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Lifetime Data Analysis, 1, 87–100 (1995)

Dependent Masking and System Life Data Analysis: Bayesian Inference for Two-Component Systems

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Received September 1994; accepted January 1995

Abstract. Data from field operations of a system is often used to estimate the reliability of components. Under ideal circumstances, this system field data contains the time to failure along with information on the exact component responsible for the system failure. However, in many cases, the exact component causing the failure of the system cannot be identified, and is considered to be masked. Previously developed models for estimation of component reliability from masked system life data have been based upon the assumption that masking occurs independently of the true cause of system failure. In this paper we develop a Bayesian methodology for estimating component reliabilities from masked system life data when the probability of masking is dependent upon the true cause of system failure. The Bayesian approach is illustrated for the case of a two-component system of exponentially distributed components.

Keywords: Bayes inference, dependent masking, posterior mean, reliability.

1. Introduction

Reliability assessments for components are often based upon the collection of field data from the actual operation of systems. Under ideal circumstances, this system field data contains the time to failure along with information on the exact component responsible for the system failure. However, in many cases, the exact component causing the failure of system cannot be identified. Instead, it may only be known that the failing component is one of a subset of components that are considered potentially responsible for the failure. When this occurs, the cause of failure is *masked*. Masking can occur in field data for a variety of reasons such as lack of proper diagnostic equipment, cost and time constraints associated with failure analysis, recording errors, and the destructive nature of certain component failures that make exact diagnosis impossible.

A variety of papers have been written that address the problem of estimating the reliability of components from masked system life data. Under the simplifying assumption that components have exponentially distributed life, Miyakawa (1984) considers a two-component series system where he derives closed-form expressions for the MLEs when some of the sample observations are masked. Under the same exponential assumption, Usher and Hodgson (1988) extend Miyakawa's results to a three-component system. Doganaksoy (1991) presents various means of finding confidence intervals for the three-component, exponential life case. For biologically related examples of masked data analysis, see Dinse (1982, 1986) and Gross (1970).

These previously developed models are based upon the assumption that masking occurs independently of the true cause of system failure. That is, the probability of observing a particular masked set does not depend upon which component failed in the system. However, in some cases, this assumption may not hold. For instance, consider the case of a circuit card with two components where, under certain environmental conditions, the failure of either component can result in a fire and complete destruction of the circuit card. If the card is destroyed, then the cause of failure cannot be identified. Dependence occurs when the probability of the card's destruction differs based upon which component fails. Moreover this probability of destruction given that a particular component failed may depend on time but it seems reasonable to assume that the ratio of these masking probabilities will not be a function of time.

Guess, Usher, and Hodgson (1991) first described the relevance of the independence assumption and its effect on the development of the likelihood function. Lin and Guess (1994) investigate the effect of dependency between the masking set and the true cause of failure. They suggest a simple means of checking for this independence or dependence via subsampling for the case of exponentially distributed components.

In this paper we develop a Bayesian methodology for estimating component reliability from masked system life data when the probability of masking is dependent upon the true cause of failure. The Bayesian approach is preferable here because in most engineering settings there is considerable prior knowledge and expertise regarding the reliability of the components that are under study. Thus, this approach allows for the explicit consideration of this prior knowledge in the estimation process. Here we use noninformative priors to deal with the case that only weak prior information is available. This will provide a baseline to look at the effect of more informative priors. Our procedure is illustrated for the case of a two-component system where component lives are exponentially distributed. We assume that nonmasked failures are correctly classified.

2. The Effect of Dependent Masking

Let T_i be the random life for the ith system where $i = 1, \ldots, n$ in a sample of n systems each consisting of J components in series. Let T_{ij} be the random life of the jth component in the ith system where $j = 1, \ldots, J$. Note that

$$T_i = \min(T_{i1}, \ldots, T_{iJ})$$

for $i = 1, \ldots, n$. We assume that the T_{ij}'s are independent. (The dependency among the lifetimes of the components could be modeled using a dependent multivariate distribution and a competing risk model. See, for example, Barlow and Proschan, 1981; Basu and Klein, 1982.) For each fixed j, the $T_{1j}, \ldots, T_{nj}$ would represent a sample of size n from component j's life distribution F_j.

We require the very mild condition that F_j has a density f_j indexed by a parameter vector θ_j. For each j, a different number of parameters in θ_j is allowed if needed. Let $\bar{F}_j(t) = 1 - F_j(t)$ be the reliability of component j at time t. Let K_i be the index of component causing the failure of system i. Due to the life distribution being continuous, the cause of failure K_i is unique. Note that K_i is a random variable and that K_i may or may not be observed, i.e., the component causing system failure may be masked. Before the sample is taken there is the *minimum random subset*, M_i, of components known to contain the true cause of failure of system i. In short, $K_i \in M_i$ and the set M_i is minimum.

After the sample data is obtained, we have $M_i = S_i \subset \{1, 2, \ldots, J\}$ and $T_i = t_i$ where $i = 1, \ldots, n$. Note that as t_i is the realized sample value of T_i, so is S_i the realized sample set of M_i. If $S_i = \{j\}$ then we know that $K_j = j$, and hence, the cause of failure is not masked. If, for example, $S_i = \{1, 2\}$, we have that $K_i \in S_i$, but the true value of K_i is masked. The observed data here is $(t_1, S_1), \ldots, (t_n, S_n)$. The full likelihood for this data is

$$L_F = \prod_{i=1}^{n} \left\{ \sum_{j \in S_i} \left[f_j(t_i) \prod_{s=1, s \neq j}^{J} \bar{F}_s(t_i) \cdot P(M_i = S_i \mid T_i = t_i, K_i = j) \right] \right\}. \qquad (1)$$

Note that the term $f_j(t_i) \prod_{s=1, s \neq j} \bar{F}_s(t_i)$ is from system i failing at time t_i due to the cause j (component j).

The expression $P(M_i = S_i \mid T_i = t_i, K_i = j)$ represents the conditional probability that the observed minimum random subset is S_i given that system i failed at time t_i and the true cause was component j. For $S_i = \{j\}$, this expression is the conditional probability that the cause of failure is known. For S_i containing more than j, it yields the conditional probability of masking with the set S_i. For the observation (t_i, S_i), we sum over all possible failure causes j in S_i. The product is then over each of the observations to yield the full likelihood L_F.

For industrial problems, masking typically occurs due to constraints of time and the expense of failure analysis. Schedules often dictate that compete failure analysis (to determine the true cause of failure) be curtailed. In such settings it seems reasonable to assume, for fixed $j' \in S_i$ that

$$P(M_i = S_i \mid T_i = t_i, K_i = j') = P(M_i = S_i \mid T_i = t_i, K_i = j) \text{ for all } j \in S_i. \qquad (2)$$

We call these masking probabilities independent over the causes $j \in S_i$. Furthermore, if $P(M_i = S_i \mid T_i = t_i, K_i = j)$ does not depend on the life distribution parameters, the reduced partial likelihood is

$$L_R = \prod_{i=1}^{n} \left\{ \sum_{j \in S_i} \left(f_j(t_i) \prod_{s=1, s \neq j}^{J} \bar{F}_s(t_i) \right) \right\}. \qquad (3)$$

Maximizing L_R with respect to the life parameters is now equivalent to using L_F. This is similar to the usual derivation of a time censored (and not masked) data partial likelihood. For further discussion of this situation, see Reiser et al. (1995).

In this paper, we replace the assumption (2) by, for any $j \in S_i$,

$$P(M_i = S_i \mid T_i = t_i, K_i = j) \neq P(M_i = S_i \mid T_i = t_i, K_i = j') \text{ for some } j' \in S_i. \qquad (4)$$

and provide a Bayesian approach to analyze such a situation. These masking probabilities are, of course, dependent on S_i. If we think of the circuit card example discussed in the

Introduction, it seems entirely natural to assume proportional probabilities for $j,\ j' \in S_i$ and $j \neq j'$ or more formally that

$$P(M_i = S_i \mid T_i = t_i, K_i = j) = c \times P(M_i = S_i \mid T_i = t_i, K_i = j') \tag{5}$$

where $c \geq 0$ is implicitly a function of j and j', but not of t_i. Note that for $c \neq 1$ there is dependent masking over S_i, while for $c = 1$, the special case of independent masking over S_i of (2) holds. For $c = 0$ and $P(M_i = S_i \mid T_i = t_i, K_i = j) > 0$ a very extreme form of dependent masking would occur.

3. Component Reliability

Consider a two-component system, $J = 2$ with n systems put on test. Let n_1 and n_2 be the number of system failures for which the failure cause is known to be component 1 and 2 respectively, while n_{12} denotes the number of failed systems where the cause is not directly known. We shall illustrate the general Bayesian approach using the exponential distribution for component lifetimes. The method can be extended to other distributions, although the computation will often be complicated.

Set

$$\begin{aligned} P(M_i = \{1\} \mid T_i = t_i, K_i = 1) &= p_1(t_i) \\ P(M_i = \{2\} \mid T_i = t_i, K_i = 2) &= p_2(t_i) \\ P(M_i = \{1,2\} \mid T_i = t_i, K_i = 1) &= p_3(t_i) \\ P(M_i = \{1,2\} \mid T_i = t_i, K_i = 2) &= p_4(t_i). \end{aligned}$$

Following the discussion in Section 2, we denote $c = p_4(t_i)/p_3(t_i)$. Note that c is not a function of t_i. Then (1) reduces to

$$L_R(c) = \lambda_1^{n_1}\lambda_2^{n_2}(\lambda_1 + \lambda_2 c)^{n_{12}} e^{-(\lambda_1+\lambda_2)T}, \tag{6}$$

where $T = \sum_{i=1}^{n} t_i$ is the total lifetime. Using the standard noninformative prior $\pi(\lambda_1, \lambda_2) \propto 1/\lambda_1\lambda_2$ results in the posterior

$$\begin{aligned} p(\lambda_1, \lambda_2 \mid c, data) &= k\, e^{-T(\lambda_1+\lambda_2)}\lambda_1^{n_1-1}\lambda_2^{n_2-1}(\lambda_1+\lambda_2 c)^{n_{12}} \\ &= k\, e^{-T(\lambda_1+\lambda_2)}\lambda_1^{n_1-1}\lambda_2^{n_2-1}\left\{\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}\lambda_1^{j}(\lambda_2 c)^{n_{12}-j}\right\} \\ &= k\, e^{-T(\lambda_1+\lambda_2)}\left\{\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}\lambda_1^{n_1+j-1}\lambda_2^{n_2+n_{12}-j-1}c^{n_{12}-j}\right\} \end{aligned} \tag{7}$$

where $k^{-1} = T^{-n}\sum_{j=0}^{n_{12}}\left\{\binom{n_{12}}{j}c^{n_{12}-j}\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)\right\}$ is obtained through integration.

Remark. If strong prior information on the λ_i is available it will frequently be possible to model this by use of a gamma prior. In this situation the results of this paper need to be slightly modified and more complex notation needs to be introduced. For simplicity we work with the noninformative prior.

For c given, we can assume that $0 \leq c \leq 1$, for otherwise we simply relabel components 1 and 2 as 2 and 1 respectively. The marginal posteriors can be derived by integration in (7), to be

$$\begin{aligned} p(\lambda_1 \mid c, data) &= k\, e^{-\lambda_1 T}\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}c^{n_{12}-j}\lambda_1^{n_1+j-1}\frac{\Gamma(n_2+n_{12}-j)}{T^{n_2+n_{12}-j}} \\ p(\lambda_2 \mid c, data) &= k\, e^{-\lambda_2 T}\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}c^{n_{12}-j}\lambda_2^{n_2+n_{12}-j-1}\frac{\Gamma(n_1+j)}{T^{n_1+j}} \end{aligned} \tag{8}$$

Figure 1 plots such marginal posteriors for some selected values of the data $(n_1, n_2, n_{12}, T) = (3, 3, 9, 1), (9, 9, 3, 1), (1, 9, 3, 1)$ and $(9, 1, 3, 1)$ as given in Lin and Guess (1994). Note from Figure 1 that the posteriors can be seriously affected by changes in c.

The posterior means can be evaluated as

$$\begin{aligned} E(\lambda_1 \mid c, data) &= k\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}c^{n_{12}-j}\frac{\Gamma(n_2+n_{12}-j)}{T^{n_2+n_{12}-j}}\frac{\Gamma(n_1+j+1)}{T^{n_1+j+1}} \\ &= k\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}c^{n_{12}-j}\frac{\Gamma(n_2+n_{12}-j)\Gamma(n_1+j+1)}{T^{n+1}} \\ E(\lambda_2 \mid c, data) &= k\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}c^{n_{12}-j}\frac{\Gamma(n_2+n_{12}-j+1)\Gamma(n_1+j)}{T^{n+1}} \end{aligned} \tag{9}$$

Figure 2 plots these posterior means for the cases in Figure 1. The results obtained here are consistent with the MLE approach given in Lin and Guess (1994) and indicate that although in certain cases the posterior means can be robust with respect to c (see the lower graphs in Figure 2), there are other cases where the choice of c is very influential (see the upper graphs in Figure 2). It appears difficult to characterize these different situations which are highly data dependent.

For known c, inference on the λ_i follows from (8) as we have shown. In general, however, c will be an unknown value ($0 \leq c \leq \infty$). This situation is not discussed by Lin and Guess (1994). In many cases, based on our knowledge of the components, we should know whether $c < 1$ (i.e., $p_3(t_i) > p_4(t_i)$) or $c > 1$ (i.e., $p_3(t_i) < p_4(t_i)$). Consider the case where prior information includes the knowledge that $c < 1$. In the absence of further knowledge on

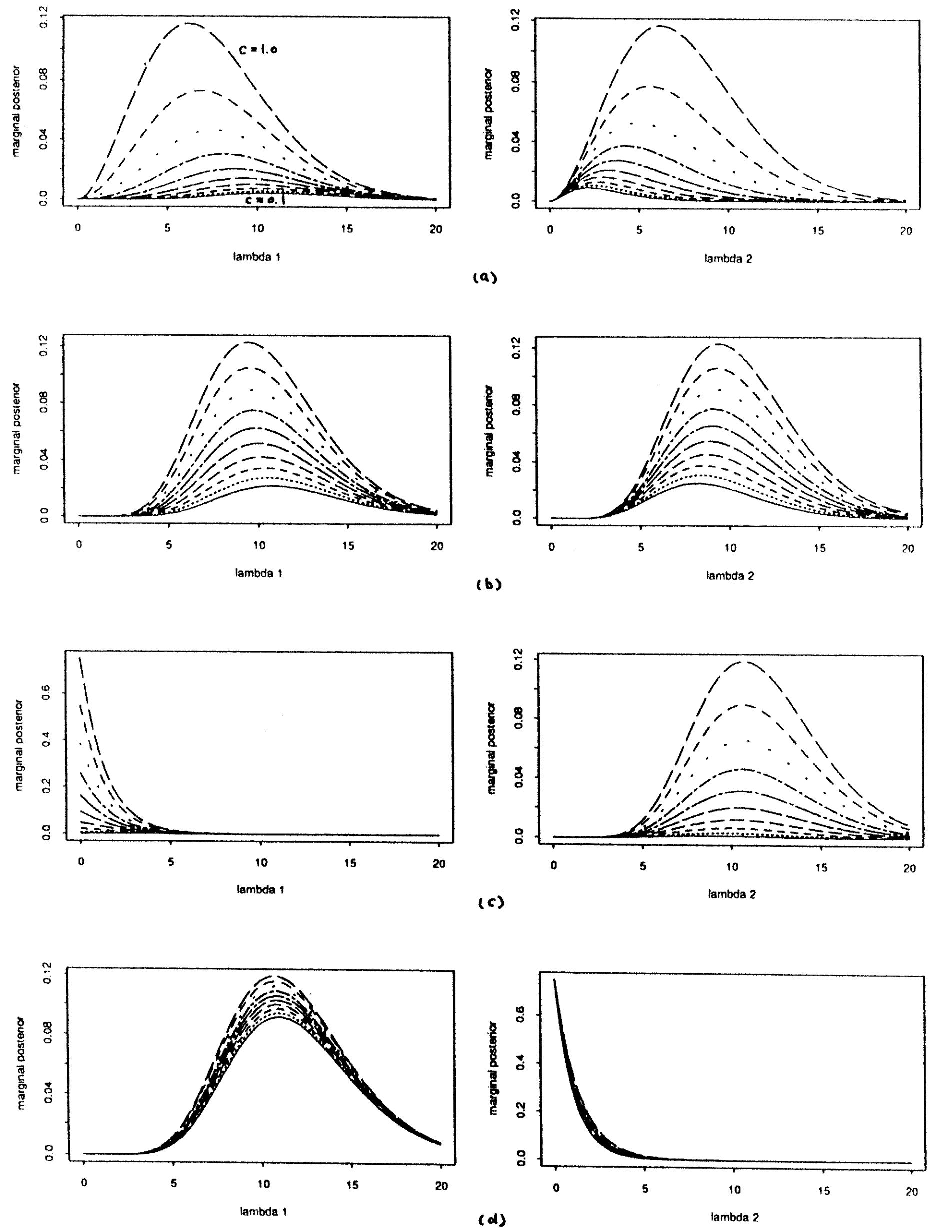

Figure 1. Marginal posteriors for various degrees of dependency ($c = 0.1, 0.2, \ldots, 1.0$), (a) $(n1, n2, n12, T) = (3, 3, 9, 1)$; (b) $(n1, n2, n12, T) = (9, 9, 3, 1)$; (c) $(n1, n2, n12, T) = (1, 9, 3, 1)$; $(n1, n2, n12, T) = (9, 1, 3, 1)$.

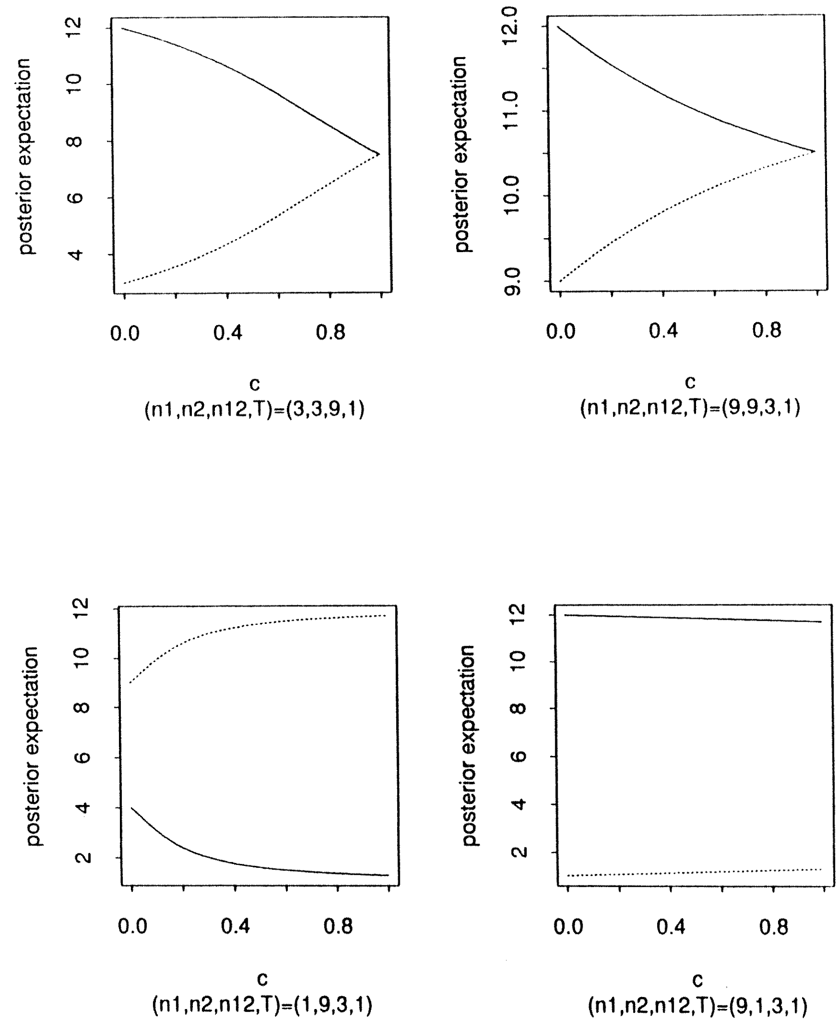

Figure 2. Bayes estimates (posterior means), -- λ_1; — λ_2.

c one general approach would be to assume a uniform prior, $U(0, 1)$, for c. The resulting posterior is then

$$p(\lambda_1, \lambda_2 \mid data) = k^* e^{-(\lambda_1+\lambda_2)T} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \lambda_1^{n_1+j-1} \lambda_2^{n_2+n_{12}-j-1} \int_0^1 c^{n_{12}-j}\, dc$$

$$= k^* e^{-(\lambda_1+\lambda_2)T} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \lambda_1^{n_1+j-1} \lambda_2^{n_2+n_{12}-j-1} \frac{1}{n_{12}-j+1}, \quad (10)$$

where

$$k^{*-1} = \frac{1}{T^n} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)}{n_{12}-j+1}.$$

Therefore, the marginal posteriors can be shown to be

$$p(\lambda_1 \mid data) = k^* e^{-\lambda_1 T} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\lambda_1^{n_1+j-1}}{n_{12}-j+1} \frac{\Gamma(n_2+n_{12}-j)}{T^{n_2+n_{12}-j}},$$

$$p(\lambda_2 \mid data) = k^* e^{-\lambda_2 T} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\lambda_2^{n_2+n_{12}-j-1}}{n_{12}-j+1} \frac{\Gamma(n_1+j)}{T^{n_1+j}}. \quad (11)$$

Figure 3 plots these marginal posteriors for the cases in Figure 1. Note that for the first two graphs, the posteriors indicate that λ_2 is less than λ_1 even though the data are symmetrical with respect to the components. This is a consequence of the prior assumption that $c<1$ which means that a masked case is more likely to be a result of a failure in Component 1. Results paralleling (10) and (11) can be obtained for $p_4(t_i) > p_3(t_i)$, i.e., $c > 1$, by switching the labels on the components.

From (11), the posterior means can be obtained as:

$$E(\lambda_1 \mid data) = k^* \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\Gamma(n_2+n_{12}-j)\Gamma(n_1+j+1)}{(n_{12}-j+1)T^{n+1}}$$

$$E(\lambda_2 \mid data) = k^* \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\Gamma(n_1+j)\Gamma(n_2+n_{12}-j+1)}{(n_{12}-j+1)T^{n+1}}. \quad (12)$$

Furthermore, posterior probability bounds on the λ_i can be obtained from (11). We are generally interested in an upper bound on λ_i. Consequently, an upper $1-\alpha$ bound can be obtained by solving for u in

$$\int_0^u p(\lambda_i \mid data)\, d\lambda_i = 1 - \alpha.$$

This equation can be solved numerically by iteration.

For the more general case where it is unknown whether $p_4(t_i)$ is greater or less than $p_3(t_i)$, it seems reasonable to give a prior weight of $1/2$ to both these cases resulting in a posterior

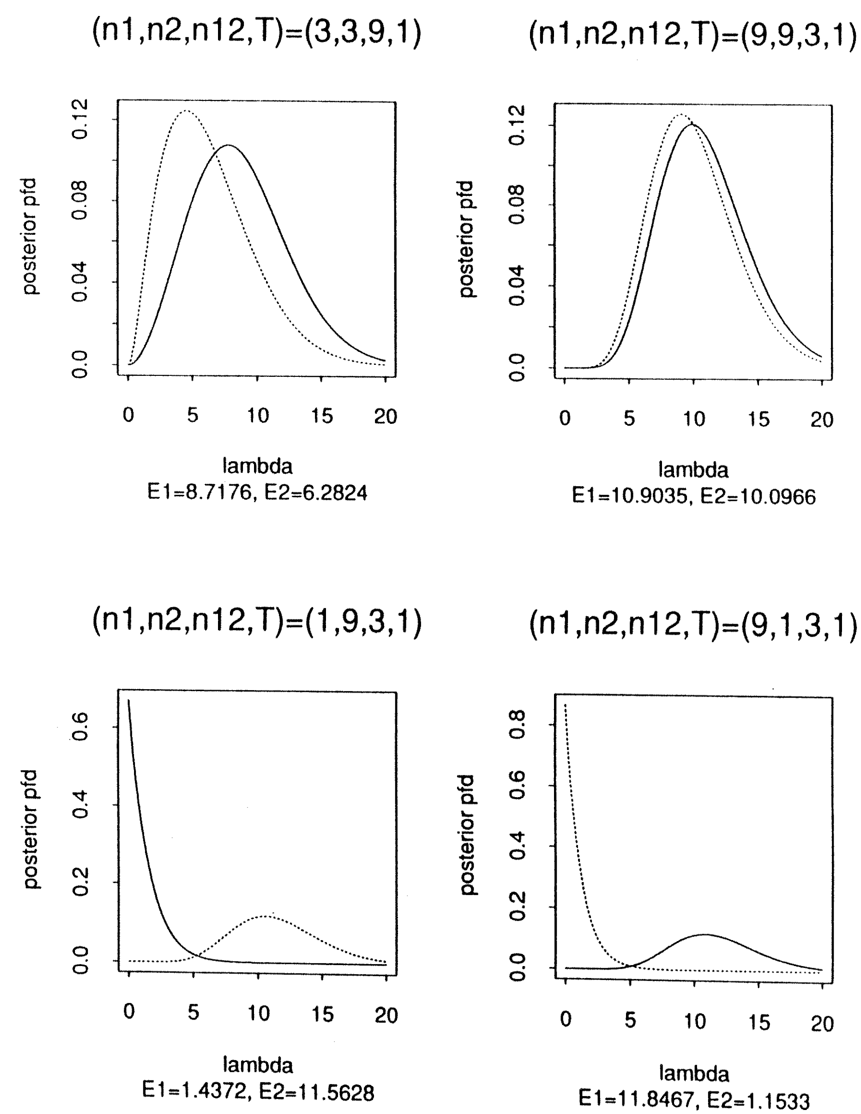

Figure 3. Marginal posteriors when c is $U(0, 1)$ distributed, -- λ_1; — λ_2.

which is proportional to

$$e^{-(\lambda_1+\lambda_2)T}\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}\lambda_1^{n_1+j-1}\lambda_2^{n_2+n_{12}-j-i}\frac{1}{n_{12}-j+1}$$
$$+\,e^{-(\lambda_1+\lambda_2)T}\sum_{j=0}^{n_{12}}\binom{n_{12}}{j}\lambda_1^{n_1+n_{12}-j-1}\lambda_2^{n_2+j-i}\frac{1}{n_{12}-j+1}\,. \qquad (13)$$

The first expression in (13) comes directly from (10), while the second one is due to switching component labels in (10). After some algebra, we can rewrite (13) as

$$p(\lambda_1,\lambda_2\mid data) = \frac{K}{T_n}\sum_{j=0}^{n_{12}}\left[\binom{n_{12}+2}{j+1}g(\lambda_1\mid T;n_1+j)\right.$$
$$\left. g(\lambda_2\mid T;n_2+n_{12}-j)\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)\right] \qquad (14)$$

where $g(x\mid a;b)=(a^b/\Gamma(b))x^{b-1}e^{-ax}$.

It follows immediately that

$$K^{-1}=\frac{1}{T_n}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)$$

and that the marginal posteriors are

$$p(\lambda_1\mid data) = \frac{K}{T^n}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}g(\lambda_1\mid T;n_1+j)\Gamma(n_1+j)\Gamma(n_2+n_{12}-j) \qquad (15)$$

$$p(\lambda_2\mid data) = \frac{K}{T^n}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}g(\lambda_2\mid T;n_2+n_{12}-j)\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)$$

which have posterior means

$$E(\lambda_1\mid data) = \frac{K}{T_n}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}\frac{n_1+j}{T}\Gamma(n_1+j)\Gamma(n_2+n_{12}-j)$$
$$= \frac{K}{T^{n+1}}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}\Gamma(n_1+j+1)\Gamma(n_2+n_{12}-j)$$

and

$$E(\lambda_2\mid data)=\frac{K}{T^{n+1}}\sum_{j=0}^{n_{12}}\binom{n_{12}+2}{j+1}\Gamma(n_1+j)\Gamma(n_2+n_{12}-j+1).$$

Figure 4 plots these marginal posteriors and means for the cases on Figure 1. Probability bounds can be computed numerically as discussed previously. For the first two cases, the posteriors for λ_1 and λ_2 coincide due to the symmetry of the data. This contrasts with the corresponding results in Figure 3 discussed above.

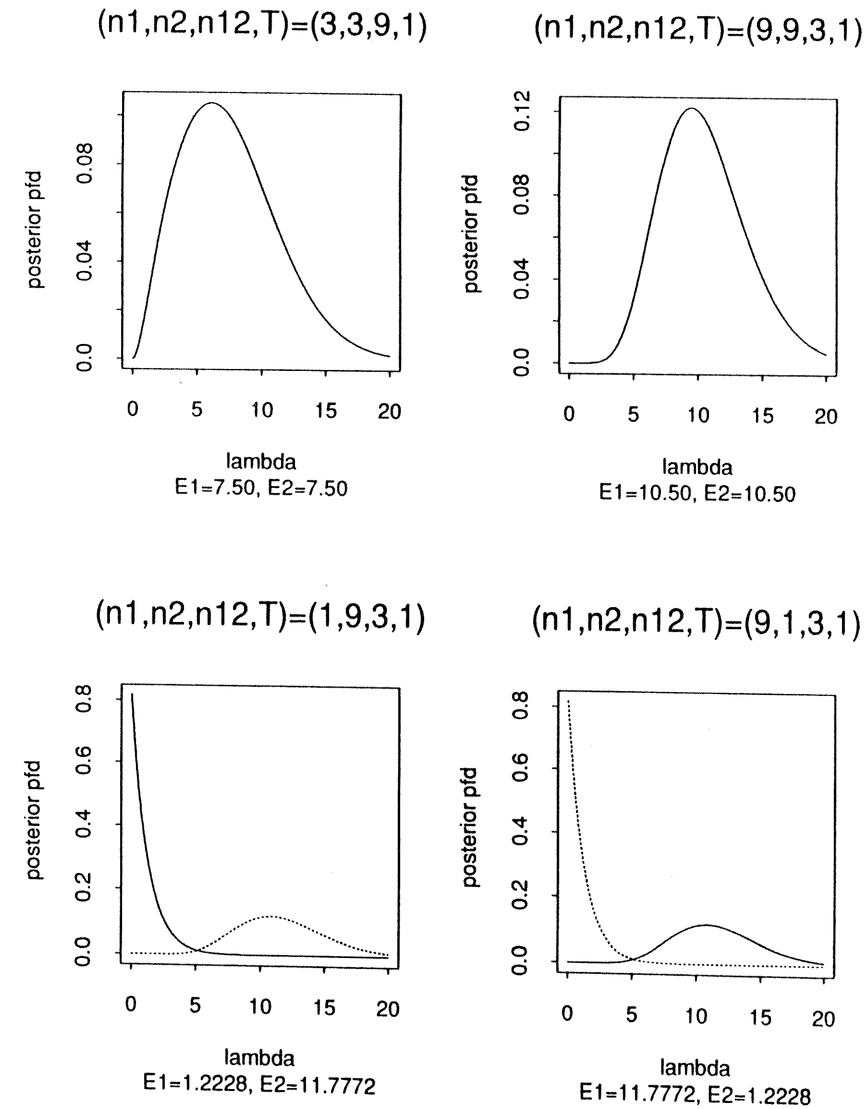

Figure 4. Marginal posteriors for $c = 1/2$, ... λ_1; — λ_2.

4. System Reliability

For any specific system G, the failure time of component j, as mentioned above, is T_{Gj}. The reliability of system G at time t is

$$\begin{aligned} P(T_G \geq t) &= P(T_{Gj} \geq t, \quad \text{for all } j = 1, \ldots, J) \\ &= e^{-(\lambda_1+\lambda_2)t} \\ &= e^{-\delta t} = R(\delta; t), \quad \text{where } \delta = \lambda_1 + \lambda_2. \end{aligned} \tag{16}$$

It is desirable to do inference on δ, for we can then carry out inference on the system reliability very easily. Now the posterior distribution for δ can also be derived from (6) as follows. Consider the transformation

$$\begin{aligned} \delta &= \lambda_1 + \lambda_2 \\ \gamma &= \lambda_2 \end{aligned}$$

in Equation (6), which results in

$$p(\delta, \gamma \mid c, data) = ke^{-\delta T}(\delta - \gamma)^{n_1-1}\gamma^{n_2-1}[\delta - \gamma(1-c)]^{n_{12}}. \tag{17}$$

Thus,

$$\begin{aligned} p(\delta \mid c, data) &= \int_0^\delta p(\delta, \gamma \mid c, data)\, d\gamma \\ &= ke^{-\delta T}\delta^{n_1+n_2+n_{12}-1}(1-c)^{n_{12}} \\ &\quad \times \int_0^\delta \left(1 - \frac{\gamma}{\delta}\right)^{n_1-1}\left(\frac{\gamma}{\delta}\right)^{n_2-1}\left(\frac{1}{1-c} - \frac{\gamma}{\delta}\right)^{n_{12}} d\frac{\gamma}{\delta} \end{aligned} \tag{18}$$

By replacing $v = \gamma/\delta$, we have

$$p(\delta \mid c, data) = ke^{-\delta T}\delta^{n_1+n_2+n_{12}-1}(1-c)^{n_{12}}\int_0^1 (1-v)^{n_1-1}v^{n_2-1}\left(\frac{1}{1-c}\right)^{n_{12}} dv.$$

Now, expand

$$\left(\frac{1}{1-c} - v\right)^{n_{12}} = \left[\left(\frac{c}{1-c}\right) + (1-v)\right]^{n_{12}}, \text{ we obtain}$$

$$\begin{aligned} p(\delta \mid c, data) &= ke^{-\delta T}\delta^{n_1+n_2+n_{12}-1}\sum_{t=0}^{n_{12}}\binom{n_{12}}{t}(1-c)^t c^{n_{12}-t} \\ &\quad \times \int_0^1 v^{n_2-1}(1-v)^{n_1+t-1}\, dv \\ &= ke^{-\delta T}\delta^{n_1+n_2+n_{12}-1}\sum_{t=0}^{n_{12}}\frac{n_{12}!}{t!(n_{12}-t)!} \\ &\quad \times (1-c)^t c^{n_{12}-t}\frac{(n_2-1)!(n_1+t-1)!}{(n_1+n_2+t-1)!}. \end{aligned}$$

The summation is of known quantities, so that $2\delta T$ follows a $\chi^2_{2(n_1+n_2+n_{12})}$ distribution.

Note that the marginal posterior distribution for δ is independent of c. In fact, it immediately follows that $E(\delta \mid data) = (n_1+n_2+n_{12})/T = n/T$, and a two-sided $1-\alpha$ posterior probability interval for δ is given by

$$\left[\frac{1}{2T}\chi^2_{2n:\alpha/2}, \frac{1}{2T}\chi^2_{2n:1-\alpha/2}\right]. \tag{19}$$

It is intuitively clear that for inference on the system reliability, both the type (dependent or independent) and amount of masking are irrelevant.

5. Concluding Remarks

This paper illustrates the use of a Bayesian approach for inference on masked system lifetime data. In many situations, the system life data is right censored. It can readily be seen that under censoring the results of Sections 3 and 4 still hold with the change that $n = n_1 + n_2 + n_{12}$ is now the total number of observed systems which are not censored rather than the total number on test, and T is the total time on test.

Other possible extensions are currently under investigation. These include more complicated: (1) systems with J (≥ 3) components, (2) life distributions and (3) prior distributions. Although a purely numerical approach is always in principle possible, it can be difficult to implement while the feasibility of an analytical solution is unclear. For example, consider a three-component system in general as follows:

$$\begin{aligned} P(M_i &= \{1,2,3\} \mid T_i = t_i, K_i = 1) = p_1(t_i) \\ P(M_i &= \{1,2,3\} \mid T_i = t_i, K_i = 2) = c_1 p_1(t_i) \\ P(M_i &= \{1,2,3\} \mid T_i = t_i, K_i = 3) = c_2 p_1(t_i) \\ P(M_i &= \{1,2\} \mid T_i = t_i, K_i = 1) = p_2(t_i) \\ P(M_i &= \{1,2\} \mid T_i = t_i, K_i = 2) = dp_2(t_i) \\ P(M_i &= \{1,3\} \mid T_i = t_i, K_i = 1) = p_3(t_i) \\ P(M_i &= \{1,3\} \mid T_i = t_i, K_i = 3) = ep_3(t_i) \\ P(M_i &= \{2,3\} \mid T_i = t_i, K_i = 2) = p_4(t_i) \\ P(M_i &= \{2,3\} \mid T_i = t_i, K_i = 3) = fp_4(t_i) \\ P(M_i &= \{1\} \mid T_i = t_i, K_i = 1) = p_5(t_i) \\ P(M_i &= \{2\} \mid T_i = t_i, K_i = 2) = p_6(t_i) \\ P(M_i &= \{3\} \mid T_i = t_i, K_i = 3) = p_7(t_i). \end{aligned}$$

Unless additional prior knowledge is available, the straightforward extension presents obvious difficulties.