

Sequential analysis of an accelerated life model

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Introduction

The inverse Gaussian (IG) density is a flexible density that competes with the extreme-value, gamma, lognormal, and Weibull densities as a random process model. Of interest herein are methods that have been developed for the IG density which have application in quality and reliability engineering. Notably, IG models have proved their usefulness in accelerated life testing where early failure times predominate (Chhikara and Folks, 1989). With regard to reliability and life testing, the interest in the IG density can be traced to the fact that it is the first passage time distribution of a positive barrier in a Brownian motion process with positive drift (Chhikara and Folks, 1976). Among recent developments for the IG density with quality and reliability engineering implications are control charts for IG distributed quality characteristics (Edgeman, 1989); analysis of reciprocals, an alternative to one-way analysis of variance for data originating from positively skewed distributions (Edgeman, 1990b; Edgeman, 1992); and derivation of the sequential probability ratio tests (SPRT) on which sequential sampling plans and acumulative sum (CUSUM) charts for the IG model are based (Edgeman and Salzberg, 1991).

Two beguiling facts about this distribution fuel our interest. First, it has an exponential family structure and the associated sampling distributions are well-known. Second, it is a reproductive exponential family, of which there exist only two other univariate families, namely the gamma and normal densities. The IG probability density function with mean μ and variance μ^3/λ is given by

$$f(x; \mu, \lambda) = (1/(2\pi x^3))^{1/2} \exp(-\lambda(x-\mu)^2/(2x\mu^2))$$

where each of x , μ , and λ are positive and λ is a shape parameter as well as a reciprocal measure of dispersion. The shape of this unimodal density depends only on the value of $\phi = \lambda/\mu$. Maximum likelihood estimators of μ and $1/\lambda$ are the sample mean, \bar{X} , and $1/\hat{\lambda}$, respectively, where, based on a random sample of n items from $f(x; \mu, \lambda)$

$$1/\hat{\lambda} = (1/n) \sum_{i=1}^n (1/X_i - \hat{X})$$

It is well-known (Tweedie, 1957) that $n/\hat{\lambda}$ is distributed according to chi-square with $(n - 1)$ degrees of freedom and that \bar{X} is distributed as IG with parameters μ and $n\lambda$. Statistical hypothesis tests and confidence interval results for this distribution can be found in Chhikara and Folks (1989).

A sequential sampling plan for the process mean

Driving sequential sampling plans for the mean of an IG process is the SPRT of Wald (1947). These tests pit one simple hypothesis against a second simple hypothesis. As an example, we may be concerned about a change in the process mean from a level μ_0 to a level μ_1 and hence desire to test the null hypothesis $H_0: \mu = \mu_0$ against the alternative hypothesis $H_1: \mu = \mu_1$ at significance level α . Further, we are willing to allow a shift in the mean from μ_0 to μ_1 to go undetected with probability β .

Standard acceptance sampling terminology refers to α and β as the producer's risk (PR) and the consumer's risk (CR), respectively, μ_0 is referred to as the producer's quality level (PQL), and μ_1 is referred to as the consumer's quality level (CQL). In general we will take the CQL to be either an upper or lower specification limit (USL or LSL) for the process parameter. If there is only one specification limit, then the sampling plan is referred to as a one-sided plan. If there are both upper and lower specification limits, then a two-sided plan is obtained by superimposing two one-sided plans.

Application of sequential sampling plans is based on the notion that the observations X_1, X_2, \dots are acquired one-at-a-time as is typically the case in life testing applications, making sequential sampling plans an attractive choice. When the n^{th} observation is made, $n = 1, 2, \dots$, one of three lot disposition decisions is made:

- (1) accept H_0 and, hence, accept the lot from which the sample originated;
- (2) reject H_0 in favour of H_1 and, hence, reject the lot from which the sample was obtained;
- (3) postpone the lot disposition decision until information from the following observation(s) is sufficiently compelling.

Though administration of sequential sampling plans tends to be more difficult than matched single sampling plans, they have the advantage that the average sample number (ASN) required to make a lot disposition decision is smaller. In reliability sampling this offers an advantage, since the cost of obtaining observations is often very high (Mann, Schafer, and Singpurwalla, 1974). Sequential sampling is often used when there are cost constraints, time limitations abound, and observations (e.g. failure times) occur one-at-a-time.

It will be assumed that the parameter λ is of known value, as might reasonably be the case if control charts have been used to monitor the process for a prolonged period. Let $R = (1 - \beta)/\alpha$ and let $A = \beta/(1 - \alpha)$ where it is assumed that $0 < \beta < 0.5 < (1 - \alpha) < 1$. Further, define L_0 and L_1 to be the joint

density functions of the random sample under H_0 and H_1 , respectively, and let the likelihood ratio, L_1/L_0 , be denoted by Λ . That is

$$L_0 = (\lambda^n / ((2\pi)^n \prod_{i=1}^n x_i^3)^{1/2}) \exp(-\lambda \sum_{i=1}^n (x_i - \mu_0)^2 / (2x_i \mu_0^2))$$

$$L_1 = (\lambda^n / ((2\pi)^n \prod_{i=1}^n x_i^3)^{1/2}) \exp(-\lambda \sum_{i=1}^n (x_i - \mu_1)^2 / (2x_i \mu_1^2))$$

and, after some simplification,

$$\Lambda = \exp[-(\lambda/2) \{ \mu_1^{-2} \sum_{i=1}^n (x_i - \mu_i)^2 / x_i - \mu_0^{-2} \sum_{i=1}^n (x_i - \mu_0)^2 / x_i \}]$$

After each observation is obtained the SPRT will conclude H_0 and accept the lot if $\Lambda \leq A$. Similarly, the SPRT will conclude H_1 and reject the lot if $\Lambda \geq R$. If $A < R$, then sampling will continue and a lot disposition decision will be postponed. Straight-forward algebraic manipulation leads to the following results.

- (1) Accept the lot if $Y \leq Y_1 = -h_1 + Sn$.
- (2) Reject the lot if $Y \geq Y_2 = h_2 + Sn$.
- (3) Continue to sample if $Y_1 < Y < Y_2$.

In this development, Y_1 and Y_2 are parallel lines that are referred to as the acceptance line and the rejection line, respectively. The intercepts of these lines are given by $-h_1$ and h_2 while S is the common slope of the two lines. Expressions for Y , S , h_1 , and h_2 follow.

$$Y = \sum_{i=1}^n x_i$$

$$S = 2\mu_0\mu_1 / (\mu_0 + \mu_1)$$

$$h_1 = \ln[1 - \alpha / \beta] \cdot (2\mu_0^2\mu_1^2) / (\lambda\{\mu_1^2 - \mu_0^2\})$$

$$h_2 = \ln[1 - \beta / \alpha] \cdot (2\mu_0^2\mu_1^2) / (\lambda\{\mu_1^2 - \mu_0^2\})$$

These results were developed for the case where $\mu_1 > \mu_0$, that is, where μ_1 is an USL. The solution set when $\mu_1 < \mu_0$ (that is, where μ_1 is a lower specification limit) is analogously derived and leads to the results:

- (1) Accept the lot if $Y \geq Y'_1 = h_1 + Sn$.
- (2) Reject the lot if $Y \leq Y'_2 = -h_2 + Sn$.
- (3) Continue to sample if $Y'_2 < Y < Y'_1$

where Y, S, h_1 and h_2 are as previously stated. The two-sided case simply superimposes the two one-sided plans, leading to a test of strength $(2\alpha, \beta)$. Clearly if a two-sided test of strength (α, β) is desired, the one-sided results should be developed by (numerically) replacing α with $\alpha/2$. Derivation of a sequential sampling plan for the mean of an IG process with λ unknown is a much more complicated situation and such results have yet to be derived.

Operating characteristic and average sample number curves

The operating characteristic (OC) curve for a sampling plan will pass approximately through the two points with co-ordinates $(\mu_0, 1 - \alpha)$ and (μ_1, β) . In general however, it is difficult to compute the OC curve directly. Fortunately, Wald (1947) provides a good approximation. If the probability of accepting a lot of items, given a process mean of μ is denoted by $P(\mu)$, then the OC curve passes through the point $(\mu, P(\mu))$. Again, only the case $\mu_1 > \mu_0$ is considered. Wald proved that for arbitrary μ that $P(\mu)$ is approximated by

$$P(\mu) = (R^{h(\mu)} - 1) / (R^{h(\mu)} - A^{h(\mu)}) \tag{1}$$

where $h(\mu)$ is the unique solution to the integral equation

$$\int_0^\infty f(x; \mu, \lambda) \cdot [f(x; \mu_1, \lambda) / f(x; \mu_0, \lambda)]^{h(\mu)} dx = 1$$

In the integral equation above, the terms $f(x; \mu, \lambda)$, $f(x; \mu_0, \lambda)$, and $f(x; \mu_1, \lambda)$ represent IG density functions with shape parameter λ and process means given by μ, μ_0 , and μ_1 , respectively. As derived by Edgeman and Salzberg (1991), the solution for $h(\mu)$ is given by

$$h(\mu) = [2\mu_1\mu_0/\mu - (\mu_0 + \mu_1)] / (\mu_1 - \mu_0)$$

To obtain a rough sketch of the OC curve, the values $h = \infty, 1, -1$, and $-\infty$ may be used. The corresponding points on the OC curve are $(0, 1)$, $(\mu_0, 1 - \alpha)$, (μ_1, β) , and $(\infty, 0)$. A fifth point that is near the centre of the OC curve has co-ordinates $(S, h_2/(h_2 + h_1))$ where

$$h_2 / h_2 + h_1 = \frac{\ln(1 - \beta) / \alpha}{\ln(1 - \beta) / \alpha + \ln((1 - \alpha) / \beta)}$$

The average sample number (ASN) is the expected number of observations required to make a lot disposition decision. Wald (1947) provides the following approximation to the ASN which depends on the true value of the process mean:

$$ASN(\mu) = [P(\mu) \cdot \ln(A) + (1 - P(\mu)) \cdot \ln(R)] / \epsilon_\mu(Z) \tag{2}$$

where $\epsilon_\mu(Z)$ is the mathematical expectation of Z and Z is given by

$$\begin{aligned}
 Z &= \ln[f(x; \mu_1, \lambda) / f(x; \mu_0, \lambda)] \\
 &= \ln\{\exp[-(\lambda/(2x)) \cdot \{(x - \mu_1)^2 / \mu_1^2 - (x - \mu_0)^2 / \mu_0^2\}]\} \\
 &= (\lambda/2)[(x\{(\mu_1^2 \mu_0^2) / (\mu_0 \mu_1)^2\} - 2(\mu_1 \mu_0) / (\mu_0 \mu_1))]
 \end{aligned}$$

Since the mathematical expectation of the random variable X is simply μ , we obtain the result

$$\varepsilon_\mu(Z) = (\lambda/2)[\mu(\mu_1^2 - \mu_0^2) / (\mu_0 \mu_1)^2 - 2(\mu_1 - \mu_0) / (\mu_0 \mu_1)]$$

which in turn may be substituted into (2) for $ASN(\mu)$. The ASN curve is then obtained by evaluating $ASN(\mu)$ for various values of μ .

An example

Suppose that we are interested in the time to failure of a particular very large scale integrated (VLSI) circuit. Jorgensen (1982) suggests that the reciprocal of time to failure can sometimes be modelled by an IG density with suitable values of μ and λ . It will be assumed that an appropriate goodness-of-fit test has been applied to historic data from the process and has determined that the IG density adequately models the reciprocal accelerated life test time to failure. Generally of concern for such phenomena would be a decrease in the mean time to failure (MTTF) and hence an increase in reciprocal MTTF.

Suppose that the PQL for an accelerated life test of the VLSI circuits is given by 32 hours to failure and that the CQL is 25 hours to failure. Further, let the agreed on values of the PR and the CR, that is, α and β respectively, be 0.05 and 0.10. Hence an increase in the reciprocal MTTF from $(1/32) = 0.03125$ to $(1/25) = 0.04$ will be allowed to go undetected with probability 0.10. Also, if the MTTF of the equipment is 32 hours, then a lot from the manufacturer will be accepted with probability 0.95. For the sake of illustration, it will be assumed that $\lambda = 0.1$.

In determination of the acceptance and rejection lines for the plan $\mu_1 = 0.04$ and $\mu_0 = 0.03125$. Substituting these values of μ_0 , μ_1 , λ , α , and β into the appropriate equations yields $S = 0.03509$, $h_1 = 0.11285$, $h_2 = 0.14488$, $Y_1 = -0.11285 + 0.03509(n)$, and $Y_2 = 0.14488 + 0.03509(n)$. Note also that the summation to be kept, Y , is the sum of the reciprocal accelerated life test times to failure.

To illustrate the determination of $P(\mu)$ and $ASN(\mu)$ for a specific value of μ , suppose that the mean accelerated life test time to failure is in fact equal to 20 hours, so that $\mu = (1/20) = 0.05$. Then $h(\mu) = -2.42857$ and substitution of this value into (1) yields $P(\mu) = 0.00422$. Similarly, $\varepsilon_{0.05}(Z)$ is easily found to be 0.2975 and thus $ASN(0.05)$ is found to be 9.64 when $\varepsilon_{0.05}(Z)$ is substituted into the equation for $ASN(\mu)$, equation (2). A more complete representation of the OC and ASN curves for specified α and β can be obtained by substitution of various values of μ into the expressions for $P(\mu)$ and $ASN(\mu)$.

A modified CUSUM control chart

The SPRT results previously derived lead, in effect, to an acceptance sampling plan. Cumulative sum (CUSUM) control charts can be thought of as "... (roughly) equivalent to the application of the sequential probability ratio test in reverse" (Johnson, 1961). Such charts allow for rapid identification of shifts in the process mean as compared to standard Shewhart control charts. In general CUSUM charts are good at detecting relatively small shifts in the process mean, say on the order of 0.5σ to 1.5σ , whereas standard Shewhart charts tend to be good for detecting shifts in the mean that are of the order of 2σ .

Traditional CUSUM control charts consist of a sequential plot to which a "V-mask" is applied point-by-point to assess the significance of the plot against a specified acceptable process level, μ_0 . A CUSUM chart cannot "accept" *per se*, but simply "does not reject" during continued sampling. Rejection occurs if any of the previously plotted sample points lie outside the angle defined by the "notch" of the V-mask when the vertex of the angle of the notch is positioned at a horizontal lead distance, d , from the most recently plotted point. The V-mask is determined by d and by the angle of the notch, 2θ , where

$$d = h_2/S$$

and

$$\tan(\theta) = S$$

These results assume that the vertical and horizontal scales of the chart are plotted 1:1. If the scales are plotted $k:1$ then d remains as previously stated, but the angle of the notch is adjusted such that

$$\tan(\theta) = S/k.$$

A variation of the traditional CUSUM chart which eliminates the need for the V-mask and simply plots with horizontal limits analogous to those of traditional Shewhart charts simply tabulates

$$Y'' = \sum_{i=1}^n (X_i - S)$$

If μ_1 is an USL, that is, if $\mu_1 > \mu_0$, then an out-of-control signal is detected when

$$Y'' \geq h_2$$

Alternatively, dividing each side of the preceding result by n after each observation is obtained, we can see that the criteria detects an out-of-control signal so that the chart becomes more sensitive as evidence accumulates. If $\mu_1 < \mu_0$ so that μ_1 is a LSL, then an out-of-control signal is detected if

$$Y'' \leq -h_2$$

If there are both upper and lower specification limits, equidistant from μ_0 , then an out-of-control signal is detected whenever $|Y''| > h_2$. The case where there are both upper and lower specification limits which are non-equidistant from μ_0 is not presented here but is derived in a straight-forward manner from previously stated results.

As a final note on CUSUM charts, tabulated values commonly do not involve individual values, X_i , but sample means, \bar{X}_i , each based on a random sample of m items from the process. Adjustments in the results presented earlier are trivial, with λ replaced by $m\lambda$ at each appearance. The impact of these substitutions are that h_1, h_2 and ASN are each divided by m , whereas $\epsilon_\mu(Z)$ is multiplied by m .

A simple goodness of fit test

Does the IG distribution adequately model process behaviour? Unless one is willing to naively plunge into process analysis this issue must be addressed. Numerous goodness-of-fit tests for the IG distribution have been developed, one of which will be highlighted in this section.

Let X_1, X_2, \dots, X_n be a random sample of n observations from the process and let \bar{X} and $1/\hat{\lambda}$ be determined using previously cited formulas. Without loss of generality, we will presume that these values are in ascending order. The goodness-of-fit test to follow is simply a Kolmogorov-Smirnov test modified to use parameter estimates (Edgeman, 1990a). An algorithm for the test follows.

- (1) For each data value determine:

$$D_i^+ = |i/n - \Phi(Z_i)| \text{ and } D_i^- = |(i-1)/n - \Phi(Z_i)|$$

where $\Phi(Z_i)$ is the standard normal cumulative probability distribution function evaluated using the standardized value

$$Z_i = (X_i - \bar{X}) / (\bar{X} \sqrt{\bar{X}} / \hat{\lambda})$$

- (2) Let D_n^+ be the largest of the D_i^+ and let D_n^- be the largest of the D_i^- .
- (3) Let L_n be the larger of D_n^+ and D_n^- .
- (4) Let $L_n^* = L_n (\sqrt{n} - 0.01 + 0.85/\sqrt{n})$
- (5) The IG distribution is rejected as an adequate process model if L_n^* surpasses the upper tail percentage point (UTPP) of the distribution of L_n^* at significance level α . A table of these percentage points for several values of α is presented below.

α : 0.150, 0.100, 0.050, 0.025, 0.010;

UTPP: 0.775, 0.819, 0.895, 0.995, 1.035.

Conclusions

The inverse Gaussian distribution has received much recent attention in quality and reliability engineering, in part because of the role that it plays in accelerated life testing applications where early failure times predominate. Given the amount of time such testing absorbs and costs that may border on prohibitive it is desirable to reach statistically sound lot disposition decisions or process control evaluations in a fashion that is both timely and economical. Sequential sampling plans and modified cumulative sum control charts support such goals. Foundational to these sampling plans and control charts is the sequential probability ratio test which makes possible such evaluations after each observation. The sequential probability ratio test for an inverse Gaussian process mean was developed in this paper and its applications to sequential sampling plans and modified cumulative sum control charts discussed. These results are of use to practising quality and reliability engineers who seek to be good stewards of stakeholder resources.

References

- Chhikara, R.S. and Folks, J.L. (1976), "Optimum test procedures for the mean of first passage time distribution in Brownian motion with positive drift (inverse Gaussian distribution)", *Technometrics*, Vol. 18 No. 2, pp. 189-93.
- Chhikara, R.S. and Folks, J.L. (1989), *The Inverse Gaussian Distribution: Theory, Methodology, and Applications*, Marcel Dekker, New York, NY, 1989.
- Edgeman, R.L. (1989), "Inverse Gaussian control charts", *Australian Journal of Statistics* Vol. 31 No. 1, pp. 78-84.
- Edgeman, R.L. (1990a), "Assessing the inverse Gaussian distribution assumption", *IEEE Transactions on Reliability*, Vol. R39 No. 3, pp. 352-5.
- Edgeman, R.L. (1990b), "An alternative to analysis of variance for reliability data", *Quality and Reliability Engineering International*, Vol. 6 No. 4, pp. 203-07.
- Edgeman, R.L. (1992), "Testing for differences in means of skewed processes", *International Journal of Modelling and Simulation*, Vol. 12 No. 1, pp. 20-21.
- Edgeman, R.L. and Salzberg, P.M. (1991), "A sequential sampling plan for the inverse Gaussian mean", *Statistische Hefte*, Vol. 32 No. 1, pp. 45-53.
- Johnson, N.L. (1961), "A simple theoretical approach to cumulative sum control charts", *Journal of the American Statistical Association*, Vol. 56 No. 4, pp. 835-40.
- Jorgensen, B. (1982), *Statistical Properties of the Generalized Inverse Gaussian Distribution*, Springer-Verlag, New York, NY.
- Mann, N., Schafer, R.E. and Singpurwalla, N.D. (1974), *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley, New York, NY.
- Tweedie, M.C.K. (1957), "Statistical properties of the inverse Gaussian distribution, 1", *Annals of Mathematical Statistics*, Vol. 28 No. 2, pp. 362-77.
- Wald, A. (1947), *Sequential Analysis*, John Wiley, New York, NY.