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Screening Properties of Certain Two-Level Designs

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Abstract: Two-level screening designs are appropriate for situations where a large number of factors (q) is examined but relatively few (k) of these are expected to be important. It is not known *which* of the q factors will be the important ones, that is, it is not known which k dimensions of the experimental space will be of further interest. After the results of the design have received a first analysis, the design will be projected into the k dimensions of interest. These projections are investigated for Plackett and Burman type-screening designs with $q \leq 23$ factors, and $k = 3, 4$, and 5.

1 Introduction

Suppose we have a "large" number (q) of factors (or inputs) $x_1, x_2, \ldots, x_q$, say, to examine in an experimental situation. Typically, many possible factors are suggested for investigation, but it is often anticipated that only a "small" subset of these (k, say) will be "real" or "effective". This usually means that it is believed that some sizeable main effects or two factor interactions may exist, but that three- or higher-order-factor interactions are unlikely to occur. Thus, it is believed that a model for the response variable y of form

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \beta_{12} x_1 x_2 + \cdots + \beta_{k-1,k} x_{k-1} x_k + \varepsilon \tag{1}$$

is appropriate, where k is much smaller than q and, furthermore, that perhaps even a smaller subset of the specific terms in (1) are actually needed to describe the behaviour of the response. A design suitable for screening out the k relevant factors from the q total factors is called a screening design. See Box, Hunter and Hunter (1978, pp. 545–546).

Screening designs are typically used in the initial stages of an experimental investigation. (Sometimes, several responses are measured in each experiment.) Because of their relative simplicity of use, two-level screening designs are very popular in practice. For example, the 2^{q-p} fractional factorial designs (that is, a 2^{-p} fraction of a 2^q two-level factorial design; see Box, Hunter, and Hunter, 1978) and the Plackett and Burman (1946) designs are widely used. When such an

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n-run screening design is employed, it is not expected that every factor will show up as important, merely a subset. This permits the use of fractionated designs with complicated alias structures. After the initial analysis, the whole design is then projected into a lower dimensional space which contains only the k apparently important factors.

In this paper, we discuss the projection properties of these two-level designs. This knowledge allows us to see what additional runs can be of value, after the results of the initial screening are available. It also provides some insight into how to allocate the design variables to the factors that are thought, a priori, to be important. The well-known 2^{q-p} series is briefly discussed in Section 2. In Section 3, projections of the Plackett and Burman design into three dimensions ($k = 3$) are discussed. Section 4 discusses considerations in adding runs. The extensions to $k = 4$ and 5 are given in Sections 5 and 6. Non-equivalent Hadamard matrices of sizes 16 and 20 are discussed in Section 7.

2 Projections of 2^{q-p} Designs

When a 2^{q-p} screening design is used, all projections are either standard two-level full factorials or fractional factorials. For $k = 2$ and $n \geq 4$, the projection is always a 2^2 design, $n/4$ times over. For $k = 3$ and $n \geq 8$, there are two types of projections: A 2^3 design, $n/8$ times over, or a 2_{III}^{3-1} design ($I = \pm 123$), $n/4$ times over. (We employ throughout the standard notation introduced by Box and Hunter (1961) in which I represents an n-run column of plus signs and, for example, 123 represents a column of signs determined by taking the product of the signs $\pm$ in columns 1, 2, and 3 of the two-level factorial design, where $-$ and $+$ denote the two levels of the factor allocated to any column.)

For $k = 4$ and $n \geq 16$, there are three possibilities: A 2^4 design, $n/16$ times over, or a 2_{IV}^{4-1} design ($I = \pm 1234$), $n/8$ times over, or a 2_{III}^{4-1} design ($I = \pm 123$), $n/8$ times over.

For $k = 5$, the possibilities for the projected designs are given in Table 1.

The extension to $k \geq 6$ is similar and straightforward. However, if the projected design remains resolution III, estimated main effects are confounded with

Table 1. Projections of a 2^{q-p} design into five dimensions

Type	Generators	Times over
2^5	—	$n/32$
2_V^{5-1}	$I = \pm 12345$	$n/16$
2_{IV}^{5-1}	$I = \pm 1234$	$n/16$
2_{III}^{5-1}	$I = \pm 123$	$n/16$
2_{III}^{5-2}	$I = \pm 124 = \pm 1235$	$n/8$

two-factor interactions. The usual advice given in such circumstances to eliminate such blurring is to "fold over" the design (Box, Hunter, and Hunter, 1978, pp. 340, 399), that is, repeat the projected design with all signs reversed. Foldover always converts a resolution III design into a resolution IV design (see Box, Hunter, and Hunter, 1978, p. 398). It also doubles the size of the experiment, however, which can be disadvantageous.

3 Plackett and Burman Screening Designs

Table 2 shows a 12-run Plackett and Burman design, obtained as follows.

(a) Write down the set of signs $+ + - + + + - - - + -$, provided by Plackett and Burman (1946).
(b) Permute the signs in 11 rows total, by taking the sign from the right hand side and moving it to the left hand side.
(c) Add a 12th row of all minus signs.

For $n \leq 24$, all of the Plackett and Burman designs can be obtained by such a cyclic permutation. The signs for the first rows are:

$n = 8$: $+ + + - + - -$

$n = 12$: $+ + - + + + - - - + -$

$n = 16$: $+ + + + - + - + + - - + - - -$

$n = 20$: $+ + - - + + + + - + - + - - - - + + -$

$n = 24$: $+ + + + + - + - + + - - + + - - + - + - - - -$.

For $n = 8$ and 16, we obtain a standard 2^{q-p} design, so these cases are covered by Section 2. For the projection of the 12-run design in any k of the 11 factor dimensions, we select k columns and examine the design that results by ignoring the other $11 - k$ columns. For example, suppose $k = 3$ and we select the 1, 2, 3 columns. The reduced 12-point design consists of a 2^3 design plus a 2^{3-1} design with $I = -123$, shown in Figure 1. This very desirable arrangement provides complete coverage of all the factorial effects plus additional pure error information obtained at four different locations well spread out over the experimental region. Moreover, *no matter which three factors are designated as the survivor columns*, a similar design is always obtained, that is, a 2^3 plus a 2^{3-1} with

Table 2. A 12-run Plackett and Burman design

Run	Factors										
No.	1	2	3	4	5	6	7	8	9	10	11
1	+	+	−	+	+	+	−	−	−	+	−
2	−	+	+	−	+	+	+	−	−	−	+
3	+	−	+	+	−	+	+	+	−	−	−
4	−	+	−	+	+	−	+	+	+	−	−
5	−	−	+	−	+	+	−	+	+	+	−
6	−	−	−	+	−	+	+	−	+	+	+
7	+	−	−	−	+	−	+	+	−	+	+
8	+	+	−	−	−	+	−	+	+	−	+
9	+	+	+	−	−	−	+	−	+	+	−
10	−	+	+	+	−	−	−	+	−	+	+
11	+	−	+	+	+	−	−	−	+	−	+
12	−	−	−	−	−	−	−	−	−	−	−

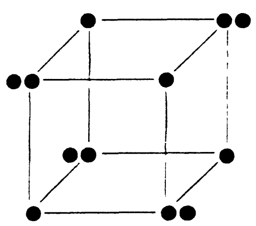

Fig. 1. A 12-run, 11 factor Plackett and Burman design projected into (any) three dimensions

$I = \pm ABC$ where A, B, C represent any three of the eleven factors. See, also, Box and Bisgaard (1993).

For $k = 3$, and $n = 20$, two types of projections can occur:

1. A 2 : 3 type. (This means two full 2^3 factorials and an additional 2^{3-1}. At the corners of the cube there are either two or three points.)
2. A 1 : 4 type. (This means a 2^3 factorial and three identical 2^{3-1} designs. At each corner of the cube there is either one point or there are four points.)

These and other possibilities for $k = 3$ are illustrated in Figure 2. The notation "$(r : s)$" means r points lie at four of the "2^{3-1} locations" and s points lie at the other four. 2^{3-1} locations are always defined by $I = \pm$ the relevant three factor interaction. Another point to note is that, in some cases, we can proceed from a three column n runs projection to a three column $(n + 4)$ runs projection by simply adding a 2^{3-1} design. For $n = 20$, a (2 : 3) can be converted into either a (3 : 3) or a (2 : 4) with $n + 4 = 24$, depending on which 2^{3-1} is added. Similarly, a

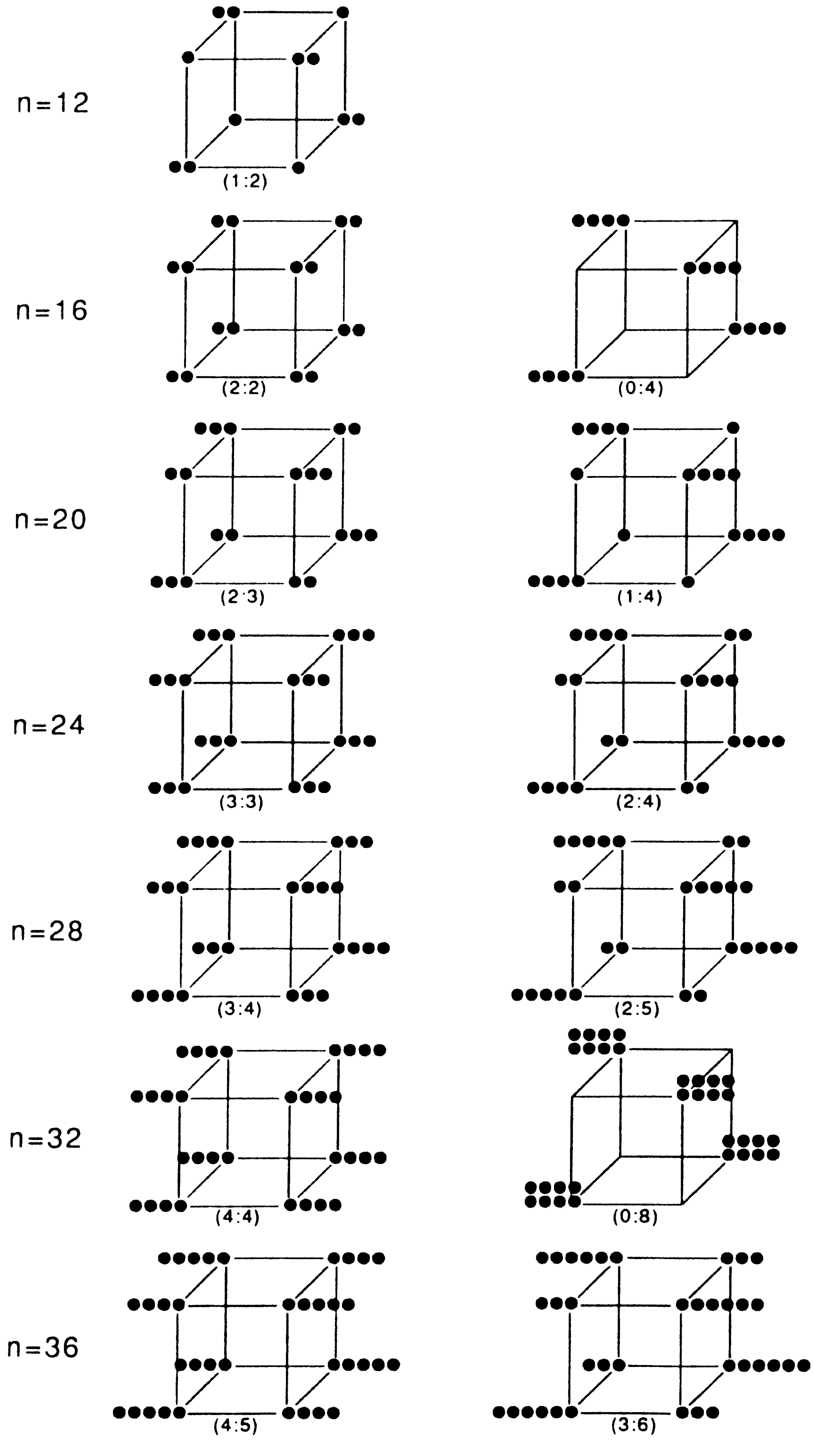

Fig. 2. Projection of *n*-run Plackett and Burman designs into $k = 3$ dimensions

Table 3. Projections of 12-, 16-, ..., 36-run Plackett and Burman designs into three dimensions

n	Sample columns to generate the design	Type of projection	Frequency of occurrence	Proportion of occurrence
12	(1, 2, 3)	1 : 2	165	1.0
16	(1, 2, 3)	2 : 2	420	0.923
	(1, 2, 13)	0 : 4	35	0.077
20	(1, 2, 3)	2 : 3	912	0.941
	(1, 2, 9)	1 : 4	57	0.059
24	(1, 2, 3)	3 : 3	1012	0.571
	(1, 2, 5)	2 : 4	759	0.429
28	(1, 2, 3)	3 : 4	2574	0.880
	(1, 2, 11)	2 : 5	351	0.120
32	(1, 2, 3)	4 : 4	4340	0.966
	(1, 2, 15)	0 : 8	155	0.034
36	(1, 2, 3)	4 : 5	5355	0.818
	(1, 2, 4)	3 : 6	1190	0.182

(1 : 4) can become a (2 : 4) or a (1 : 5); however, the latter is not a three-column projection of a 24-run Plackett and Burman design.

Table 3 provides sample columns from the original Plackett and Burman design construction that generate the various projections of Figure 2, and also gives the frequencies (and consequent proportions) of occurrences of the various types of designs. If the experimenter has some prior knowledge of the three factors likely to be significant and wishes to emerge with a specific type of projection, the prior factors would be assigned to the three columns indicated. If columns are randomly assigned to factors, we can determine from the frequencies (or proportions) of occurrence how likely it is that a particular projection will arise. For example, for $n = 20$, there are $\binom{n-1}{k} = \binom{19}{3} = 969$ 20-run projections, 912 of which are of type 2 : 3 and 57 of which are of type 1 : 4. So it is most likely (0.941) that the 2 : 3 type will occur.

4 Addition of Runs: Some Considerations

The projections into three dimensions of the Plackett and Burman designs for $n \leq 36$ considered in the foregoing section and illustrated in Figure 2, always provide a neat structure consisting of 2^3 and 2^{3-1} designs. When projected into higher dimensions, however, irregular fractions of the 2^{q-p} typically arise. These can be analyzed directly via regression methods *if* the runs permit estimation of an appropriate linear model. If not, one can consider supplementing the information available by making additional factorial runs to convert the projected design into a neater format with some desirable properties, for example, into a regular 2^{q-p} type design. What is an optimal way to add runs? This is a difficult question to answer uniquely, because the answer depends on what we know or can assume about the results available from the initial design, and what model it is desired to estimate. We discuss this briefly below, considering examples of projections into four and five dimensions of the $n = 12$-, 20-, and 24- run Plackett and Burman designs in later sections.

The n-run Plackett and Burman designs are saturated for $(n - 1)$ factors, and can also be run in a "near-saturated" form with somewhat fewer than $(n - 1)$ factors. (See Draper and Lin, 1990.) Their alias patterns are extremely complicated. (See Lin and Draper, 1993.) Thus, the interpretation of the larger contrasts in terms of the specific factors chosen for the design, is often ambiguous. For example, suppose that $n = 12$, and that $k = 4$ contrasts appear to be larger than the others. Let us assume these contrasts are associated with factors 1, 2, 3, and 4. Because of the aliasing, we are in fact estimating, for $i = 1, 2, 3, 4$,

$$\text{Main effect} \quad i + \frac{1}{3}\{\Sigma \pm \text{(all two factor interactions } \textit{not} \text{ involving factor } i)\} \ .$$

Thus it might be that we are observing

(i) the additive main effects of factors 1–4 only.
(ii) some main effects, and some interactions which "appear" to be main effects.
(iii) interaction effects which "appear" to be main effects.

Moreover, it could also be true that

(iv) interactions estimated in one or more other columns are large and opposite in sign to the corresponding large column main effect(s) thus cancelling and providing a low and non-significant contrast.

In general, it is impossible to know exactly what has occurred. It would be possible to fit a linear model involving an intercept, four main effect terms (1–4) and six two-factor interaction terms (12, 13, ..., 34). This would not provide an orthogonal representation, but would be reasonably efficient. One can also proceed on the basis of plausibility arguments (such as "it is likely (but not certain) that main effects will be larger than two-factor interactions") and hope to clarify the consequent conclusions by making additional factorial runs. (We elaborate on this issue in Sections 5 and 6, when specific projections are presented.)

5 Projections into $k = 4$ Dimensions

For $n = 12$, one could complete a 2_{IV}^{4-1} design by adding one run; there are five additional runs as well. For example, suppose we use columns 1–4 of Table 2. The eight runs 1, 3, 5, 6, 7, 9, 10, and 11 all have a negative product of signs, but runs 3 and 11 are identical, $(+ - + +)$. The new run $(- + - -)$ completes the 2_{IV}^{4-1} with $I = -1234$. This additional point is always uniquely determined as the foldover complement of the duplicate point. An alternative to this one run addition would be to fit a "main effects plus two factor interactions" model to all the initially available data, as outlined in the last paragraph of the foregoing section.

Only five runs need be added to complete a full 2^4 design, an efficient result. The specific runs needed are determined by the specific columns selected for further examination. For example, suppose we again use columns 1–4 of Table 2. To get the full 2^4 we add four more runs with a positive sign product, $I = +1234$, namely $(+ - + -)$, $(+ - - +)$, $(- - + +)$, and $(+ + + +)$ in addition to the run $(- + - -)$ used to complete the 2_{IV}^{4-1} design above.

What levels should be chosen for the other seven variables? If the assumption that factors 5–11 are inactive is true, the choices should not be crucial. It would be possible, however, to set the signs exactly as in their mirror image runs in the original design. This offers the great advantage (pointed out by a referee) of providing five pairs of runs folded over in the four retained factors, but identical in the other seven factors. This immediately gives five new estimates of each of the four main effects of interest, unbiased by *any* two-factor interactions. We see that this particular choice of the levels of the (supposedly) nonsignificant factors makes a valuable contribution to resolving the possibilities arising from the initial design. (Not that, in our example, two choices of levels are possible for the mirror image run of the "duplicates", runs 3 and 11.)

Addition of other non-factorial runs is also possible, for example, axial points $(\pm\alpha, 0, 0, 0), \ldots, (0, 0, 0, \pm\alpha)$ where $\alpha \neq 1$ could be added. This would permit further exploration of the underlying response surface by providing estimates of the coefficients of the pure quadratic terms $\beta_{ii}x_i^2$, $i = 1, \ldots, 4$ when the new response values are analyzed with the original two-level design results.

Table 4 shows all the projection possibilities into four dimensions for the 12-, 20- and 24-run cases. For $n = 20$, only three types of projections exist (apart from sign changes in the columns, permutations of the columns, and rearrangements of the rows). At the left of Table 4 we show the 16 runs of a 2^4 design. Design 20-4.1 has 15 of these runs plus five replicates of runs 1, 6, 7, 12, and 13. Run 5 does not occur in the projection. Thus, addition of the run $(- - + -)$ completes a 2^4 design. Similarly, Designs 20-4.2 and 20.4.3 require the addition of four more runs each to provide a complete 2^4. Which projection is actually attained depends on the specific four factors retained after analysis.

For $n = 24$, two types of projections, 24-4.1 and 24-4.4 provide a full 2^4 design plus a 2_R^{4-1}; for the former $R = IV$, and for the latter $R = III$. Designs 24-4.2 and 24-4.3 both require two additional runs to complete a full 2^4 design.

Table 4. Projections of 12-, 20-, and 24-run Plackett and Burman designs into four dimensions. The numbers in the columns indicate multiplicities of the various 2^4 design points shown at left

					12-run	20-run case			24-run case			
Run No.	1	2	3	4	Design 12-4.1	Design 20-4.1	Design 20-4.2	Design 20-4.3	Design 24-4.1	Design 24-4.2	Design 24-4.3	Design 24-4.4
1	−	−	−	−	1	2	2	2	2	1	1	1
2	+	−	−	−	1	1	0	0	1	1	1	2
3	−	+	−	−	1	1	2	0	1	1	3	2
4	+	+	−	−	0	1	1	3	2	3	1	1
5	−	−	+	−	0	0	0	1	1	3	2	1
6	+	−	+	−	1	2	3	2	2	1	2	2
7	−	+	+	−	1	2	1	2	2	1	0	2
8	+	+	+	−	1	1	1	0	1	1	2	1
9	−	−	−	+	1	1	1	1	1	2	2	2
10	+	−	−	+	0	1	2	2	2	2	2	1
11	−	+	−	+	0	1	0	2	2	2	0	1
12	+	+	−	+	2	2	2	0	1	0	2	2
13	−	−	+	+	1	2	2	1	2	0	1	2
14	+	−	+	+	1	1	0	1	1	2	1	1
15	−	+	+	+	1	1	2	1	1	2	3	1
16	+	+	+	+	0	1	1	2	2	2	1	2
Runs needed to complete a 2^4 design					5	1	4	4	0	2	2	0
Column needed in addition to columns (1, 2, 3) to generate the projection					4	4	6	16	4	8	6	5
Frequency of occurrence					330	2736	912	228	759	2024	3036	3036
Proportion of occurrence					1	.706	.235	.059	.086	.228	.343	.343

A sample way in which each of the projections can be generated is shown in Table 4. Columns (1, 2, 3) of the original Plackett and Burman design are combined with one further column whose number is shown. If the experimenter has some prior knowledge of the four factors likely to be significant and wishes to emerge with a specific projection, the prior factors would be assigned to columns (1, 2, 3) plus the fourth choice indicated. If columns are randomly assigned to factors, we can determine from the frequencies (or proportions) of occurrence how likely it is that a particular projection will arise. For example, there are $\binom{n-1}{k} = \binom{19}{4} = 3876$ 20-run projections of which (2736, 912, 228) are, respectively, of types (20-4.1, 20-4.2, 20-4.3). The respective proportions are therefore (.706, .235, .059); fortunately, the "best" projection is most likely to occur. The opposite is true for the 24-run case.

6 Projections into $k = 5$ Dimensions

For $n = 12$ two types of designs are possible, one with a repeat run pair ("type 5.1," say) and one with a mirror image pair ("type 5.2"); see Draper (1985, Table 2). A number of possibilities exist for supplementing these designs. To determine which of the two design types has been obtained via projection, one must check to see if the specific design has a repeat run pair, or a mirror image run pair, an easy thing to do.

For example, if we choose columns 1–5 of Table 2, we see that runs 7, $(+----+)$, and 10, $(-+++-)$, are mirror image runs. Thus we have a design of type 5.2 which we could convert to a standard form, in which the mirror image runs are $(-----)$ and $(+++++)$, by changing the signs in either columns 1 and 5 or in 2, 3, and 4 and perhaps rearranging the columns appropriately. Even without making those changes, it is clear that the product of signs in the columns 1, 2, 3, 4, and 5 is "$-$" for runs 1, 5, 7, 8, 11, and 12, and "$+$" for the remaining six runs. A 2_V^{5-1} can thus be produced in two alternative ways, by adding 10 runs with the same signed products in each case. One case is indicated in Table 5, in the 12-5.2 column, as we explain in the next paragraph.

Table 5 shows the 16 runs (1–16) of a 2^{5-1} ($I = 12345$) design, plus six additional runs. Design 12-5.1 consists of runs 4, 6–8, 10–15, 17, 17 while Design 12-5.2 contains runs 7, 8, 10, 12, 13, 16, 17–22. (These are the aforementioned standard forms. Out of 462 possible projections, 66 are of type 5.1 and 396 are of type 5.2.) Thus, to complete a 2^{5-1} with $I = 12345$ starting with Design 12-5.1, one must add the six "0" runs in the 12-5.1 column. To achieve a similar result with Design 12-5.2, 10 additional "0" runs are needed.

In examining the possibilities for Design 12-5.2, we discovered that the 32 runs of a 2^5 design can be divided as follows:

(a) Into a 12-run portion and a 20-run portion so that the two portions are the projections into five dimensions of (respectively) 12-run (11 factors) and 20-run (19 factors) Plackett and Berman designs.
(b) Into 8-run and 24-run portions which are projections into five dimensions of 8 and 24 run Plackett and Burman designs.

In all such cases, the model appropriate to the completed 2_R^{q-p} design could be fitted by least squares, and the runs already made in addition to the 2_R^{q-p} runs will provide some residual degrees of freedom in an analysis of variance table.

Table 5. Projections of a 12-run Plackett and Burman design into five dimensions. The numbers indicate frequencies of the various 2^5 design points specified at left

Run No.	I = 12345 for Runs 1–16: 1	2	3	4	5	Design Number: 12-5.1	12-5.2
1	−	−	−	−	+	0	0
2	+	−	−	−	−	0	0
3	−	+	−	−	−	0	0
4	+	+	−	−	+	1	0
5	−	−	+	−	−	0	0
6	+	−	+	−	+	1	0
7	−	+	+	−	+	1	1
8	+	+	+	−	−	1	1
9	−	−	−	+	−	0	0
10	+	−	−	+	+	1	1
11	−	+	−	+	+	1	0
12	+	+	−	+	−	1	1
13	−	−	+	+	+	1	1
14	+	−	+	+	−	1	0
15	−	+	+	+	−	1	0
16	+	+	+	+	+	0	1
Additional runs for which $I = -12345$							
17	−	−	−	−	−	2	1
18	+	−	+	−	−		1
19	+	−	−	−	+		1
20	−	+	−	+	−		1
21	−	+	−	−	+		1
22	−	−	+	+	−		1
Runs needed to complete a 2_V^{5-1} design						6	10

Table 6 shows all the nine possible different projected designs for $n = 20$, designated 20-5.1 through 20-5.9, while Table 7 shows the nine projections, 24-5.1 through 24-5.9, that occur for $n = 24$. Both tables are displayed with a split of the runs into the two pieces for which $I = 12345$ or $I = -12345$. (The run numbers apply to a full 2^5 design in standard order.) This shows which, and how many , runs are needed to complete either of the 2^{5-1} designs, or to complete a 2^5. Both Tables 6 and 7 provide a sample way to generate the projections described. Again, if the experimenter has some prior knowledge of the five factors likely to be significant and wishes to emerge with a specific projection, the prior factors would be assigned to the indicated columns. If columns are randomly assigned to factors, the frequencies of occurrence of the various projections will be as indicated in Tables 6 and 7.

Table 6. Projections of 20-run Plackett and Burman designs into five dimensions. The numbers indicate frequencies of the various 2^5 design points shown at left

Run						Design Number									
No.	1	2	3	4	5	20-5.1	20-5.2	20-5.3	20-5.4	20-5.5	20-5.6	20-5.7	20-5.8	20-5.9	20-5.10
17	−	−	−	−	+	1	1	1	0	1	1	0	0	0	2
2	+	−	−	−	−	1	0	0	1	0	0	0	0	0	0
3	−	+	−	−	−	1	1	1	0	0	0	0	1	0	0
20	+	+	−	−	+	1	1	0	0	0	0	0	1	1	1
5	−	−	+	−	−	0	0	0	0	0	0	0	0	0	1
22	+	−	+	−	+	1	0	1	2	1	1	1	2	2	0
23	−	+	+	−	+	1	2	2	2	1	0	1	1	0	2
8	+	+	+	−	−	0	1	1	1	1	0	0	1	1	0
9	−	−	−	+	−	0	1	1	0	0	1	0	0	1	2
26	+	−	−	+	+	1	1	1	1	0	0	1	2	1	1
27	−	+	−	+	+	0	0	1	0	0	0	1	0	0	1
12	+	+	−	+	−	1	1	1	0	1	0	2	2	1	1
29	−	−	+	+	+	1	2	0	1	0	2	1	0	2	0
14	+	−	+	+	−	1	1	0	1	0	1	1	0	0	0
15	−	+	+	+	−	0	1	0	1	0	0	1	0	1	2
32	+	+	+	+	+	0	1	0	0	1	0	1	0	0	1
Runs needed to complete a 2_V^{5-1} ($I = 12345$) design						6	4	7	8	10	11	7	9	8	6

1	−	−	−	−	−	1	1	1	2	1	1	2	2	2	0
18	+	−	−	−	+	0	1	1	0	1	1	1	0	0	0
19	−	+	−	−	+	0	0	0	1	1	1	1	1	2	0
4	+	+	−	−	−	0	0	1	1	1	1	1	0	0	2
21	−	−	+	−	+	0	0	0	0	0	0	0	0	0	0
6	+	−	+	−	−	1	2	1	0	1	1	1	1	1	2
7	−	+	+	−	−	1	0	0	0	1	2	1	0	1	0
24	+	+	+	−	+	1	0	0	0	0	1	1	0	0	0
25	−	−	−	+	+	1	0	0	1	1	0	1	1	0	0
10	+	−	−	+	−	0	0	0	0	1	1	0	0	1	0
11	−	+	−	+	−	1	1	0	1	1	1	0	0	0	0
28	+	+	−	+	+	1	1	1	2	1	2	0	0	1	0
13	−	−	+	+	−	1	0	2	1	2	0	1	2	0	0
30	+	−	+	+	+	0	0	1	0	1	0	0	0	0	2
31	−	+	+	+	+	1	0	1	0	1	1	0	2	1	0
16	+	+	+	+	−	1	0	1	1	0	1	0	1	1	0
Runs needed to complete a 2_V^{5-1} ($I = -12345$) design						6	11	7	8	3	4	7	9	8	13
Runs needed to complete a 2^5 design						12	15	14	16	13	15	14	18	16	19
Columns needed in addition to columns (1, 2, 3) to generate the projection						4, 5	4, 6	4, 9	4, 11	4, 14	4, 15	4, 16	6, 9	6, 10	—
Frequency of occurrence						1881	1368	3078	1026	1368	684	1539	171	513	—
Proportion of occurrence						.162	.118	.265	.088	.118	.059	.132	.015	.044	—

Table 7. Projections of 24-run Plackett and Burman designs into five dimensions. The numbers indicate frequencies of the various 2^5 design points shown at left

Run No.	1	2	3	4	5	Design Number 24-5.1	24-5.2	24-5.3	24-5.4	24-5.5	24-5.6	24-5.7	24-5.8	24-5.9
17	−	−	−	−	+	1	1	1	1	1	0	0	0	0
2	+	−	−	−	−	1	0	0	0	0	1	1	0	0
3	−	+	−	−	−	1	0	0	0	1	1	0	1	1
20	+	+	−	−	+	2	1	0	1	1	0	0	0	2
5	−	−	+	−	−	0	1	1	1	0	1	1	1	1
22	+	−	+	−	+	1	0	1	2	1	2	1	0	2
23	−	+	+	−	+	0	1	1	0	1	1	0	1	1
8	+	+	+	−	−	0	0	0	1	1	0	1	1	1
9	−	−	−	+	−	0	0	0	1	1	0	1	1	1
26	+	−	−	+	+	1	0	1	0	1	1	1	1	2
27	−	+	−	+	+	1	0	1	2	1	1	0	0	1
12	+	+	−	+	−	1	0	1	1	0	2	1	3	1
29	−	−	+	+	+	1	2	0	2	1	0	1	1	2
14	+	−	+	+	−	1	0	0	1	1	1	0	0	0
15	−	+	+	+	−	0	1	0	1	0	0	3	1	1
32	+	+	+	+	+	1	1	1	2	1	1	1	1	0
Runs needed to complete a 2_V^{5-1} ($I = 12345$) design						5	9	8	4	4	6	6	6	4

1	−	−	−	−	−	1	1	1	1	1	1	1	1	2
18	+	−	−	−	+	0	1	1	1	1	1	0	2	0
19	−	+	−	−	+	0	1	1	1	0	1	3	2	1
4	+	+	−	−	−	0	1	2	1	1	1	1	0	0
21	−	−	+	−	+	1	0	0	0	1	0	1	0	0
6	+	−	+	−	−	1	2	1	0	1	0	1	2	1
7	−	+	+	−	−	2	1	1	2	1	1	0	0	0
24	+	+	+	−	+	1	1	1	0	0	1	1	1	0
25	−	−	−	+	+	1	1	1	0	0	2	1	1	0
10	+	−	−	+	−	1	2	1	2	1	0	1	0	1
11	−	+	−	+	−	1	2	1	0	1	0	0	0	0
28	+	+	−	+	+	0	1	0	0	1	0	1	0	0
13	−	−	+	+	−	1	0	2	0	1	2	0	1	0
30	+	−	+	+	+	0	1	1	0	0	0	1	1	0
31	−	+	+	+	+	1	0	1	0	1	1	0	1	1
16	+	+	+	+	−	1	1	1	0	1	1	0	0	2
Runs needed to complete a 2_V^{5-1} ($I = -12345$) design						5	3	2	10	4	6	6	7	10
Runs needed to complete a 2^5 design						10	12	10	14	8	12	12	13	14
Columns needed in addition to columns (1, 2) to generate the projection						3, 4, 5	3, 4, 7	3, 4, 8	3, 4, 9	3, 4, 14	3, 5, 8	3, 6, 9	4, 5, 10	4, 9, 15
Frequency of occurrence						2277	10,626	6072	5060	1012	6831	759	506	506
Proportion of occurrence						.068	.316	.180	.150	.030	.203	.023	.015	.015

Note that the 12 runs that may be added to Design 20-5.1 to complete a 2^5 design consist of five specific columns from a 12-run Plackettand and Burman design. This is the 12 : 20 split mentioned above.

For Design 20-5.10, see Section 7.

7 Non-Equivalent Hadamard Matrices for $n = 16, 20$

By adding a column of 1's to a Plackett and Burman design, we obtain a Hadamard matrix H which satisfies $H'H = nI$. For $n = 12$, H is unique, but for higher n this is not true. Non-equivalent Hadamard matrices have different projection properties. We illustrate using the cases $n = 16$ and $n = 20$.

There are five non-equivalent Hadamard matrices for $n = 16$; see Hall (1961). Only one of these corresponds to a Plackett and Burman design, that is, only one (called H16-1 here, and I by Hall) provides a 2^{k-p} 16-run design of the type whose projections were studied in Section 2. We now briefly discuss the projection patterns of the other four types, which we designate as H16-2, H16-3, H16-4, and H16-5. These designs are, respectively designs II, III, IV, and V in Hall (1961, pages 23–24).

The various projections into $k = 3, 4$, and 5 dimensions have their frequencies and proportions listed in Table 8. For $k = 3$, there are three different possible projections. Two of these, illustrated in Figure 2, arise from *all* the five Hadamard matrices. In addition, H16-5 produces projected designs of type 1 : 3, not show in Figure 2.

For $k = 4$, five projections occur, as shown in Figure 3. Two of these, (a) and (c), arise from *all* of the Hadamard matrices. See Table 8. In all parts of Figure 3, the cube represents the space of three of the four factors and the fourth is represented by open dots for the lower level, and solid dots for the upper level. Note the "unbalanced" structure of designs (d) and (e). Whereas the other three designs (a), (b), and (c) are replicated 2_{III}^{4-1}, replicated 2_{IV}^{4-1} and a full 2^4 respectively, designs (d) and (e) are not of this form.

For $k = 5$, there are eight different projections; see Table 8 and Figure 4. Two of these, (a) and (b) arise from *all* of the Hadamard matrices. In all parts of Figure 4, the cube represents the space of three of the five factors. Each circle represents a run of the projected design and in each circle, the left portion is for the fourth factor and the right portion is for the fifth factor. An open half-circle represents the lower level of a factor and a solid (black) half-circle represents the upper level of a factor. (We profited here from viewing similar diagrams in Lucas, 1991.) Note the "unbalanced" structure of projections (e), (f), (g), and (h). Designs (a), (b), (c), and (d) are, respectively, a replicated 2_{III}^{5-2}, a 2_{III}^{5-1}, a 2_{IV}^{5-1}, and a 2_V^{5-1}. There are three non-equivalent Hadamard matrices for $n = 20$ (Hall, 1965). We have discussed, in Section 6, only the one that is equivalent to a Plackett and

Table 8. The frequencies and proportions for all 16-run non-equivalent Hadramard matrices projected into $k = 3$, 4, and 5 dimensions

Type	Design				
	H16-1	H16-2	H16-3	H16-4	H16-5
$k = 3$					
(a)	420 0.9231	420 0.9231	420 0.9231	420 0.9231	360 0.1758
(b)	35 0.0769	35 0.0769	35 0.0769	35 0.0769	15 0.0330
(c)					80 0.1758
$k = 4$					
(a)	560 0.3077	480 0.2637	440 0.2418	420 0.2308	200 0.1099
(b)	140 0.0769	60 0.0330	20 0.0110		20 0.0110
(c)	1120 0.6154	960 0.5275	880 0.4835	840 0.4615	640 0.3516
(d)		320 0.1758	480 0.2637	560 0.3077	480 0.2637
(e)					480 0.2637
$k = 5$					
(a)	525 0.1049	285 0.0569	165 0.0330	105 0.0210	75 0.0150
(b)	2800 0.5595	1920 0.3836	1480 0.2957	1260 0.2517	1460 0.2917
(c)	1400 0.2797	600 0.1199	200 0.0400		170 0.0340
(d)	280 0.0559	120 0.0240	40 0.0080		
(e)		1600 0.3197	2400 0.4795	2800 0.5594	1920 0.3836
(f)		480 0.0959	720 0.1439	840 0.1678	360 0.0719
(g)					960 0.1918
(h)					60 0.0120

Burman design. The other two give exactly similar projections for $k = 3$ and 4. For $k = 5$, however, there is one projection additional to the nine listed previously. It is shown as column 20-5.10 in Table 6.

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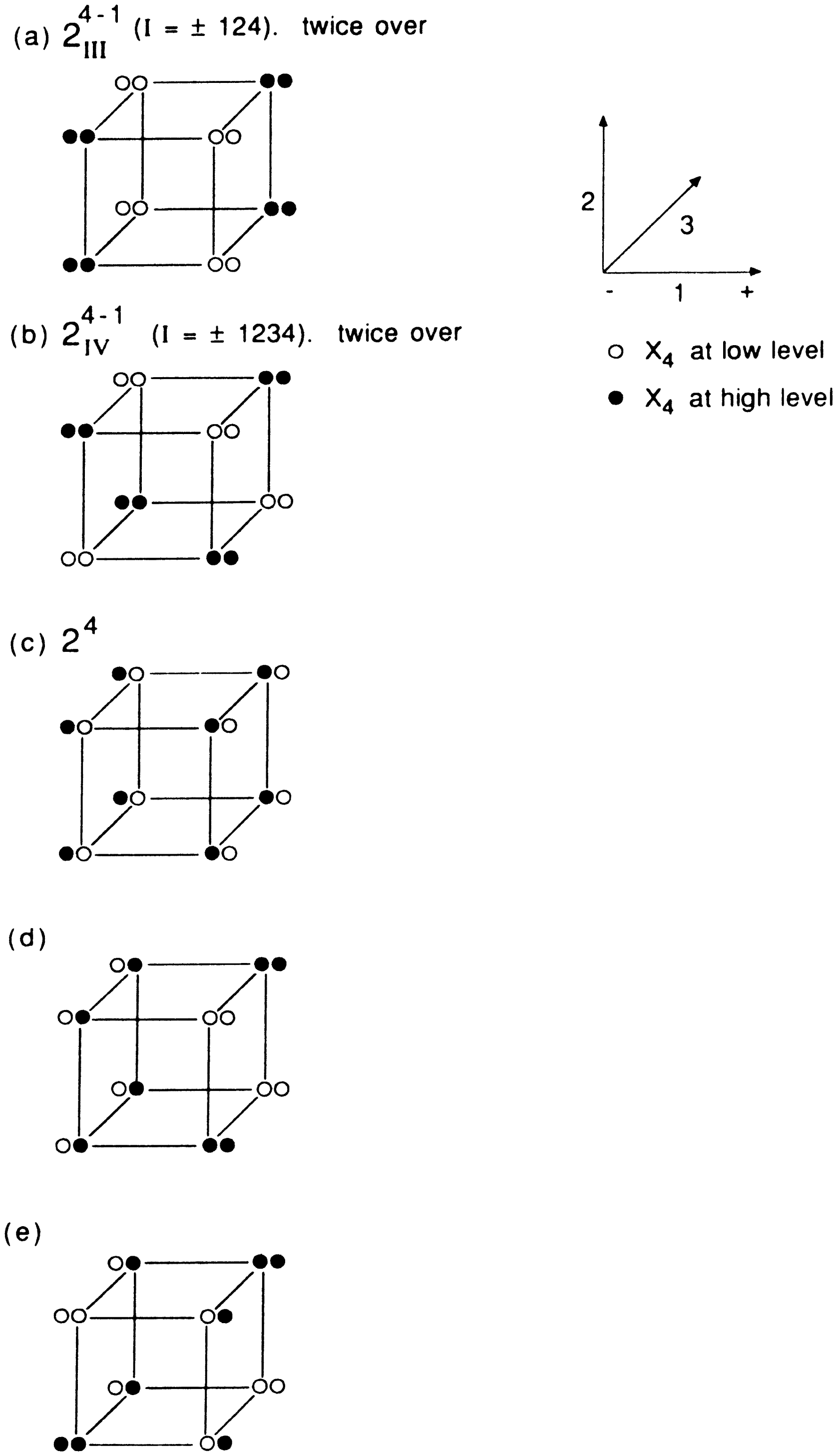

Fig. 3. Projections of 16-run Hadamard matrix type designs into four dimensions

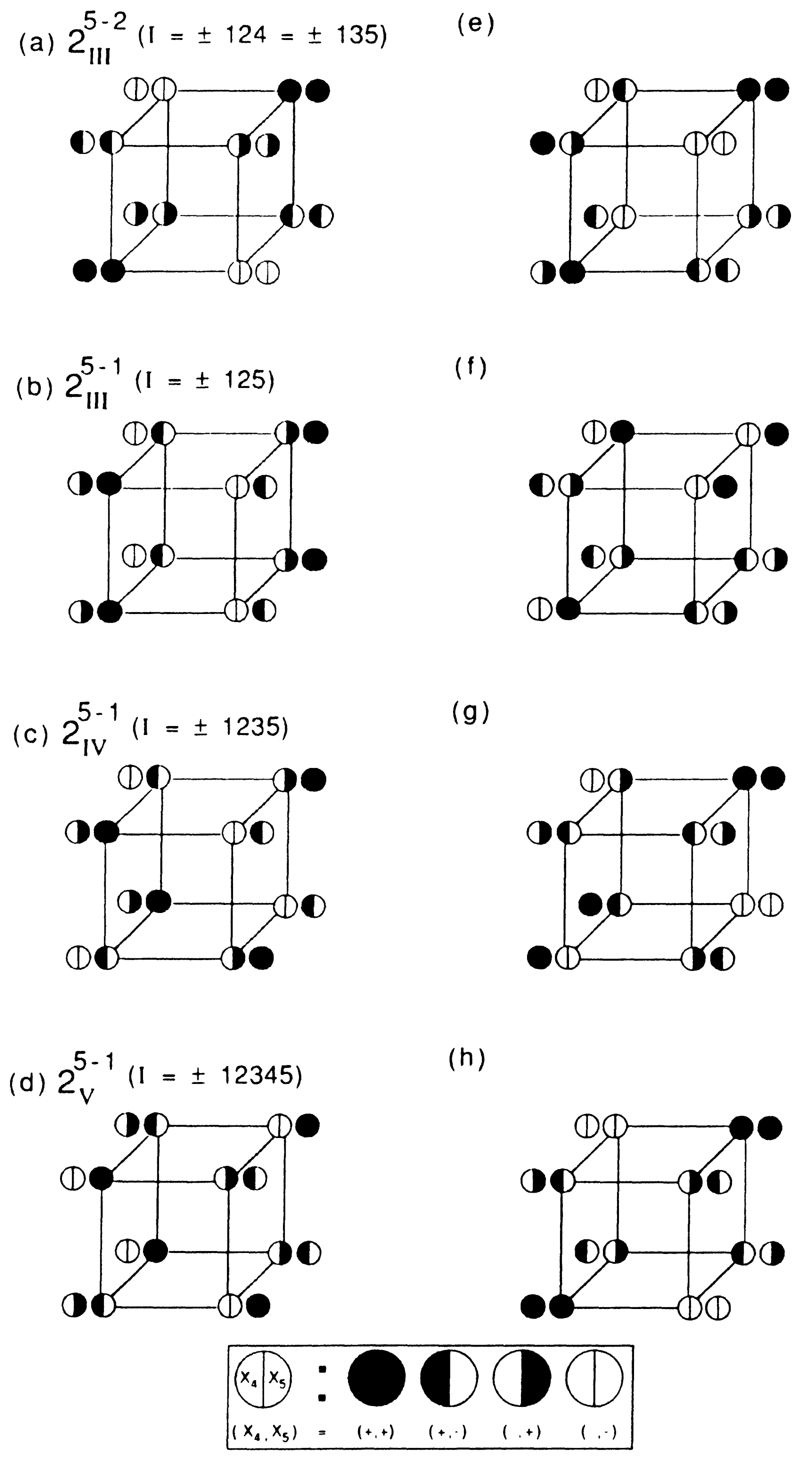

Fig. 4. Projection of 16-run Hadamard matrix type designs into five dimensions

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