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Supersaturated design including an orthogonal base

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ABSTRACT

Recently, many supersaturated designs have been proposed. A supersaturated design is a fractional factorial design in which the number of factors is greater than the number of experimental runs. The main thrust of the previous studies has been to generate more columns while avoiding large values of squared inner products among all design columns. These designs would be appropriate if the probability for each factor being active is uniformly distributed. When factors can be partitioned into two groups, namely, with high and low probabilities of each factor being active, it is desirable to maintain orthogonality among columns to be assigned to the factors in the high-probability group. We discuss a supersaturated design including an orthogonal base which is suitable for this common situation. Mathematical results on the existence of the supersaturated designs are shown, and the construction of supersaturated designs is presented. We next discuss some properties of the proposed supersaturated designs based on the squared inner products.

RÉSUMÉ

Récemment, de nombreux schémas supersaturés ont été proposés. Un schéma supersaturé est un schéma factoriel fractionnaire dans lequel le nombre de facteurs est supérieur au nombre d'itérations expérimentales. La motivation principale des études précédentes a été de générer plus de colonnes tout en évitant une valeur élevée des carrés des produits scalaires de toutes les colonnes du schéma. Ces schémas seraient appropriés si la probabilité que chaque facteur soit actif était uniformément distribuée. Lorsque les facteurs peuvent être partitionnés en deux groupes, nommément les probabilités élevées et les probabilités faibles que chaque facteur soit actif, il est désirable de maintenir que l'orthogonalité entre colonnes soit assignée aux facteurs se trouvant dans le groupe à probabilité élevée. Dans cet article, nous discutons d'un schéma supersaturé incluant une base orthogonale adéquate à une situation aussi courante. Les propriétés mathématiques concernant l'existence des schémas supersaturée sont présentées. Nous discutons ensuite de quelques propriétés des schémas supersaturés proposés fondés sur les carrés des produits scalaires.

1. INTRODUCTION

In the initial stage of developing an industrial process, experimental studies based on factorial designs are often used. If the number of factors is less than the number of experimental runs, ordinary techniques of design of experiments, such as factorial design or fractional factorial design, are utilized to improve the process. This type of design guarantees that all the factor effects can be estimated. However, when the experiment is expensive and the number of factors is large, the ordinary techniques are not applicable (mainly due to the limitation of number of experimental runs). One approach in such a situation is an application of supersaturated designs under the *effect sparsity* assumption, namely, that only a few dominant factors actually affect the response. The supersaturated design is a fractional factorial design in which the number of factors is greater than the number of experimental runs, n. Some examples of the application of supersaturated designs are shown in Lin (1991, 1993a, 1995) and Wu (1993).

The supersaturated design was introduced by Satterthwaite (1959). He described a construction of supersaturated designs as a random balance design. Booth and Cox (1962) constructed supersaturated designs in a systematic manner. In their construction, some additional columns are sequentially added to the initial design while avoiding a large value of the squared inner products among all (initial and additional) columns. They obtained seven supersaturated designs via computer search. Lin (1993a) proposed a construction method for supersaturated designs via half fractions of Plackett-Burman (1946) designs. This method has the advantage of having a simple construction method, good properties for the squared inner products, and flexibility for n, viz., supersaturated designs are available for any even number of experimental runs. Wu (1993) proposed supersaturated designs by adding the cross products of two columns in the Plackett-Burman design. This method generates more columns than the method of Lin (1993a) when n is a multiple of four except when n is a power of two. Tang and Wu (1993) showed a method for constructing supersaturated designs while considering the average squared inner products. Furthermore, Lin (1995) examined the maximum number of columns that can be accommodated when the degree of nonorthogonality is specified by computer search. These proposed supersaturated designs would be convenient in practical usage when all factors have uniform probability (uncertainties) of a factor being active, because these designs ensure small squared inner products among columns.

Watson (1961) described a method of group screening based on the Bayesian approach. In order to determine the prior distribution, we need sufficient information about the existence of factor effects for the grouping scheme. The situation discussed here does not require such information, but does require prior information to partition factors into two groups, the groups with high and low probabilities of a factor being active. When we utilize a supersaturated design in such a situation, it is desirable to maintain orthogonality among columns to be assigned the factors in the high-probability group. Note that once the few dominant factors are identified, the initial design is then projected into a much smaller dimension. It is desirable that the projective (reduced) design be an orthogonal design. Thus, factors with a higher (prior) probability of being active are assigned to the orthogonal base in order to efficiently estimate their effects. This paper gives a new class of supersaturated designs including an orthogonal base. A method of construction is discussed based on the mathematical properties when n = 8, 12, 16, 20, and 24. Furthermore, a numerical comparison is made with previous supersaturated designs.

2. PROBLEM FORMULATION

In order to maintain orthogonality between the estimate of the constant term and the estimates of effects, we consider n-dimensional column vectors with equal occurrence. Equal occurrence implies that a column vector consists of equal numbers of -1 and $+1$. When n = 8, 12, 16, 20 and 24, there exist $n-1$ mutually orthogonal column vectors with the equal occurrence property. These are $n-1$ dimensional orthogonal bases in $\mathbb{R}^n$ which can be easily derived from the Hadamard matrix. A complete orthogonal base is given by adding the vector $\mathbf{1}$ (i.e., all elements are 1's) to the column vectors.

Let $\mathbf{C}_O^n = [\mathbf{c}_1 \ \cdots \ \mathbf{c}_{n-1}]$ be an $n \times (n-1)$ matrix consisting of $n-1$ mutually orthogonal column vectors with the equal occurrence property in $\mathbb{R}^n$. Let $\mathbf{C}_+^n = [\mathbf{c}_n \ \mathbf{c}_{n+1} \ \cdots \ \mathbf{c}_{n+k-1}]$

be an $n \times k$ matrix consisting of equal-occurrence vectors. The squared inner products s_{ij}^2 are given by $(c_i^{\mathrm{T}} c_j)^2$ $(i,j = 1,\ldots,n+k-1)$. Small squared inner products s_{ij}^2 $(i,j = 1,\ldots,n+k-1)$ are required in constructing supersaturated designs. The criteria of the average squared inner products

$$\mathcal{E}s_{ij}^2 = \binom{n+k-1}{2}^{-1} \sum_{i<j} s_{ij}^2$$

and of $\max\{s_{ij}^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\}$ will be used. Here, we apply the criterion of $\max\{s_{ij}^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\}$ for the construction, and the criterion of $\mathcal{E}s_{ij}^2$ for the evaluation. Let $\mathbf{C}^n$ be the set of n-dimensional equal-occurrence vectors. The problem of constructing a supersaturated design including an orthogonal base can be presented as that of constructing $\mathbf{C}_O^n$ and $\mathbf{C}_+^n$ from the set C^n under the constraint $\max\{s_{ij}^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\} = p^2$, where p is a prespecified nonorthogonality value.

3. A CONSTRUCTION METHOD

3.1. Construction of an Orthogonal Base.

Plackett and Burman (1946) obtained two-level orthogonal designs by an application of Hadamard matrices. Hadamard matrices can be found by many statistical software packages, such as MINITAB. For details, the reader can consult Hedayat and Wallis (1978). These designs are utilized to construct an orthogonal base for the supersaturated design when $n = 8$, 12, 16, 20, and 24.

3.2. Some General Properties for Construction.

Once $\mathbf{C}_O^n = [\mathbf{c}_1 \ \ldots \ \mathbf{c}_{n-1}]$ is obtained, the problem is how to construct an $n \times k$ matrix $\mathbf{C}_+^n = [\mathbf{c}_n \ \mathbf{c}_{n+1} \ \ldots \ \mathbf{c}_{n+k-1}]$ from the set C^n such that $\max\{s_{ij}^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\} = p^2$. Since n is an integer multiple of four, the squared inner product for any two vectors in C^n is in the set $\{0^2, 4^2, 8^2, \ldots, n^2\}$ (see Lin 1993b, Appendix). The following properties are useful in constructing $\mathbf{C}_+^n$.

PROPERTY 1. *There does not exist any vector* $\mathbf{c}$ *in* C^n *which satisfies* $\max\{(\mathbf{c}^{\mathrm{T}}\mathbf{c}_1)^2, (\mathbf{c}^{\mathrm{T}}\mathbf{c}_2)^2, \ldots, (\mathbf{c}^{\mathrm{T}}\mathbf{c}_{n-1})^2\} \leq 4^2$ *for* $n \geq 16$.

Proof of Property 1. Assume that there exists a vector $\mathbf{c}$ in C^n which satisfies $\max\{(\mathbf{c}^{\mathrm{T}}\mathbf{c}_1)^2, (\mathbf{c}^{\mathrm{T}}\mathbf{c}_2)^2, \ldots, (\mathbf{c}^{\mathrm{T}}\mathbf{c}_{n-1})^2\} \leq 4^2$. Since each squared inner product $(\mathbf{c}^{\mathrm{T}}\mathbf{c}_j)^2$ is equal to either 0^2 or 4^2, we get the inequality

$$\sum_{j=1}^{n-1} (\mathbf{c}^{\mathrm{T}}\mathbf{c}_j)^2 \leq 4^2(n-1). \tag{3.1}$$

On the other hand,

$$\sum_{j=1}^{n-1} (\mathbf{c}^{\mathrm{T}}\mathbf{c}_j)^2 = \mathbf{c}^{\mathrm{T}}\mathbf{C}_O^n\mathbf{C}_O^{n\mathrm{T}}\mathbf{c} = \mathbf{c}^{\mathrm{T}}(\mathbf{D} - \mathbf{1}\mathbf{1}^{\mathrm{T}})\mathbf{c} = n^2, \tag{3.2}$$

where $\mathbf{D}$ is a diagonal matrix whose diagonal elements are equal to n. We get an inequality $n^2 \leq 4^2(n-1)$ by comparing the inequality (3.1) with (3.2). When $n \geq 16$, this is a contradiction. □

Property 1 implies that some supersaturated designs ($n \geq 16$) obtained by Lin (1995) do not include an orthogonal base, even though these designs have minimum $\mathcal{E}s_{ij}^2$. From this property, we can only construct supersaturated designs for $p = 4$ when $n = 8$, 12 and for $p = 8$ when $n = 16$, 20, 24.

PROPERTY 2. *Let* $\mathbf{C}_O^n = [\mathbf{c}_1 \ \ldots \ \mathbf{c}_{n-1}]$ *and* $\mathbf{C}_+^n = [\mathbf{c}_n \ \mathbf{c}_{n+1} \ \ldots \ \mathbf{c}_{n+k-1}]$ *be an* $n \times (n-1)$ *matrix consisting of an orthogonal base and an* $n \times k$ *matrix consisting of the vectors in* C^n *which satisfy* $\max\{(\mathbf{c}_i^\mathsf{T}\mathbf{c}_j)^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\} = p^2$, *respectively. The matrix*

$$\begin{bmatrix} \mathbf{1} & \mathbf{C}_O^n & \mathbf{C}_O^n & \mathbf{C}_+^n & \mathbf{C}_+^n \\ -\mathbf{1} & \mathbf{C}_O^n & -\mathbf{C}_O^n & \mathbf{C}_+^n & -\mathbf{C}_+^n \end{bmatrix} = [\mathbf{c}_1^*, \ldots, \mathbf{c}_{2n+2k-1}^*] \tag{3.3}$$

has the following properties:

(i) $(\mathbf{c}_i^{*\mathsf{T}}\mathbf{c}_j^*)^2 = 0$ $(i,j = 1,\ldots,2n-1,\ i \neq j)$,
(ii) $\max\{(\mathbf{c}_i^{*\mathsf{T}}\mathbf{c}_j^*)^2\} = (2p)^2$ $(i,j = 1,\ldots,2n+2k-1,\ i \neq j)$.

Proof of Property 2. (i): For $i,j = 1,\ldots,2n-1$, $i \neq j$, it is sufficient to consider the inner product matrices

$$\begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_O^n \\ \mathbf{C}_O^n \end{bmatrix}, \quad \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{C}_O^n \\ -\mathbf{C}_O^n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{C}_O^n \\ \mathbf{C}_O^n \end{bmatrix} \begin{bmatrix} \mathbf{C}_O^n \\ -\mathbf{C}_O^n \end{bmatrix}.$$

Property 2(i) follows, since all elements of $\mathbf{1}^\mathsf{T}\mathbf{C}_O^n$ and $-\mathbf{1}^\mathsf{T}\mathbf{C}_O^n$ are equal to O and since $\mathbf{C}_O^{n\mathsf{T}}\mathbf{C}_O^n = nI_{n-1}$ and $-\mathbf{C}_O^{n\mathsf{T}}\mathbf{C}_O^n = -nI_{n-1}$, where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix.

(ii): It is sufficient to consider the inner-product matrix

$$\begin{bmatrix} \mathbf{C}_+^n \\ \mathbf{C}_+^n \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{C}_+^n \\ \mathbf{C}_+^n \end{bmatrix},$$

since other inner-product matrices can be examined the same way. The squared inner products in the matrix $\mathbf{C}_+^{n\mathsf{T}}\mathbf{C}_+^n$ are at most p^2. Thus, the squared inner products in the matrix considered are at most $(2p)^2$, from the relation

$$\begin{bmatrix} \mathbf{C}_+^n \\ \mathbf{C}_+^n \end{bmatrix}^\mathsf{T} \begin{bmatrix} \mathbf{C}_+^n \\ \mathbf{C}_+^n \end{bmatrix} = \mathbf{C}_+^{n\mathsf{T}}\mathbf{C}_+^n + \mathbf{C}_+^{n\mathsf{T}}\mathbf{C}_+^n. \quad \square$$

Note that the design in (3.3) will always maintain the maximum absolute correlation of the original design $[\mathbf{C}_O^n \ \mathbf{C}_+^n]$. Thus, most of the nice properties of $[\mathbf{C}_O^n \ \mathbf{C}_+^n]$ will be carried over.

Property 2 is useful for constructing the designs of cases $n = 16$ and 24, as will be shown below.

4. SOME RESULTS

4.1. Case $n = 8$.

For $n = 8$, the squared inner product for any two vectors in C^8 is in the set $\{0^2, 4^2, 8^2\}$. The case $p^2 = 8^2$ means completely opposite signs, so that we only need to consider $p^2 = 4^2$. The number of elements in C^8 is equal to

$$\binom{8}{4} = 70.$$

TABLE 1: Supersaturated design including an orthogonal base ($n = 8$). Columns 1–7: orthogonal base.

1	2	3	4	5	6	7
1	1	1	1	1	1	1
1	1	−1	−1	1	−1	−1
1	−1	1	−1	−1	1	−1
1	−1	−1	1	−1	−1	1
−1	1	1	1	−1	−1	−1
−1	1	−1	−1	−1	1	1
−1	−1	1	−1	1	−1	1
−1	−1	−1	1	1	1	−1

8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1
1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1
−1	−1	1	1	−1	−1	1	1	−1	−1	1	1	1	1
−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	−1	−1
1	1	1	1	1	1	1	1	1	1	1	1	−1	−1
−1	−1	1	−1	1	−1	−1	−1	1	−1	1	−1	−1	−1
1	−1	−1	−1	−1	1	1	−1	−1	−1	−1	1	−1	1
−1	1	−1	1	−1	−1	−1	1	−1	1	−1	−1	1	−1

22	23	24	25	26	27	28	29	30	31	32	33	34	35
1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1
1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	1
−1	−1	1	1	1	1	1	1	1	1	1	1	1	1
−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1
1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1
−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1
−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1

Each element in the set C^8 has an opposite-sign vector in C^8 such that $\mathbf{c}_i = -\mathbf{c}_j$. Thus, the number of elements under consideration is $70/2 = 35$. Table 1 shows the 35 vectors listed in a lexicographical order. The first seven columns are mutually orthogonal. This is equivalent to an 8-run Hadamard matrix. Any supersaturated design with 8 runs and 35 columns such that $\max\{s_{ij}^2 | i,j = 1,\ldots,35,\ i \neq j\} = 4^2$ is equivalent to the design shown in Table 1. As a matter of fact, the design given in Table 1 is equivalent to Tang and Wu's (1993) design with 8 runs, although their design was based on the minimization of $\mathcal{E}s_{ij}^2$.

4.2. Case $n = 12$.

Lin and Draper (1993), Wu (1993) and Iida (1994) pointed out interesting properties of the 12-run Plackett-Burman design and discussed applications of these properties to supersaturated designs. Specifically, the squared inner product between any combination of two cross-product column vectors of any two column vectors in the design is either 0^2 or 4^2. Because of this nice property, Wu (1993) proposed 11+55-column supersaturated designs, where 55 columns are derived from the cross products of all pairs of two columns in the $n = 12$ Plackett-Burman design. As a matter of fact, the supersaturated designs with 66 columns include 11 columns as an orthogonal base. The proposed design would be appropriate from the viewpoint of $\max\{s_{ij}^2 | i,j = 1,\ldots,66,\ i \neq j\}$, because

$\max\{s_{ij}^2 | i,j = 1,\ldots,66,\ i \neq j\} = 4^2$ and it contains 66 columns. There is no room for further improvement in this case (see Tang and Wu 1993).

4.3. Case $n = 16$.

Property 1 indicates the nonexistence of a supersaturated design such that max $\{s_{ij}^2 | i,j = 1,\ldots,n+k-1,\ i \neq j\} = 4^2$. We thus focus on the supersaturated design for $p = 8$. For simplicity, we only consider designs constructed by Property 2. A brute-force computer search may yield more design columns, but the computing time is apparently infeasible. Besides, the results obtained here are somewhat satisfactory (16 runs and 71 columns).

From Property 2, supersaturated designs with $2n$ runs and $2n+2k-1$ columns can be generated from a supersaturated design with n runs and $n+k-1$ columns. In addition, Property 2(i) ensures that the orthogonality in the original design is maintained, and Property 2(ii) ensures that the maximum squared inner products are equal to $(2p)^2$. A supersaturated design with 16 runs, 71 columns and $p = 8$ can be constructed by an application of Property 2 via the supersaturated design with 8 runs and 35 columns shown in Table 1. Specifically, let $\mathbf{C}_O^8$ and $\mathbf{C}_+^8$ be the first to seventh columns and the eighth to thirty-fifth columns in Table 1, respectively. A supersaturated design with 16 runs and 71 columns can be obtained by substituting $\mathbf{C}_O^8$ and $\mathbf{C}_+^8$ into Equation (3.3).

4.4. Case $n = 20$.

From Property 1, there does not exist a supersaturated design with 20 columns and $p = 4$. An orthogonal base is listed as columns A to S in Table 2 for the Plackett-Burman design with 20 runs. Other columns are derived in the following way. Let $\mathbf{C}_*^8$ be an 8×35 matrix as derived in Section 4.1. We put the matrix into the first to thirty-fifth columns from the first to eighth rows of the matrix $\mathbf{C}_+^{20}$. Other elements in $\mathbf{C}_+^{20}$ are chosen from the matrix $\mathbf{C}_+^{12}$ discussed in Section 4.2. There are 66 candidate columns to put into the remaining part of matrix $\mathbf{C}_+^{20}$. When the 35 column vectors shown in the lower part of Table 2 are chosen for $\mathbf{C}_+^{20}$, all squared inner products are less than or equal to 8^2. There are many other possibilities for these 35 columns, of course. Table 2 presents one specific choice.

4.5. Case $n = 24$.

A supersaturated design with 24 runs and 133 columns can be obtained by the application of a supersaturated design with 12 runs and 66 columns using Property 2. This is the same approach used in the case of $n = 16$. The constructed design satisfies $\max\{s_{ij}^2 | i,j = 1,\ldots,133,\ i \neq j\} = 8^2$.

5. EVALUATION BASED ON THE SQUARED INNER PRODUCTS

5.1. Distributions of the Squared Inner Products.

For $n = 8$, the proposed supersaturated design has 7 columns for $\mathbf{C}_O^8$ and 28 columns for $\mathbf{C}_+^8$. The columns in $\mathbf{C}_O^8$ are mutually orthogonal: therefore, our interest is in the frequencies of 0^2 and 4^2 in the inner-product matrices $\mathbf{C}_O^{8\,\mathrm{T}}\mathbf{C}_+^8$ and $\mathbf{C}_+^{8\,\mathrm{T}}\mathbf{C}_+^8$. There are 196 elements in the matrix $\mathbf{C}_O^{8\,\mathrm{T}}\mathbf{C}_+^8$; among them, 84 are 0 and 112 are ± 4. Furthermore, there are 378 [$= 28 \times (28-1)/2$] elements in the inner-product matrix $\mathbf{C}_+^{8\,\mathrm{T}}\mathbf{C}_+^8$. Among

TABLE 2: Supersaturated design including an orthogonal base ($n = 20$). Columns A–S: orthogonal base.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1
−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1
1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1
1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1
−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1
−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1
−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1	−1
−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1	1
1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1	−1
−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1	1
1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1	−1
−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1	1
1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1	1
1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1	1
1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	1
1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1
−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1	−1
−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1	1
1	−1	−1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	1
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	−1
1	1	−1	−1	1	−1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	1
1	−1	1	−1	−1	1	−1	−1	−1	1	1	−1	−1	1	1	−1	−1	1
1	−1	−1	1	−1	−1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1
−1	1	1	1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	1
−1	1	−1	−1	−1	1	1	−1	−1	1	−1	1	−1	−1	−1	1	−1	1
−1	−1	1	−1	1	−1	1	1	−1	−1	−1	−1	1	1	−1	−1	−1	−1
−1	−1	−1	1	1	1	−1	−1	1	−1	1	−1	−1	−1	1	−1	1	−1
1	1	1	1	−1	1	−1	−1	−1	−1	1	−1	−1	1	1	−1	1	1
1	1	1	−1	1	−1	1	1	−1	−1	−1	1	1	−1	−1	−1	1	−1
1	1	−1	1	1	−1	−1	−1	1	1	1	−1	1	−1	−1	1	−1	1
1	−1	−1	−1	1	1	1	−1	1	−1	−1	1	1	1	1	−1	1	1
1	−1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1	−1	1	−1	1
1	−1	−1	1	−1	−1	1	1	−1	1	1	1	−1	−1	1	1	1	−1
−1	1	−1	1	−1	1	1	1	1	−1	−1	−1	1	−1	1	1	−1	1
−1	1	1	−1	−1	−1	1	−1	1	1	1	−1	1	1	−1	−1	1	−1
−1	1	−1	−1	1	1	−1	1	−1	1	1	1	−1	−1	−1	−1	−1	−1
−1	−1	1	1	1	−1	−1	1	1	−1	−1	1	−1	1	−1	1	−1	−1
−1	−1	1	1	1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	−1
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	1	1	1	1	1	1	1	1

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
−1	1	1	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	−1
1	1	1	1	1	1	1	1	1	−1	−1	−1	−1	1	1	1	1
1	1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	1	1
−1	−1	−1	−1	−1	1	1	1	1	1	1	1	1	1	1	1	1
1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1
−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1
1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1
−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1	1	−1	−1	−1

TABLE 2: *Concluded*

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
−1	1	−1	1	1	−1	−1	1	−1	−1	−1	−1	1	−1	−1	1	−1
−1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1	1	−1	1	−1
−1	−1	1	−1	−1	−1	−1	1	1	1	−1	−1	1	1	1	−1	−1
−1	−1	−1	−1	1	−1	1	−1	1	1	−1	1	−1	−1	−1	−1	−1
1	−1	1	−1	−1	1	−1	1	−1	1	−1	1	−1	−1	1	1	1
1	−1	1	1	−1	−1	−1	1	−1	−1	1	−1	−1	−1	−1	1	1
−1	1	−1	1	−1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	1
1	1	−1	−1	−1	1	−1	−1	1	−1	−1	1	1	1	1	−1	−1
1	−1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	−1	−1	1
1	−1	−1	1	1	1	−1	−1	1	1	1	−1	1	−1	1	1	−1
−1	1	−1	−1	1	−1	1	1	−1	−1	1	1	−1	1	1	−1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

TABLE 3: The distributions of the squared inner product. (Upper number: number of squared inner products. Lower number: percentage.)

		$C_O^T C_+$				$C_+^T C_+$			
$n-1$	k	Total	0	4^2	8^2	Total	0	4^2	8^2
7	28	196	84	112	—	378	210	168	—
			0.43	0.57	—		0.56	0.44	—
11	55	605	110	495	—	1485	495	990	—
			0.18	0.82	—		0.33	0.67	—
15	56	840	616	—	224	1540	1204	—	336
			0.73	—	0.27		0.78	—	0.22
19	35	665	222	299	144	595	209	299	87
			0.33	0.45	0.22		0.35	0.50	0.15
23	110	2530	1540	—	990	5995	4015	—	1980
			0.61	—	0.39		0.67	—	0.33

them, 210 are 0 and 164 are ± 4. We studied the cases of $n = 12$, 16, 20 and 24 in the same manner. The result is shown in Table 3.

There is no entry -4 or 4 for the cases $n = 16$ and 24, since these designs are derived by Property 2. Generally, when a supersaturated design with $2n$ runs and $2n+2k-1$ columns is constructed from a supersaturated design with n runs and $n+k-1$ columns through Property 2, the value of inner products between any two columns in the constructed design is either 0 or twice the original value of the inner products. That is, the maximum absolute value of the correlation coefficient between any two columns is the same, since the number of runs is doubled.

An interesting point here is the difference of the distributions between $n = 8, 16$ and $n = 12, 24$. Specifically, unlike the cases $n = 8$ and 16, the appearance of 0 is less frequent than that of 4^2 or 8^2 when $n = 12$ or 24. Such a difference may be regarded as a difference of usage of Euclidian space.

5.2. *Comparison with Previous Designs.*

Table 4 summarizes supersaturated designs of previous studies and this paper from

TABLE 4: Comparison with previous supersaturated designs. (Upper number: number of columns. Lower number: $\mathcal{E}s_{ij}^2$.)

	Lin (1993a)	Wu	Tang & Wu	The authors
8	—[a]	—[a]	35	35
			7.00	7.00
12	22	66	110	—[a]
	6.86	11.08	11.89	
16	—[a]	—[a]	45	71
			11.64	14.42
20	—[a]	—[a]	57	54
			14.29	17.02
24	46	30	69	133
	12.80	9.27	14.77	21.65

[a] No design is available.

the viewpoint of n, the number of generated columns, and the average squared inner product $\mathcal{E}s_{ij}^2$. Note that the proposed method has some similarity to the work of Tang and Wu (1993). Table 5 shows a detailed comparison with Tang and Wu's designs. It can be easily seen that Tang and Wu's designs are superior in terms of the $\mathcal{E}s_{ij}^2$ criterion, while the proposed design has a smaller $\max\{s_{ij}^2\}$. Moreover, the proposed design can accommodate more columns, as shown.

For a fair comparison, Table 5 also gives the detailed results for the submatrix in Tang and Wu (1993) of the proposed design which contains the same number of columns. As expected, Tang and Wu's design preforms well under the $\mathcal{E}s_{ij}^2$ criterion. Note also that the proposed designs contain more orthogonal pairs, which is a desirable property, as previously discussed. As a result, the proposed designs perform well in terms of the criterion

$$\mathcal{E}|s_{ij}| = \binom{n+k-1}{2}^{-1} \sum_{i<j} |s_{ij}|.$$

As compared to the $\mathcal{E}(s_{ij}^2)$ criterion, such a criterion suffers a smaller penalty from the large inner-product value.

6. CONCLUDING REMARKS

The supersaturated designs discussed in this paper have a very different aspect from other existing supersaturated designs such as Lin (1993a, 1995), Wu (1993) and Tang and Wu (1993). Specifically, previous supersaturated designs would be appropriate if there is uniform probability of each factor being active, while the designs given here would be appropriate where the factors can be partitioned into two groups, depending on their probabilities of existence of factor effects.

Concerning data analysis, the variable selection procedures in regression analysis, such as stepwise selection and ridge regression, are applied to the data collected by supersaturated designs [see Lin (1991, 1993a, 1995) and Wu (1993)]. What type of data analysis methods are most appropriate for the supersaturated designs given here is a subject for future research. For a recent article on the analysis issue, see Westfall *et*

TABLE 5: Detailed comparison with Tang and Wu's design.

	$n = 16$			$n = 24$		
	Tang & Wu	The authors		Tang & Wu	The authors	
No. of columns:	45	45	71	69	69	133
No. of pairs:	990	990	2485	2346	2346	8778
Distribution:						
0	540	780	1925	1249	1668	5808
4	360	0	0	722	0	0
8	90	210	560	324	678	2970
12				50		0
16				1		
$\mathcal{E}s_{ij}^2$:	11.64	13.58	14.42	16.94	18.50	21.65
$\mathcal{E}\lvert s_{ij}\rvert$:	2.18	1.70	1.80	2.60	2.31	2.71
max s_{ij}^2 :	8^2	8^2	8^2	16^2	8^2	8^2

al. (1995). Also, it is believed that there are possibilities for improving our results via brute-force computation.

It has come to our attention since the submission of this paper that Nguyen (1996) describes a method of constructing supersaturated designs from balanced incomplete block designs. Since his algorithm mainly deals with the $\mathcal{E}s_{ij}^2$ criterion, we believe that the designs given there should perform well in terms of $\mathcal{E}s_{ij}^2$.

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