Connections Between Two-Level Factorials and Venn Diagrams

Dennis K. J. LIN and Amy W. LAM

The Venn diagram is widely used in many textbooks to illustrate relationships in logic, algebra and probability events. Most applications are limited to $k = 2$ or 3 sets. Attempts have been made to construct Venn diagrams for many sets. This paper illustrates the connection between Venn diagrams and the well-known two-level factorial designs. Such a connection provides: (1) a simple way to construct a Venn diagram for any k sets, and (2) a graphical understanding of two-level factorials. Examples are given for $k = 2$, 3, 4, and 5.

KEY WORDS: Design point; Event; One-factor-at-a-time experiments; Run order.

1. INTRODUCTION

One of the most important tools to describe event operations in elementary probability theory is the Venn diagram. Following conventional notations the events are denoted by $A, B, C, \ldots$; the intersection of any two events A and B is denoted by AB; the intersection of any three events A, B, and C is denoted by ABC; and so forth. Furthermore, we define lower case $a = A\backslash$ (all intersections of two or more sets), namely, the set of elements in set A, but not in the intersection of A with any other set or sets, $ab = AB\backslash$ (intersections of three or more sets), and so on. The Venn diagram then partitions the sample space into 2^k disjoint regions. For $k = 2$, 3, and 4 these 2^k regions are

$$k=2:\ \{O, a, b, ab\} \qquad (1)$$

$$k=3:\ \{O, a, b, c, ab, ac, bc, abc\} \qquad (2)$$

$$k=4:\ \{O, a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\} \qquad (3)$$

where O denotes the part that does not intersect with any of these events.

On the other hand, a two-level factorial design is one in which k factors, labeled $(A, B, C, \ldots, K)$, are each allocated to two levels, conventionally called "high level" and "low level," and every possible combination of the levels is a run. We follow the classical notation for two-level designs. The high level of each factor is indicated by the presence of the corresponding lower case letter in a symbol; the low level is indicated by the absence of that letter. For example, in a three-factor experiment (factors A, B, and C) ab means that a experiment is to be run at the high level of the two factors A and B and the low level of C. The symbol (1) means an experiment with all factors at their low levels (see, for example, John 1971). For $k = 2$, 3, and 4 these 2^k runs then can be represented as

$$k=2:\ \{(1), a, b, ab\}$$

$$k=3:\ \{(1), a, b, c, ab, ac, bc, abc\}$$

$$k=4:\ \{(1), a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd\}.$$

Symbolically, these are, of course, (1), (2), and (3), respectively.

2. CONSTRUCTION METHOD AND EXAMPLES

As mentioned, the Venn diagram partitions the sample space into 2^k regions. More precisely, as pointed out by a referee, these regions must be convex, such as rectangle-type. Given such a Venn diagram for k events one can construct a Venn diagram for $k + 1$ events by drawing a continuous line that passes through each of these 2^k region exactly once, and finally returns to the starting point. Note that the notations for any two adjacent regions differ by exactly one symbol (i.e., they have one more or one less additional symbol), but are otherwise identical. For example, bc and abc are adjacent regions, but a and bc are not. (See Figures 1–3, discussed below.) In terms of two-level factorial designs this is equivalent to running the 2^k runs by altering the factor levels one at a time, that is, varying only one factor from the condition of the last preceding run (Daniel 1973). Thus a Venn diagram for $k + 1$ events can be obtained by drawing a line following the run-order sequence of a one-at-a-time 2^k design on the Venn diagram for k events.

One-at-a-time run-order sequences for 2^k design are given below for $k = 1$, 2, 3, and 4.

$$k=1:\ (1) \to a$$

$$k=2:\ (1) \to a \to ab \to b$$

$$k=3:\ (1) \to a \to ab \to b \to bc \to abc \to ac \to c$$

$$k=4:\ (1) \to a \to ab \to b \to bc \to abc \to ac \to c \to cd \to acd \to abcd \to bcd \to bd \to abd \to ad \to d.$$

These can be used to construct the Venn diagrams for $k + 1 = 2$, 3, 4, and 5, respectively, as shown in Figures 1–3. For example, to construct the Venn diagram for $k = 4$ (shown in Figure 2) one can draw a path in the order of a one-at-a-

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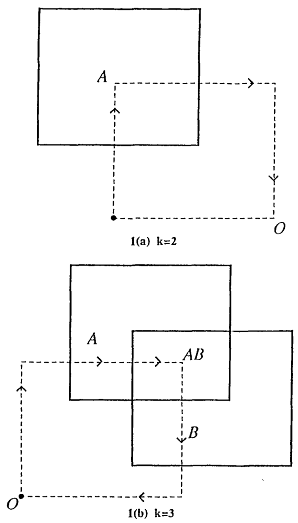

Figure 1. Constructing Venn Diagrams for k = 2 and k = 3.

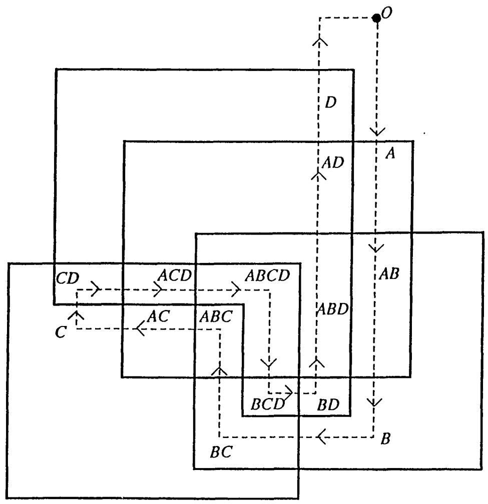

Figure 3. Constructing Venn Diagram for k = 5.

time sequence for $k = 3$ given above to the Venn diagram of $k = 3$. Namely, based on the Venn diagram formed by the solid lines for $k = 3$ one begins with point O, makes a path (indicated by the dotted line) to region A, then region AB, region $B, \ldots,$ finally region C, and then returns to point O. Such an algorithm ensures that each partition is included exactly once. In addition, one can always collapse the proper boundaries to eliminate any specific region for an empty intersection if so desired.

In general, the sequence can be generated by induction (as is the Venn diagram) as follows. Let X be the run-order sequence for $k = N - 1$ and let X^* be its reverse sequence. Then the run-order sequence for $k = N$ is formulated by (X, X^*N), where X^*N means attaching the symbol "N" to all elements of X^*. Thus for $k = 5$ the sequence is as follows:

$$k = 5\colon\ (1), a, ab, b, bc, abc, ac, c, cd, acd, abcd, bcd, bd, abd, ad, d, de, ade, abde, bde, bcde, abcde, acde, cde, ce, ace, abce, bce, be, abe, ae, e.$$

This essentially proves the existence of one-at-a-time run-order sequence. Consequently, one can construct the Venn diagram for any k.

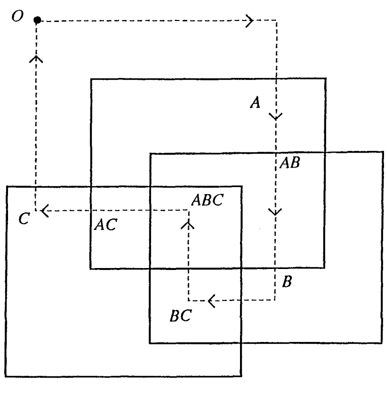

Figure 2. Constructing Venn Diagram for k = 4.

3. UNDERSTANDING TWO-LEVEL FACTORIALS VIA VENN DIAGRAMS

We now discuss how Venn diagrams give a useful graphical understanding of two-level factorials. Two important issues in the 2^k design are of particular interest: run order and blocking.

3.1 Run Order of the 2^k Design

The standard advice given to the experimenters in using 2^k design is that the order of the run should be randomized before the design is performed. Randomization of the runs, however, can lead to unfavorable sequences. For example, certain input factor combinations may be difficult to alter or certain main effects can be highly confounded with a time trend; see Cheng (1985). In certain circumstances, therefore, it may be desirable to adopt a specific run order. Two particular useful sequences that have received a great deal of attention in the design literature are: (1) the (time) trend-free sequence, and (2) the one-at-a-time sequence. Although the trend-free sequence is orthogonal (unconfounded) to a prespecified time trend (linear trend is most likely to be assumed here), the one-at-a-time sequence is shown to be most economical (Daniel 1973). The latter sequence is particularly useful when the time trend itself is not fully obvious.

Constructing a one-at-a-time sequence is a natural application of the Venn diagram because any path that goes through each region exactly once will result in a one-at-a-time sequence. The specific sequences given in Section 2 are special cases. For example, for the case $k = 3$ there are precisely three intrinsically different one-at-a-time sequences: (1)-*a*-*ab*-*b*-*bc*-*c*-*ac*-*abc*, (1)-*a*-*ab*-*b*-*bc*-*abc*-*ac*-*c*, and (1)-*a*-*ab*-*abc*-*ac*-*c*-*bc*-*b*. Without loss of generality we begin all sequences with (1). The multiple correlations of these three sequences with linear time trend are .14, .24, and .43, respectively. Although not perfectly trend-free, many one-at-a-time sequences are reasonably robust (low correlation) to the linear trend. A computer algorithm for generating all possible one-at-a-time sequences is given in Lin (1994).

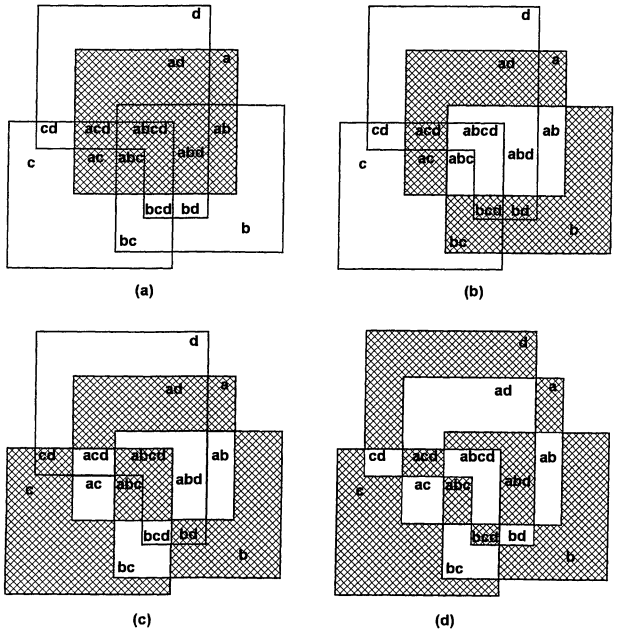

Figure 4. Venn Diagrams for k = 4. (a) Main effect A as the blocking variable. (b) Two-factor interaction AB as the blocking variable. (c) Three-factor interaction ABC as the blocking variable. (d) Four-factor interaction ABCD as the blocking variable.

3.2 Blocking 2^k Designs

With the help of Venn diagrams we can also see how to block 2^k designs into two blocks. Consider the case $k = 4$ for illustration. We discuss how to block on a main effect A, a two-factor interaction AB, a three-factor interaction ABC, and the four-factor interaction $ABCD$. Of course, the last choice is the sensible one. Figure 4 illustrates these four possible blocking schemes. Runs from those two different blocks are represented by different colors (blank and shaded).

(1) Factor A as the blocking variable. All regions associated with A form one block, while the other block consists of the other eight regions not associated with A. These blocks, $(a, ab, abd, abcd, abc, ac, acd, ad)$ and $((1), b, bd, bcd, bc, c, cd, d)$ are shown in Figure 4a. If the run order is considered within the block, the eight runs in Figure 4a may form a one-at-a-time sequence, for example, *a-ab-abd-abcd-abc-ac-acd-ad*.

(2) Interaction AB as the blocking variable. If two-factor interaction is used as the blocking variable, we see from Figure 4b that each block consists of eight runs that form two separate groups of one-at-a-time sequences, for example, *a-ad-acd-ac* and *b-bd-bcd-bc*.

(3) Interaction ABC as the blocking variable. Similar to the above case, the eight runs in both blocks form four separate pairs of one-at-a-time sequences, for example, *a-ad*, *b-bd*, *c-cd*, *abc-abcd* as shown in Figure 4c.

(4) Interaction $ABCD$ as the blocking variable. The regions in the Venn diagram fall into several layers. Figure 4d shows that the five layers, layers 1–5 say, correspond to the null set (O), the pure sets (A, B, C, D), the two-set intersections (AB, AC, AD, BC, BD, CD), the three-set intersections (ABC, ABD, ACD, BCD), and the four-set intersection $(ABCD)$. We use the even layers for one block and the odd layers for the other. The eight runs in the even layers are $a, b, c, d, abc, abd, acd$, and bcd. This is the optimal blocking scheme using $ABCD$ interaction as the blocking variable (Box, Hunter, and Hunter 1978). The eight runs in either block cannot be derived from one run to the other via one-at-a-time, however.

Of course, from a blocking perspective case 4 is more sensible than case 3, than case 2, than case 1. This seems to indicate (see Figure 4) that a better blocking scheme is to isolate as many runs from one block to another as possible.

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