comparison. The purpose of the experiment was to determine the effect of speed (x_1), pressure (x_2), and distance (x_3) on the quality of a printing process. The experiment was conducted in a 3^3 factorial design with three replicates at each design point. The data set is reproduced in Table 1.

Assuming quadratic models were adequate, response surfaces for the mean and standard deviation of the characteristic of interest were fitted as follows (Vining and Myers (1990)),

$$\begin{aligned}\hat{w}_\mu = 327.6 &+ 177.0x_1 + 109.4x_2 \\ &+ 131.5x_3 + 32.0x_1^2 - 22.4x_2^2 \\ &- 29.1x_3^2 + 66.0x_1x_2 \\ &+ 75.5x_1x_3 + 43.6x_2x_3\end{aligned}$$

$$\hat{w}_\sigma = 34.9 + 11.5x_1 + 15.3x_2 + 29.2x_3 + 4.2x_1^2 - 1.3x_2^2 + 16.8x_3^2 + 7.7x_1x_2 + 5.1x_1x_3 + 14.1x_2x_3.$$

For illustration, the exponential membership function given in (5) was employed with the following bounds: $w_\mu^{\min}$=490, $w_\mu^{\max}$=510, T=500, $w_\sigma^{\min}$= $\sqrt{1500}$, and $w_\sigma^{\max}$= $\sqrt{2100}$. To examine the effect of the membership function shape, various values of d_μ, ranging from -4.39 to 4.39, were tested with d_σ fixed at 0. (The subscripts μ and σ, represent the mean and standard deviation, respectively.) The results of the proposed approach, based on a cuboidal region $-1 \leq x_i \leq 1$ $(i = 1, 2, 3)$, are summarized and compared with those of VM and LT in Table 2.

Effect of Membership Function

Table 2 shows that the results of the VM and LT approaches are not affected by the membership function, except for the λ value from LT which was computed *a posteriori* for comparison with the proposed approach. The value of λ from the VM method turns out to be zero for all d because $\hat{w}_\sigma$ of $\sqrt{2679.70}$ is out of the allowable range of $m(z)$ for the standard deviation; that is, $\lambda = \min\{m(\hat{w}_\mu), m(\hat{w}_\sigma)\} = \min\{m(500), m(\sqrt{2679.70})\} = \min\{1.0, 0.0\} = 0.0$. If $w_\sigma^{\max}$ had been set at a value greater than $\sqrt{2679.70}$, however, the VM method would have achieved a positive level of λ.

The effect of a change in the value of d_μ can be easily seen in the results of the proposed approach. Increasing the value of d_μ, with d_σ fixed, indicates that $m(z_\mu)$ is becoming more concave, and thus the requirement on the mean is becoming less stringent. The result shows that as d_μ increases (from a negative value to zero, and then to a positive value), $|\hat{w}_\mu - T|$ increases and $\hat{w}_\sigma$ decreases because the satisfaction level on the standard deviation becomes more crucial, in a relative sense, compared to that of the mean.

The optimal objective function value, λ^*, increases as d_μ increases. This is because the member-

TABLE 2. Comparisons of Results: Proposed vs. Existing Approaches

method	$d_\mu = -4.39$	$d_\mu = -1.70$	$d_\mu = 0.00$	$d_\mu = 1.70$	$d_\mu = 4.39^{(\dagger)}$
VM					
Optimal Setting	(0.62,0.23,0.10)	(0.62,0.23,0.10)	(0.62,0.23,0.10)	(0.62,0.23,0.10)	(0.62,0.23,0.10)
$\hat{w}_\mu$	500	500	500	500	500
$\hat{w}_{\sigma^2}$	2679.70	2679.70	2679.70	2679.70	2679.70
MSE	2679.70	2679.70	2679.70	2679.70	2679.70
λ	0.00	0.00	0.00	0.00	0.00
LT					
Optimal Setting	(1.00,0.07,−0.25)	(1.00,0.07,−0.25)	(1.00,0.07,−0.25)	(1.00,0.07,−0.25)	(1.00,0.07,−0.25)
$\hat{w}_\mu$	494.44	494.44	494.44	494.44	494.44
$\hat{w}_{\sigma^2}$	1974.02	1974.02	1974.02	1974.02	1974.02
MSE$^{*(\dagger\dagger)}$	2005.14	2005.14	2005.14	2005.14	2005.14
λ	0.08	0.19	0.20	0.20	0.20
Proposed					
Optimal Setting	(1.00,0.086,−0.254)	(1.00,0.067,−0.251)	(1.00,0.055,−0.248)	(1.00,0.047,−0.247)	(1.00,0.041,−0.246)
$\hat{w}_\mu$	496.08	493.84	492.32	491.34	490.67
$\hat{w}_{\sigma^2}$	1991.74	1967.88	1951.79	1941.45	1934.48
MSE	2007.07	2005.80	2010.77	2016.47	2021.44
$\lambda^{*(\dagger\dagger)}$	0.17	0.21	0.23	0.25	0.26

(†) $d_\mu = -4.39, -1.70, 0.00, 1.70, 4.39$ corresponds to s (membership value at $z = 0.5$) = 0.1, 0.3, 0.5, 0.7, 0.9, respectively. d_σ was set at 0.00 throughout the example.

(††) The expression with "*" is the quantity that was optimized.

ship value $m(z)$ at an arbitrary $z \in (0,1)$ increases as the membership function shape becomes decreasingly convex or increasingly concave. The MSE value, however, does not display a systematic pattern as a function of d_μ because the MSE was computed at $\mathbf{x}^*$, which yields the maximum λ^*.

It should be noted that the membership function shape affects λ^*, but not $\mathbf{x}^*$, nor MSE, if $d_\mu = d_\sigma$ (e.g., $\mathbf{x}^*$ is the same for $d_\mu = d_\sigma = -4.39$ or for $d_\mu = d_\sigma = 4.39$). Assigning the same membership function shape to both $\hat{w}_\mu$ and $\hat{w}_\sigma$ implies that they have the same level of stringency within the specified range. Therefore, the membership function shape, either convex or concave, should not affect the optimal setting when a common shape is used for both objectives.

Comparison with VM and LT

The solution of the proposed approach is much closer to that of LT than VM in terms of the optimal setting $\mathbf{x}^*$ and MSE, as well as λ^*. When $d_\mu = -4.39$, the membership function is highly convex, which indicates the high stringency associated with the mean compared to a linear membership function for the standard deviation ($d_\sigma = 0.00$). The proposed approach explicitly considers such priority manifested by the membership function shape in contrast to the existing methods. As a result, the proposed approach results in a better $\hat{w}_\mu$, and the LT method, in a better $\hat{w}_\sigma$. Both $\hat{w}_\mu$ and $\hat{w}_\sigma$ of the proposed approach are 17% satisfactory, hence $\lambda^* = 0.17$. In LT's method, the satisfaction is 20% in $\hat{w}_\sigma$, but only 8% in $\hat{w}_\mu$. Thus the overall satisfaction level would be 8%, which is significantly lower than the satisfaction level from the proposed approach. Figure 3 shows the solutions from both methods on the common axes for the above mentioned case.

In all cases, as expected, the proposed and the LT methods give a higher λ^* and a lower MSE*, respectively. Furthermore, except for when $d_\mu = -4.39$, the proposed method yields a smaller $\hat{w}_\sigma$, while LT's method yields a value of $\hat{w}_\mu$ which is closer to 500.

Practical Concerns

The following steps are suggested to implement the proposed model. A simple subroutine can be constructed to perform all the computations.

Step 0: Develop the experimental design, conduct the experiments, and collect the data.

Step 1: Fit response surfaces for the mean ($\hat{w}_\mu$) and the standard deviation ($\hat{w}_\sigma$).

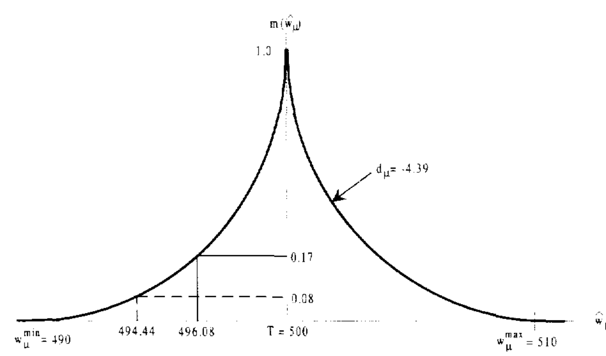
Step 2: Determine the bounds of aspirations on the mean ($w_\mu^{\min}$, $w_\mu^{\max}$, and T for the NTB case) and the standard deviation ($w_\sigma^{\min}$ and $w_\sigma^{\max}$).

Step 3: Assess the membership functions of the mean ($m(\hat{w}_\mu)$) and the standard deviation ($m(\hat{w}_\sigma)$), as given in (5)–(7).

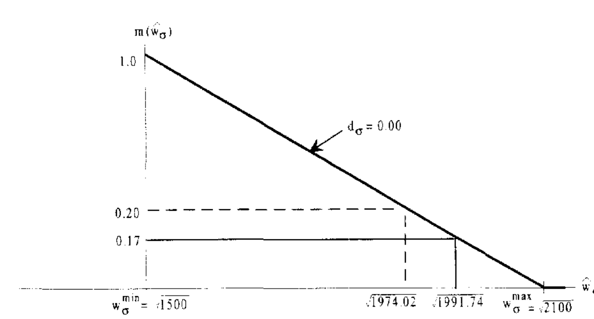
Step 4: Formulate and solve the optimization problem using the information from Steps 1 to 3, as given in (3) and (4).

Notes on Implementation Issues

Step 0 typically employs the classical first- and second-order design as discussed in Box and Draper (1987) and Myers and Montgomery (1995). When fitting models for the responses in Step 1, the predictive capability of the model is a very important consideration and should be justified by meaningful criteria, such as a high R^2 value. In order to improve the R^2 value, various model selection procedures such as stepwise regression, all possible subsets regression, C_P, and PRESS may be employed (Lin and Tu (1995)), possibly with a transformation of the data (see, e.g., Vining and Myers, (1990)).



(a) Proposed vs. LT method: $m(\hat{w}_\mu)$.



(b) Proposed vs. LT method: $m(\hat{w}_\sigma)$.

FIGURE 3. Comparison of Solutions from the methods at $(d_\mu, d_\sigma) = (-4.39, 0.00)$. Note: The axes are shown in the original units.

The bounds to be determined in Step 2 are the ones in a practical (rather than theoretical) sense, representing the extreme values of the DM's aspiration interval. The determination of the bounds is not always straightforward. One possible way to determine the bounds is to use the current operating condition (which is to be improved) as a baseline, that is, set $w_\mu^{\min}$(or $w_\mu^{\max}$) and $w_\sigma^{\max}$ at the current process mean and standard deviation, respectively.

The assessment of a membership functional form in Step 3 can be simplified by employing the exponential function given in (5), which can generate various shapes through adjustment of the exponential constant d. The value of d is chosen depending on the range of the responses and the DM's preference change within the range. Although there are no theoretical bounds on d, realistic values of d will generally have a magnitude between -10 and 10 (Kirkwood (1996)). Figure 2 shows that the membership functions are very curved at $d = -10$ or $d = 10$. Generally, a negative (positive) value of d represents the high (low) stringency of the requirement that the characteristic (mean or standard deviation) should be close to its target value.

Once all the modeling components described above are determined, an optimization problem for the dual response can be formulated as in (3) and (4). The important concept of specification limits for the mean and standard deviation can be easily incorporated here. The region of interest of a design point, $\mathbf{x}$, is defined by $\mathbf{x} \in \Omega$. If there are regulations or standards to be met, such conditions can be included as part of $\mathbf{x} \in \Omega$ to further restrict the feasible range of $\mathbf{x}$. Assuming that the fitted response models are of a second-order or higher, the formulation represents a constrained nonlinear optimization with a single objective. In principle, any general algorithm for a nonlinear problem can be used, including the Generalized Reduced Gradient (GRG) method (Del Castillo and Montgomery (1993), Del Castillo (1996)), and the Nelder-Mead simplex procedure (Copeland and Nelson (1996)). A built-in optimization routine based on the GRG algorithm, Microsoft Excel's Solver, Microsoft (1993), was used to solve the example problems in this work.

It should be noted that a gradient-based optimization method, like the GRG algorithm, may fail to reach an optimal point if the membership function (in (3) and (4)) has non-differentiable points (Del Castillo, Montgomery and McCarville (1996)). Moreover, the nonlinear optimization may result in a local optimum because the ultimate solution found can depend on the starting point supplied by the user. It is recommended that the optimization program be run with several different starting points when the optimization result is suspect.

Example 2

In a Roman-style catapult experiment, Luner (1994) used three variables, x_1 (arm length), x_2 (stop angle), and x_3 (pivot height), to predict the distance to the point where a projectile landed from the base of the catapult. The experiment is a central composite design with three replicates, as shown in Table 3. We will solve this problem using our suggested steps, discussed in the previous section. The information required for Steps 0, 1, and 2 is taken from Luner (1994).

Step 0: The results of the experiment are summarized and reproduced in Table 3.

Step 1: A weighted least squares regression analysis was performed for the mean response. The fitted second-order model is

$$\begin{aligned}\hat{w}_\mu = {} & 84.88 + 15.29x_1 + 0.24x_2 \\ & + 18.80x_3 - 0.52x_1^2 - 11.80x_2^2 \\ & + 0.39x_3^2 + 0.22x_1x_2 \\ & + 3.60x_1x_3 - 4.42x_2x_3. \end{aligned} \tag{10}$$

The fitted second-order model for standard deviation is

$$\begin{aligned}\hat{w}_\sigma = {} & 4.53 + 1.84x_1 + 4.28x_2 \\ & + 3.73x_3 + 1.16x_1^2 + 4.40x_2^2 \\ & + 0.94x_3^2 + 1.20x_1x_2 \\ & + 0.73x_1x_3 + 3.49x_2x_3. \end{aligned} \tag{11}$$

Step 2: The target value for the mean is 80. A mean value between 79 and 81 is deemed acceptable, thus $w_\mu^{\min}$ and $w_\mu^{\max}$ were set at 79 and 81, respectively. It is desired that the standard deviation be minimized and not exceed 3.5. Therefore, $w_\sigma^{\min}$ and $w_\sigma^{\max}$ were set at 0 and 3.5, respectively (see, e.g., Luner (1994, page 701)).

Step 3: Since it is presumed that the satisfaction level with respect to the mean changes linearly with the deviation from the target value, a linear membership function is employed for the mean, that is,

$$m(\hat{w}_\mu) = \begin{cases} 0 & \text{if } \hat{w}_\mu < 79 \text{ or } \hat{w}_\mu > 81 \\ 1 - |80 - \hat{w}_\mu| & \text{if } 79 \le \hat{w}_\mu \le 81. \end{cases}$$

TABLE 3. The Catapult Study Data

	u	x_1	x_2	x_3	y_{u1}	y_{u2}	y_{u3}	$\bar{y}_u$	s_u
Factorial Points	1	−1	−1	−1	39	34	42	38.3	4.0
	2	−1	−1	1	80	71	91	80.7	10.0
	3	−1	1	−1	52	44	45	47.0	4.4
	4	−1	1	1	97	68	60	75.0	19.5
	5	1	−1	−1	60	53	68	60.3	7.5
	6	1	−1	1	113	104	127	114.7	11.6
	7	1	1	−1	78	64	65	69.0	7.8
	8	1	1	1	130	79	75	94.7	30.7
Axial Points	9	−1.682	0	0	59	51	60	56.7	4.9
	10	1.682	0	0	115	102	117	111.3	8.1
	11	0	−1.682	0	50	43	57	50.0	7.0
	12	0	1.682	0	88	49	43	60.0	24.4
	13	0	0	−1.682	54	50	60	54.7	5.0
	14	0	0	1.682	122	109	119	116.7	6.8
Center Points	15	0	0	0	87	78	89	84.7	5.9
	16	0	0	0	86	79	85	83.3	3.8
	17	0	0	0	88	81	87	85.3	3.8
	18	0	0	0	89	82	87	86.0	3.6
	19	0	0	0	86	79	88	84.3	4.7
	20	0	0	0	88	79	90	85.7	5.9

This membership function corresponds to the case $d = 0$ in $m(z)$, given in (5). The requirement on the standard deviation is assumed to be less stringent than that on the mean. Hence a concave-shaped exponential membership function with $d = 1.70$ (corresponding to $s = 0.70$ in (9)) is chosen, that is,

$$m(\hat{w}_\sigma) = \frac{e^{1.7} - e^{1.7z_\sigma}}{e^{1.7} - 1},$$

where

$$z_\sigma = \frac{\hat{w}_\sigma - w_\sigma^{\min}}{w_\sigma^{\max} - w_\sigma^{\min}} = \frac{\hat{w}_\sigma - 0}{3.5 - 0} = \frac{\hat{w}_\sigma}{3.5}$$

for $0 \le \hat{w}_\sigma \le 3.5$.

Step 4: The complete formulation for the catapult study problem is as follows

$$\underset{\mathbf{x}}{\text{Maximize}}\ \lambda$$

$$\text{subject to}\quad 1 - |80 - \hat{w}_\mu| \ge \lambda,$$

$$\frac{e^{1.7} - e^{1.7z_\sigma}}{e^{1.7} - 1} \ge \lambda,$$

$$79 \le \hat{w}_\mu \le 81,$$

$$0 \le \hat{w}_\sigma \le 3.5,$$

$$-1 \le x_i \le 1, \quad i = 1, 2, 3$$

where $\hat{w}_\mu$ and $\hat{w}_\sigma$ are given as a function of x_i's ($i = 1, 2, 3$) in (10) and (11), respectively, and $z_\sigma = \hat{w}_\sigma/3.5$.

The optimal design point turns out to be $\mathbf{x}^* = (0.12, -0.27, -0.32)$, where $\hat{w}_\mu$=79.23, $\hat{w}_\sigma$=3.06, and the resulting $\lambda^* = 0.23$. Different membership functions (i.e., different d_μ and/or d_σ values in Step 3) would have resulted in different $\mathbf{x}^*$. The sensitivity of the results to the changes in d_μ and d_σ or to the ratio (d_μ/d_σ) is now under investigation. This should provide some useful insights as to under what conditions the membership function shape has a critical effect, and thus the assessment of d_μ and d_σ could be done more carefully.

Conclusion

A fuzzy modeling approach to optimize the dual response system has been presented. The proposed approach aims to identify a set of process parameter conditions to simultaneously maximize the degree of satisfaction with respect to the mean and the standard deviation responses. In using two real examples, it was shown that the proposed method can model the decision maker's preference on the estimated responses very flexibly and achieves a better balance between bias and variance. Comparisons with other existing methods were also discussed.

An explicit procedure was presented that allows one to map steps for the application of the proposed procedure. In particular, Steps 0 and 1 (data collection and empirical model building) are well known subjects and were not further pursued here. Steps 2 and 3 (membership function assessment and problem formulation) have been discussed in detail. Step 4 (optimization) requires a constrained nonlinear optimization solver, in general. We have used Microsoft Excel in this paper, although other algorithms in this area, as previously mentioned, can be used as well.

Acknowledgments

The authors thank the editor and the reviewers for their helpful comments and support.

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Key Words: *Central Composite Designs, Membership Functions, Three-level Factorials*.