Designing Outer Array Points

YUCHUNG WANG

Academia Sinica, Taipei and Rutgers University, Camden, NJ 08102

DENNIS K. J. LIN

University of Tennessee, Knoxville, TN 37996

KAI-TAI FANG

Academia Sinica, Bejing and Hong Kong Baptist University, Hong Kong

Taguchi's product-array design consists of two portions: an inner array containing the design factors and an outer array containing noise factors. The function of the outer array is very different from that of the inner array, however. The outer array is most likely to sample or simulate the distribution of the noise factors, while the inner array is designated to facilitate the optimization. Since performance for each inner point is evaluated via its corresponding outer array points, the outer array plays an important role in robust design. We show here that the optimal representative point method via quantizer is superior to using other methods (including orthogonal array) to design outer array points. All optimal representative points are tabulated for practical use. The usage of these tables is demonstrated by examples.

Introduction

When a designed product or manufacturing process is put to mass production, there are inevitably some uncontrollable or hard-to-control variables which affect the quality of the product. These variables are called *noise variables*. Taguchi (1987) emphasizes that statistical testing of a product or process should be carried out at the design stage. He advocates moving the investigation of the impact of noise variables upstream in the design stage, instead of downstream in the production stage. One novel idea of Taguchi is the product-array design.

A product-array design consists of two parts: the inner array for controllable design variables and the outer array for uncontrollable noise variables. All the product configurations being experimented on are subject to combinations of noise variables which degrade or deviate the product performance. Data analysis of the experiment reveals the nominal settings for the design variables at which the impact of noise variables is minimized. For the inner array design, an orthogonal array (essentially a fractional factorial) is commonly prescribed because of its economy and efficiency. Fractional factorials have been used for many years, certainly since Yates (1935).

The objective of the outer array is to efficiently draw information about the joint distribution of noise variables, rather than to compare levels of the variables as in an inner array. For examples, to assess the polysilicon deposition process for silicon wafers, as in Phadke (1989), the location factor as top, bottom and center within a wafer, are selected as the outer array factors. In a case study for the Ina Tile company (Taguchi (1986)) to aid uniform expansion of tile clay, the outer array factors are at different locations inside the baking kiln. To assess the uniformity of thickness in a protective paint coating over the back of a glass mirror, an outer array factor consists of different spots scattered over the back of the glass. Box and Jones (1992) found the best recipe for a new cake mix that would make the taste of the baked cake insensitive to the outer array factors of baking temperature and baking time. In Taguchi's (1987, p. 98) Wheatstone bridge example, the outer

Dr. Wang is an Associate Research Fellow in the Institute of Statistical Sciences and an Associate Professor of Statistics.

Dr. Lin is an Associate Professor of Statistics. He is a member of ASQC.

Dr. Fang is a Research Professor in the Institute of Applied Mathematics and a Professor of Mathematics.

array factors represent the component variation (see Example 4). In all of the above situations, the noise variables are either impractical to identify or impossible to control in the production stage.

If the goal of robust design is to estimate the response distribution in terms of the inner array variables, the performance measure for each inner array point is critical. Such a performance measure is mostly evaluated through the outer array points. Thus the outer array design plays an important role in robust design. With an orthogonal array (OA) it is useful to investigate and compare relatively few levels of the factors. But as a sampling plan an orthogonal array is not efficient, particularly in designing outer array points. The outer array design that is based on an orthogonal array does not take advantage of the information available from the noise distribution. Such distribution information is very likely available from the supplier because this information is typically used to rate and classify components. The outer array design should take into account the distribution of the noise factors to achieve a better estimate of variation.

Recently, Wang, Fang, and Lin (1992) introduce the concept of a quantizer to design representative points for the given distribution in which optimal representative points partition the noise space according to the variability of the noise variables using a sample that can reproduce the underlying distribution with little loss of fidelity. They demonstrate that the use of the optimal representative point to design the outer array points has certain advantages over other approaches. In this paper, we will illustrate how to apply this new approach to real problems through the use of examples. A brief description of the optimality criterion and construction method are given below. The reader is referred to Wang, Fang, and Lin (1992) for its detailed derivation and theoretical implications.

Optimality Criterion and Construction Algorithm

Let Ξ be the s-dimensional domain of a random variable N, and $F(N)$ be the cumulative distribution function (cdf) of N over Ξ. A k-run quantizer $Q = (N^{(k)}, U)$ of N consists of

1. a set of k output vectors $N^{(k)} = (N_1, N_2, \ldots, N_k)$;
2. a partition $U = (S_1, S_2, \ldots, S_k)$ of the s-dimensional region Ξ with k disjoint and exhaustive regions; and
3. a mapping $Q : \Xi \rightarrow U$ defined by $Q(N) = N_i$, if $N \in S_i$.

A quantizer defines a discrete random variable in Ξ and approximates $F(N)$ by its empirical cdf. The output vector $\{N_i, 1 \leq i \leq k\}$ is called the *representative points*. Note that to design representative points is equivalent to finding an "optimal" quantizer when k is prespecified.

The l_p-discrepancy of Q, relative to F, is defined as

$$D_F^p(k) = D_F^p(k; Q) = \left[\int_{\Xi} \| F_Q(\mathbf{t}) - F(\mathbf{t}) \|^p dt\right]^{1/p}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_s)^T$, $F_Q(\mathbf{t}) = n(\mathbf{t}; Q)/k$ is the empirical cdf of Q, and $n(\mathbf{t}, Q)$ denotes the number of points $N_j = (N_{j1}, \ldots, N_{js})^T$ in Q satisfying $N_{jl} \leq t_l$ for $l = 1, \ldots, s$. The discrepancy is the Kolmogrov-Smirnov distance between the empirical distribution of Q and the uniform distribution over C^s. The l_∞-discrepancy is shortened to "discrepancy", and for the case F with uniform cdf on the s-dimensional unit hypercube $\Xi = C^s = [0,1]^s$ is denoted by $D(k; Q)$. A discrepancy optimal quantizer Q^* satisfies

$$D(k; Q^*) = \inf_Q D(k; Q).$$

Discrepancy optimal representative points are useful for approximating performance measure of integral type (such as variance).

Construction Algorithm

For $s = 1$ dimension, it can be shown that the discrepancy optimal quantizer in [0,1], Q^*, is

$$Q^* = \left\{\frac{2i-1}{2k},\ i = 1, 2, \ldots, k\right\}.$$

In this case, $D(k; Q^*) = (2k)^{-1}$. The discrepancy of the optimal quantizer relative to any cdf $F(N)$ is

$$Q^* = \left\{F^{-1}\left(\frac{2i-1}{2k}\right),\ i = 1, 2, \ldots, k\right\}.$$

The case of the univariate normal cdf has been extensively used in the Princeton study for robustness (see Andrews, Bickel, Hampel, Huber, Rogers, and Tukey (1972)).

For a higher dimension ($s \geq 2$), we adapt the good lattice points (GLP) method proposed by Wang and Fang (1981). Let $(k; h_1, \ldots, h_s)$ be an integer vector satisfying $1 = h_1 < h_2 < \cdots < h_s$. Let

$$q_{ij} = \{ih_j - 1 \pmod{k}\} + 1$$

and

$$x_{ij} = (2q_{ij} - 1)(2k)^{-1}$$

for $i = 1, \ldots, k$ and $j = 1, \ldots, s$. The set $P = \{\mathbf{x}_i = (x_{i1}, \ldots, x_{is})^T,\ i = 1, \ldots, s\}$ is called *lattice points* of the generating vector $(k; h_1, \ldots, h_s)$ and is used to design the representative points in C^s.

A large-sample justification of GLP and its existence was given in Korobov (1959). It is also proved that the GLP has the discrepancy $D(k) = O(k^{-1}(\log k)^s \log\log k)$ which is very close to the theoretical bounds (see Wang, Fang, and Lin (1992)). The GLP method is available for most practical values of k. However, theoretical limitation results in some missing entries in certain combinations of n and k, as will be shown later in Table 2.

One-Dimension Cases

Table 1 shows the optimal representative points in [0, 1] for uniform, normal, and exponential distributions. In practice, one should choose the representative points according to the specific value of n (the number of runs) and the prior distribution, and then convert these points by proper scale. If no prior knowledge about the distribution is available, a uniform distribution is most likely to be assumed. We illustrate the procedure using the polysilicon deposition example as follows.

Example 1: Polysilicon Deposition Process

The process to deposit polysilicon on a silicon wafer described in Phadke (1989, p.68) involved mounting wafers on two quartz carriers and placing them in a hot-wall, reduced-pressure reactor. Silane and nitrogen gases were introduced at one end and pumped out at the other end of the reactor. The silane gas pyrolizes, and a polysilicon layer is deposited on top of the oxide layer on the wafers. Each quartz carrier accommodated twenty-five wafers, and each occupied half of the reactor. A total of fifty wafers were deposited simultaneously along the flow of the silane gas from the gas inlet to the gas outlet. Gas flow patterns caused different silane gas concentrations along the length of the reactor. The sampling locations were selected to approximate the distribution of the gas flow pattern.

In the original experiment described in Phadke (1989), three measurements were taken along the quartz carrier at locations 3, 23, and 48. However, an optimal one-dimensional quantizer provides a better way to select the representative points. If the uniform distribution is assumed as in Phadke, then Table 1 shows that the set $\{1/6, 1/2, 5/6\}$ is the optimal

TABLE 1. The One-Dimensional Optimal Representative Points in [0,1]

n	Uniform	Normal	Exponential
2	0.2500	0.3876	0.0575
	0.7500	0.6124	0.2773
3	0.1667	0.3388	0.0365
	0.5000	0.5000	0.1386
	0.8333	0.6612	0.3584
4	0.1250	0.3083	0.0267
	0.3750	0.4470	0.0940
	0.6250	0.5530	0.1962
	0.8750	0.6917	0.4159
5	0.1000	0.2864	0.0211
	0.3000	0.4127	0.0713
	0.5000	0.5000	0.1386
	0.7000	0.5873	0.2408
	0.9000	0.7136	0.4605
6	0.0833	0.2695	0.0174
	0.2500	0.3876	0.0575
	0.4167	0.4650	0.1078
	0.5833	0.5350	0.1751
	0.7500	0.6124	0.2773
	0.9167	0.7305	0.4970
7	0.0714	0.2557	0.0148
	0.2143	0.3681	0.0482
	0.3571	0.4391	0.0884
	0.5000	0.5000	0.1386
	0.6429	0.5609	0.2059
	0.7857	0.6319	0.3081
	0.9286	0.7443	0.5278
8	0.0625	0.2443	0.0129
	0.1875	0.3522	0.0415
	0.3125	0.4186	0.0749
	0.4375	0.4738	0.1151
	0.5625	0.5262	0.1653
	0.6875	0.5814	0.2326
	0.8125	0.6478	0.3348
	0.9375	0.7557	0.5545
9	0.0556	0.2344	0.0114
	0.1667	0.3388	0.0365
	0.2778	0.4018	0.0651
	0.3889	0.4530	0.0985
	0.5000	0.5000	0.1386
	0.6111	0.5470	0.1889
	0.7222	0.5982	0.2562
	0.8333	0.6612	0.3584
	0.9444	0.7656	0.5781

choice on [0, 1]. Accordingly, locations 8 ($= 50 \times 1/6$), 25 ($= 50 \times 3/6$), and 42 ($= 50 \times 5/6$) should be measured. See Figure 1(a) for a graphical comparison. Note that in Figure 1, we see that the optimal representative points will minimize the maximum distance from the "true" cdf. The design given here is better than Phadke's original choice.

The evidence is even more striking when the underlying distribution is normal or exponential (see Figure 1(b) and 1(c), respectively). If a normal distribution is assumed, the locations 17 ($= 50 \times 0.3388$), 25 ($= 50 \times 0.5$), and 33 ($= 50 \times 0.6612$) should be measured. If gas flow decays exponentially, then the noise distribution is a truncated exponential, and locations 2 ($= 50 \times 0.0365$), 7 ($= 50 \times 0.1386$), and 18 ($= 50 \times 0.3584$) are recommended.

Two- and Higher- Dimension Cases

Table 2 lists the optimal representative points in a hypercube as $n = 4, \ldots, 15$ – assuming that the

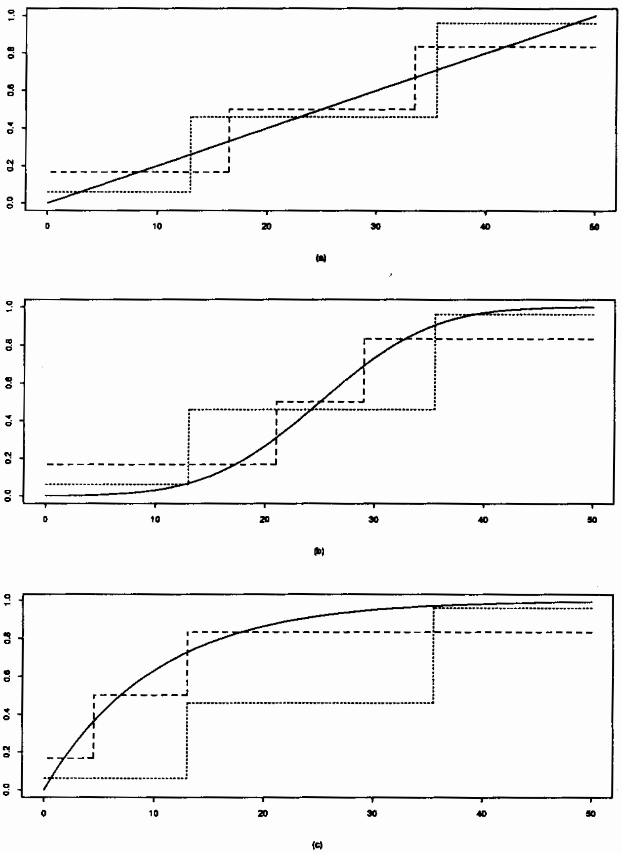

FIGURE 1. Empirical cdf's for $k = 3$ Representative Points: (a) Uniform, (b) Normal, and (c) Exponential. The Solid Line is the True Curve; the Dotted Line is Phadke's Design; and the Dashed Line is the Optimal Design.

noise factors are uniformly distributed, which is the most common assumption when the true distribution is unknown. The cases for $n = 16, \ldots, 25$ are given in the Appendix. Given the values of n and k (the number of runs and the number of factors, respectively), one can easily obtain the optimal representative points directly from the table. For example, the four-run optimal representative points in $[0,1]^2$ (i.e., $(n,k) = (4,2)$) are {(0.125, 0.375), (0.375, 0.875), (0.625, 0.125), (0.875, 0.625)}; and the five optimal representative points for a three-factor outer array in $[0,1]^3$ are {(0.1, 0.3, 0.7), (0.3, 0.7, 0.5), (0.5, 0.1, 0.3), (0.7, 0.5, 0.1), (0.9, 0.9, 0.9)}. For $k = 2$ dimensions, Figure 2(a) plots these four points as well as the 2^2 factorial design points for a visual comparison. Figure 2(b) shows the five point case: five optimal representative points versus 2^2 plus a center point. Note that the representative point is available for any number of n, unlike an orthogonal array whose run size is usually limited (e.g., a multiple of four in two-level cases).

Example 2: Cake Mix

Consider Box and Jones's (1992) example of the search for the best recipe for a cake mix. The inner array points were run with three design factors, flour, shortening, and egg powder, in a 2^3 factorial design. A 2^2 factorial was used to design the outer array points, mainly considering the variation that results from two noise factors: the baking temperature and the baking time. As mentioned, a better way to design the four-run outer array in this case would be {(0.125, 0.375), (0.375, 0.875), (0.625, 0.125), (0.875, 0.625)}. Table 2 can be used for designing any number of outer array points. For example, if six runs are considered, the six points should be {(0.0833, 0.4167), (0.25, 0.9167), (0.4167, 0.25), (0.5833, 0.75), (0.75, 0.0833), (0.9167, 0.5833)}.

Example 3: Computer Experiments

In this example, we choose a simple (but typical) problem in computer experiments where the response y is a known deterministic function of the design factors. The goal here is to find the nominal settings for the design factors such that the response is *insensitive* to component variation. The Wheatstone bridge described in Taguchi (1987) is a well-known problem of this general type.

Consider the distance traveled by an object ejected from a cannon. Elementary physics gives us

$$y = \frac{1}{g}\left[\frac{G}{m}\right]^2 \sin(2\alpha)$$

TABLE 2. The Optimal Representative Points in $[0,1]^k$ Assuming a Uniform Distribution for $n = 4, \ldots, 15$ (all values need to be multiplied by 10^{-4})

Number of Runs, n	Number of Factors, k											
	1	2	3	4	5	6	7	8	9	10	11	12
4	1250	3750	8750	6250								
	3750	8750	6250	1250								
	6250	1250	3750	8750								
	8750	6250	1250	3750								
5	1000	3000	7000	5000								
	3000	7000	5000	1000								
	5000	1000	3000	7000								
	7000	5000	1000	3000								
	9000	9000	9000	9000								
6	833	4167	2500	9167	5833	7500						
	2500	9167	5833	7500	833	4167						
	4167	2500	9167	5833	7500	833						
	5833	7500	833	4167	2500	9167						
	7500	833	4167	2500	9167	5833						
	9167	5833	7500	833	4167	2500						
7	714	3571	2143	7857	5000	6429						
	2143	7857	5000	6429	714	3571						
	3571	2143	7857	5000	6429	714						
	5000	6429	714	3571	2143	7857						
	6429	714	3571	2143	7857	5000						
	7857	5000	6429	714	3571	2143						
	9286	9286	9286	9286	9286	9286						
8	625	4375	8125	1875	9375	5625						
	1875	9375	5625	4375	8125	625						
	3125	3125	3125	6875	6875	6875						
	4375	8125	625	9375	5625	1875						
	5625	1875	9375	625	4375	8125						
	6875	6875	6875	3125	3125	3125						
	8125	625	4375	5625	1875	9375						
	9375	5625	1875	8125	625	4375						
9	556	3889	7222	1667	8333	5000						
	1667	8333	5000	3889	7222	556						
	2778	2778	2778	6111	6111	6111						
	3889	7222	556	8333	5000	1667						
	5000	1667	8333	556	3889	7222						
	6111	6111	6111	2778	2778	2778						
	7222	556	3889	5000	1667	8333						
	8333	5000	1667	7222	556	3889						
	9444	9444	9444	9444	9444	9444						
10	500	6500	4500	1500	2500	9500	3500	7500	8500	5500		
	1500	2500	9500	3500	5500	8500	7500	4500	6500	500		
	2500	9500	3500	5500	8500	7500	500	1500	4500	6500		
	3500	5500	8500	7500	500	6500	4500	9500	2500	1500		
	4500	1500	2500	9500	3500	5500	8500	6500	500	7500		
	5500	8500	7500	500	6500	4500	1500	3500	9500	2500		
	6500	4500	1500	2500	9500	3500	5500	500	7500	8500		
	7500	500	6500	4500	1500	2500	9500	8500	5500	3500		
	8500	7500	500	6500	4500	1500	2500	5500	3500	9500		
	9500	3500	5500	8500	7500	500	6500	2500	1500	4500		

TABLE 2. — *Continued*

Number of Runs, n	Number of Factors, k											
	1	2	3	4	5	6	7	8	9	10	11	12
11	455	5909	4091	1364	2273	8636	3182	6818	7727	5000		
	1364	2273	8636	3182	5000	7727	6818	4091	5909	455		
	2273	8636	3182	5000	7727	6818	455	1364	4091	5909		
	3182	5000	7727	6818	455	5909	4091	8636	2273	1364		
	4091	1364	2273	8636	3182	5000	7727	5909	455	6818		
	5000	7727	6818	455	5909	4091	1364	3182	8636	2273		
	5909	4091	1364	2273	8636	3182	5000	455	6818	7727		
	6818	455	5909	4091	1364	2273	8636	7727	5000	3182		
	7727	6818	455	5909	4091	1364	2273	5000	3182	8636		
	8636	3182	5000	7727	6818	455	5909	2273	1364	4091		
	9545	9545	9545	9545	9545	9545	9545	9545	9545	9545		
12	417	4583	6250	7917	7083	1250	9583	3750	2083	2917	8750	5417
	1250	9583	2083	5417	3750	2917	8750	7917	4583	6250	7083	417
	2083	3750	8750	2917	417	4583	7917	1250	7083	9583	5417	6250
	2917	8750	4583	417	7917	6250	7083	5417	9583	2083	3750	1250
	3750	2917	417	8750	4583	7917	6250	9583	1250	5417	2083	7083
	4583	7917	7083	6250	1250	9583	5417	2917	3750	8750	417	2083
	5417	2083	2917	3750	8750	417	4583	7083	6250	1250	9583	7917
	6250	7083	9583	1250	5417	2083	3750	417	8750	4583	7917	2917
	7083	1250	5417	9583	2083	3750	2917	4583	417	7917	6250	8750
	7917	6250	1250	7083	9583	5417	2083	8750	2917	417	4583	3750
	8750	417	7917	4583	6250	7083	1250	2083	5417	3750	2917	9583
	9583	5417	3750	2083	2917	8750	417	6250	7917	7083	1250	4583
13	385	4231	5769	7308	6538	1154	8846	3462	1923	2692	8077	5000
	1154	8846	1923	5000	3462	2692	8077	7308	4231	5769	6538	385
	1923	3462	8077	2692	385	4231	7308	1154	6538	8846	5000	5769
	2692	8077	4231	385	7308	5769	6538	5000	8846	1923	3462	1154
	3462	2692	385	8077	4231	7308	5769	8846	1154	5000	1923	6538
	4231	7308	6538	5769	1154	8846	5000	2692	3462	8077	385	1923
	5000	1923	2692	3462	8077	385	4231	6538	5769	1154	8846	7308
	5769	6538	8846	1154	5000	1923	3462	385	8077	4231	7308	2692
	6538	1154	5000	8846	1923	3462	2692	4231	385	7308	5769	8077
	7308	5769	1154	6538	8846	5000	1923	8077	2692	385	4231	3462
	8077	385	7308	4231	5769	6538	1154	1923	5000	3462	2692	8846
	8846	5000	3462	1923	2692	8077	385	5769	7308	6538	1154	4231
	9615	9615	9615	9615	9615	9615	9615	9615	9615	9615	9615	9615
14	357	2500	4643	1071	7500	5357	8929	9643				
	1071	5357	9643	2500	4643	357	7500	8929				
	1786	8214	3929	3929	1786	6071	6071	8214				
	2500	357	8929	5357	9643	1071	4643	7500				
	3214	3214	3214	6786	6786	6786	3214	6786				
	3929	6071	8214	8214	3929	1786	1786	6071				
	4643	8929	2500	9643	1071	7500	357	5357				
	5357	1071	7500	357	8929	2500	9643	4643				
	6071	3929	1786	1786	6071	8214	8214	3929				
	6786	6786	6786	3214	3214	3214	6786	3214				
	7500	9643	1071	4643	357	8929	5357	2500				
	8214	1786	6071	6071	8214	3929	3929	1786				
	8929	4643	357	7500	5357	9643	2500	1071				
	9643	7500	5357	8929	2500	4643	1071	357				

TABLE 2. — *Continued*

Number of Runs, n	Number of Factors, k 1	2	3	4	5	6	7	8	9	10	11	12
15	333	2333	4333	1000	7000	5000	8333	9000				
	1000	5000	9000	2333	4333	333	7000	8333				
	1667	7667	3667	3667	1667	5667	5667	7667				
	2333	333	8333	5000	9000	1000	4333	7000				
	3000	3000	3000	6333	6333	6333	3000	6333				
	3667	5667	7667	7667	3667	1667	1667	5667				
	4333	8333	2333	9000	1000	7000	333	5000				
	5000	1000	7000	333	8333	2333	9000	4333				
	5667	3667	1667	1667	5667	7667	7667	3667				
	6333	6333	6333	3000	3000	3000	6333	3000				
	7000	9000	1000	4333	333	8333	5000	2333				
	7667	1667	5667	5667	7667	3667	3667	1667				
	8333	4333	333	7000	5000	9000	2333	1000				
	9000	7000	5000	8333	2333	4333	1000	333				
	9667	9667	9667	9667	9667	9667	9667	9667				

where y is the distance, g is the gravity constant, G is the momentum, m is the fixed mass (0.2 kg in this study), and α is the angle. The goal is to seek the best combination of G and α so that the the distance y is close to a given target value (2000 meters in this study) with the least variation transmitted from each component. The inner array factors are, of course, G and α. Suppose that the tolerance for each component is Δm, ΔG, and $\Delta\alpha$. These are the outer array factors. Now, if we take three levels for all inner array factors, a 3^2 full factorial design is appropriate. For the outer array, if each factor is at three levels (e.g., $m-\Delta m$, m, $m+\Delta m$), the current orthogonal array leads to a 3^{3-1} fractional factorial design. After converting into $[0,1]$ these nine orthogonal array points are $\{(0,0,0.5), (0.5, 0,1), (1,0,0), (0,0.5,1), (0.5,0.5,0), (1,0.5,0.5), (0,1,0), (0.5,1,0.5), (1,1,1)\}$. However, if nine points are to be run, a better choice would be $\{(0.0566, 0.4375, 0.8125), (0.1667, 0.8333, 0.5), (0.2778, 0.2778, 0.2778), (0.3889, 0.7222, 0.0566), (0.5, 0.1667, 0.8333), (0.6111, 0.6111, 0.6111), (0.7222, 0.0556, 0.3889), (0.8333, 0.5, 0.1667), (0.9444, 0.9444, 0.9444)\}$as shown in Table 2. The benefit of using optimal representative points is even more significant for larger values of k. We next illustrate the advantages of our approach using the well-known Wheatstone bridge as an example.

Example 4: Wheatstone Bridge

The circuit layout is displayed in Figure 3 (see also Taguchi (1987, p. 98)). The goal here is to select the nominal values of the parameters A, C, D, E, and F so that the unknown resistance y can be measured *precisely* (i.e., with least variability). The sources of variation are the bridge components A, B, C, D, E, F, and the reading of the galvanometer X. Taguchi proposed to use the L_{36} design as the inner array to accommodate the five 3-level design factors (A, C, D, E, and F). By assuming the tolerance for resistors A, B, C, D, and F to be $\pm 0.3\%$ and the current X to be ± 0.2 milliamperes, the same OA (L_{36}) is used for the outer array. The three levels for each noise factor in the OA (L_{36}) are selected at a value below or above a fixed portion from the nominal value and at the nominal value itself, where the nominal value is specified by the inner array point. If the distributions for all bridge components (factors A - F) are assumed to be distributed independently uniform, Table 2 can be used to choose the best representative points in the noise space. Likewise, if independent normality is assumed, a different set of representative points will be selected (as will be shown below).

Since small size for the outer array is desirable, the GLP quantizer is sufficient. The GLP quantizer has the additional benefit of fixing levels at $\{(2j-1)(2k)^{-1},\ 1 \le j \le k\}$ for all noise factors. Operationally, this is much more convenient. We adapted a GLP quantizer recommended by Shaw (1988) with $(s,k) = (7,29)$ and $\rho = 4$. These 29 representative points were then mapped into the appropriate space to form the representative points under multivariate normal or uniform distribution.

For each inner array point, Table 3 shows the mean and standard deviations which are computed corresponding with the outer array points. Mean

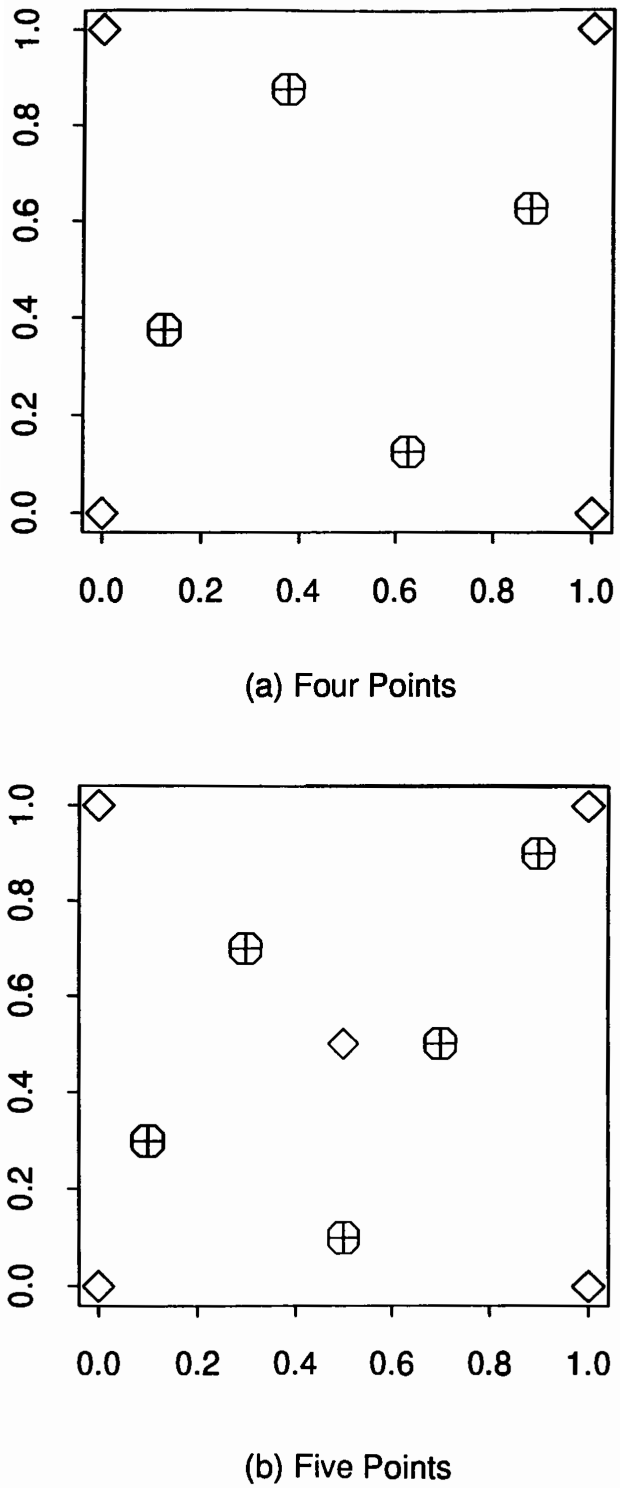

FIGURE 2. A Visual Comparison of Four and Five Point Designs in a Square.

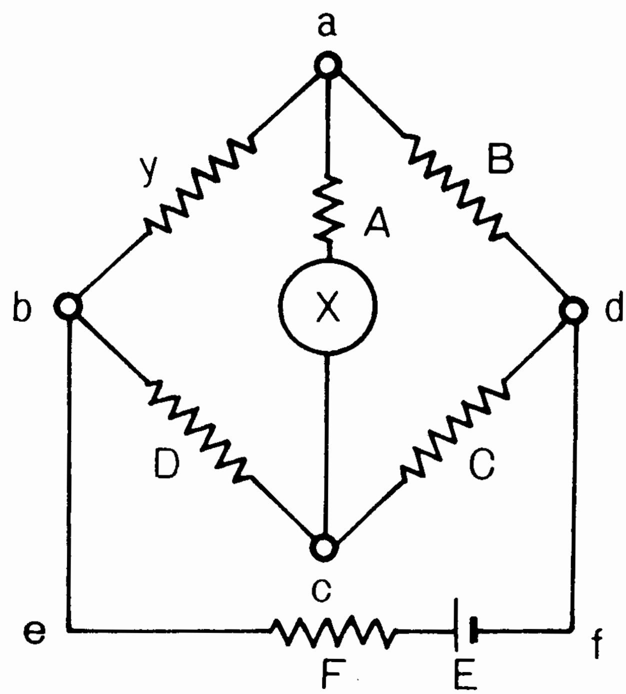

FIGURE 3. The Wheatstone Bridge.

and standard deviations are given in Table 3 as the typical performance measures of interest for illustration. In principle, any other performance measures (such as signal-to-noise ratio) can be included. Three columns are listed. The values in the first column are obtained by Taguchi's original 36 points (OA (L_{36})), the values in the second column are obtained by the optimal representative points with only n = 29 points, and the values in the third column are obtained by 15,019 GLP representative points as a benchmark for the comparison. These 15,019 representative points, given in Hua and Wang (1981), have a very small discrepancy, $D(k) = 5 \times 10^{-5}$, namely, very precise integral approximation to the "true" values.

In all cases, the n = 29 optimal representative points provide a consistently accurate approximation to the benchmark values for mean and standard deviation for all inner points. For the normal noise, Taguchi's OA-approach provides a fairly good estimate for the mean but not for the standard deviation. Therefore, the signal-to-noise ratios can be far from the true values. For example, in runs 27, 28, and 36, the OA-approach gives the signal-to-noise ratio of -4.17, 21.99 and 1.82 while the correct value is -7.20, 17.07 and -1.72 respectively. (Note also that the n = 29 representative points approach gives -7.20, 18.26, and -1.79.) When the noise factors are uniformly distributed, the OA-approach leads to poor estimation for both mean and standard deviation (consequently signal-to-noise ratio would be poorly

TABLE 3. Means and Standard Deviations for All 36 Inner Array Points in the Wheatstone Bridge Example

Inner Array	Normal Noise						Uniform Noise					
	Mean			Std. Dev.			Mean			Std. Dev.		
1	2.00[a]	2.00[b]	2.00[c]	0.050	0.092	0.104	2.00	2.03	2.03	0.263	0.183	0.196
2	2.00	2.00	2.00	0.094	0.179	0.192	2.00	2.06	2.06	0.483	0.385	0.365
3	2.00	2.00	2.00	0.326	0.640	0.657	2.00	2.19	2.20	1.648	1.256	1.250
4	2.00	2.00	2.00	0.031	0.056	0.068	2.00	2.02	2.02	0.173	0.112	0.127
5	2.00	2.00	2.00	0.076	0.142	0.155	2.00	2.04	2.05	0.390	0.283	0.294
6	2.00	2.00	2.00	0.885	1.754	1.780	2.01	2.52	2.53	4.466	3.434	3.390
7	10.00	10.00	10.00	0.600	1.149	1.214	10.01	10.34	10.36	3.059	2.269	2.311
8	0.08	0.08	0.08	0.158	0.316	0.319	0.08	0.17	0.18	0.800	0.618	0.608
9	10.00	10.00	10.00	0.271	0.502	0.565	10.01	10.15	10.16	1.431	1.004	1.070
10	0.40	0.40	0.40	0.259	0.514	0.521	0.40	0.55	0.56	1.305	1.006	0.991
11	50.00	49.98	50.00	4.192	8.109	8.464	50.07	52.36	52.34	21.342	15.992	16.132
12	0.40	0.40	0.40	0.048	0.094	0.097	0.40	0.43	0.43	0.243	0.184	0.184
13	0.08	0.08	0.08	0.002	0.003	0.003	0.08	0.08	0.08	0.009	0.006	0.007
14	10.00	9.99	10.00	9.608	19.094	19.329	10.07	15.51	15.80	48.305	37.339	36.818
15	10.00	9.99	10.00	1.220	2.382	2.459	10.02	10.70	10.74	6.189	4.685	4.686
16	0.40	0.40	0.40	0.021	0.041	0.043	0.40	0.41	0.41	0.109	0.081	0.082
17	50.00	49.98	50.00	5.386	10.483	10.354	50.08	53.05	53.26	27.347	20.631	20.696
18	0.40	0.40	0.40	0.208	0.412	0.418	0.40	0.52	0.53	1.049	0.808	0.797
19	50.00	50.00	50.00	0.566	1.063	1.360	50.03	50.31	50.33	3.456	2.133	2.507
20	0.40	0.40	0.40	0.081	0.160	0.164	0.40	0.45	0.45	0.410	0.314	0.311
21	0.40	0.40	0.40	0.216	0.429	0.435	0.40	0.53	0.53	1.090	0.840	0.828
22	10.00	10.00	10.00	0.045	0.145	0.175	10.01	10.01	10.01	0.441	0.268	0.305
23	0.08	0.08	0.08	0.183	0.365	0.368	0.08	0.11	0.19	0.923	0.714	0.701
24	10.00	9.99	10.00	9.360	18.600	18.830	10.06	15.37	15.65	47.255	36.374	35.869
25	0.40	0.40	0.40	0.046	0.089	0.092	0.40	0.43	0.43	0.230	0.175	0.175
26	0.40	0.40	0.40	0.004	0.007	0.009	0.40	0.40	0.40	0.024	0.014	0.087
27	50.00	49.53	49.99	130.505	259.650	262.611	51.21	123.64	128.71	659.798	508.011	500.521
28	2.00	2.00	2.00	0.013	0.030	0.039	2.00	2.01	2.01	0.100	0.057	0.070
29	2.00	2.00	2.00	0.022	0.041	0.053	2.00	2.01	2.01	0.135	0.083	0.098
30	2.00	2.03	2.00	14.447	28.791	29.060	2.07	10.41	10.71	72.868	56.267	55.339
31	2.00	2.00	2.00	0.143	0.275	0.289	2.00	2.08	2.09	0.726	0.543	0.549
32	2.00	2.00	2.00	0.013	0.030	0.039	2.00	2.01	2.01	0.099	0.057	0.069
33	2.00	2.01	2.00	2.240	4.456	4.506	2.01	3.31	3.35	11.300	8.714	8.580
34	10.00	10.00	10.00	0.050	0.144	0.180	10.01	10.02	10.02	0.454	0.269	0.315
35	10.00	10.00	10.00	6.575	13.051	13.227	10.04	13.79	13.97	33.188	25.530	25.189
36	0.08	0.08	0.08	0.060	0.119	0.121	0.08	0.12	0.12	0.303	0.234	0.230

a: First Columns = OA(L_{36})
b: Second Columns = Optimal Representing Points
c: Third Columns = True Values

estimated as well). This is partially because the uniform distribution has more dispersion than the normal distribution.

Concern has been raised by one referee that "... the ordering of the runs is not consistent. For example, runs 9 and 10 with values 0.502 and 0.514 reverse the true orderings whose corresponding values are 0.565 and 0.521 for the normal case. ..." Note that the design given here ensures a small discrepancy (i.e., a precise approximation to the true performance in each inner array point). Even when the ordering is not consistent, the difference is insignificant. Indeed, this can be easily seen from Table 3. Consequently, such an ordering effect to the choice of factor level is in general irrelevant.

Remarks

If the underlying distribution is believed to be other than uniform distribution, a simple transformation of $\mathbf{x}_j$'s is needed to obtain the optimal representative points. Suppose each of the noise variables is independent of each other, and that the joint distribution of noise variables $\mathbf{N} = (N_1, \ldots, N_k)$ is known to be $F(\mathbf{N}) = \prod_{i=1}^{k} F_i(N_i)$, where $F_i(N_i)$ is the marginal distribution for N_i. Also suppose that $\{\mathbf{x}_j = (x_{j1}, \ldots, x_{jk}),\ 1 \leq j \leq n\}$

are the optimal representative points relative to the k-dimensional unform distribution in $[0,1]^k$. Then $\{\mathbf{y}_j = (F_1^{-1}(x_{j1}), \ldots, F_k^{-1}(x_{jk})),\ 1 \le j \le n\}$ will be the optimal representative points relative to $F(\mathbf{N})$. See Wang, Fang, and Lin (1992).

Designing Outer Array Points on a Disk, Ball or Spherical Surface

The optimal representative points over a hypercube can be easily transformed in order to be readily used for other shapes of experimental regions. In this section, three specific types of experimental regions are discussed: a two-dimensional disk, a three-dimensional ball and a three-dimensional spherical surface. These transformations are motivated to preserve their minimal discrepancy as previously described. The distribution on the disk, ball or spherical surface is assumed to be uniform. For mathematical details, see Wang and Fang (1981).

Designing Outer Array Points Over a Disk

A set of two-dimensional ($k = 2$) optimal outer array points in a square can be obtained from Table 2, denoted by $(c_1, c_2)'$ here. The transformation

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{c_1}\cos(2\pi c_2) \\ \sqrt{c_1}\sin(2\pi c_2) \end{bmatrix}$$

will transfer all representative points in a square into a disk, with its optimality retained. For example, the four optimal outer array points in a disk can be obtained by applying the above transformation to the four points: $\{(0.125, 0.375), (0.375, 0.875), (0.625, 0.125), (0.875, 0.625)\}$. The resulting outer array points in the disk are: $\{(-0.2500, 0.2500), (0.4330, -0.4330), (0.5590, 0.5590), (-0.6614, -0.6614)\}$. These four points are displayed in Figure 4(a). The optimal five-run outer array points are plotted in Figure 4(b) as another example.

The results given here can be applied to, for example, the EPI problem (see Kackar and Shoemaker (1986)). If five points are sampled, Figure 4(b) will serve as a better than original design in accurately approximating the distribution in each layer.

Designing Outer Array Points Over a Ball

Given n, the optimal outer array points in a cube can be obtained from Table 2, denoted by $(c_1, c_2, c_3)'$ here. The transformation

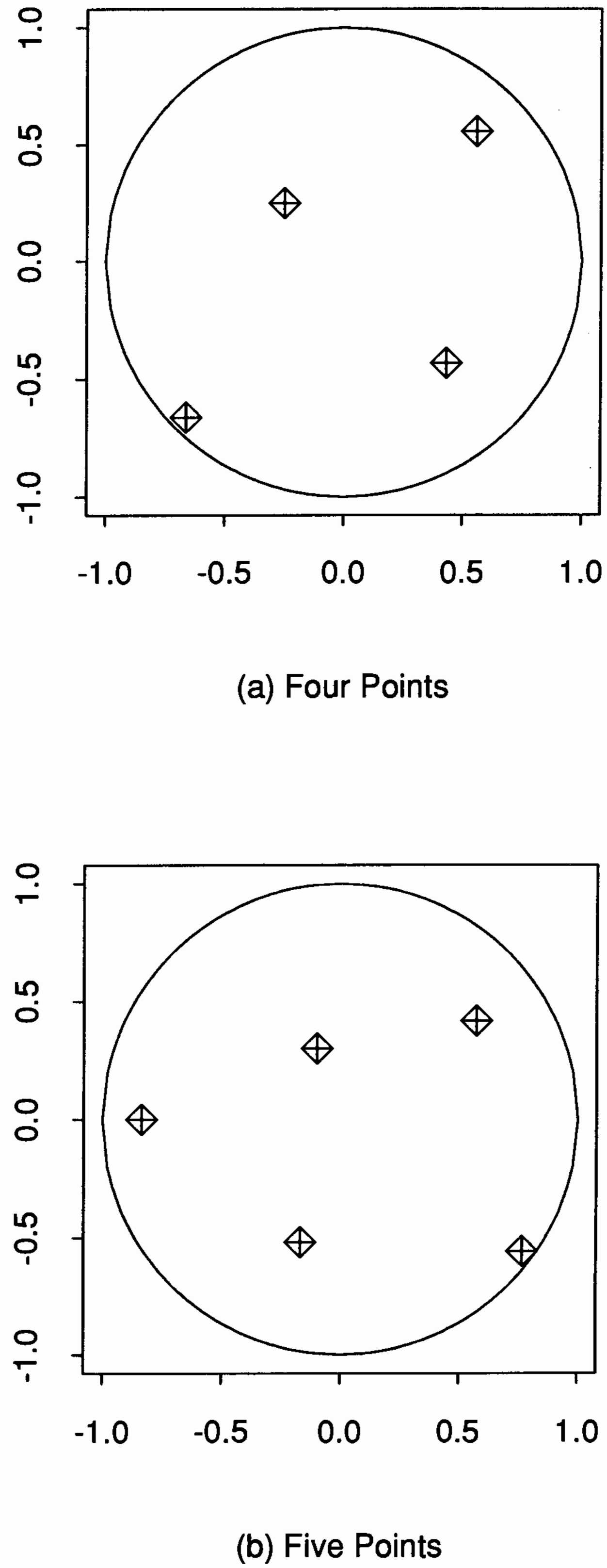

FIGURE 4. Four and Five Optimal Representative Points in a Disk.

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightarrow \begin{bmatrix} c_1^{1/3}(1-2c_2) \\ c_1^{1/3}\cos(2\pi c_3)\ 2\sqrt{c_2(1-c_2)} \\ c_1^{1/3}\sin(2\pi c_3)\ 2\sqrt{c_2(1-c_2)} \end{bmatrix}$$

is recommended. Such a transformation is proven to retain its optimality in a 3-dimensional ball. A seven-run outer array of points, for example, can be obtained as follows:

(1) Write down the seven points, $(n,k) = (7,3)$, from Table 2.

(2) Transform these seven points into a ball,

$$\begin{bmatrix} 0.0714 & 0.2143 & 0.3571 \\ 0.2143 & 0.5000 & 0.7857 \\ 0.3571 & 0.7857 & 0.2143 \\ 0.5000 & 0.0714 & 0.6429 \\ 0.6429 & 0.3571 & 0.0714 \\ 0.7857 & 0.6429 & 0.5000 \\ 0.9286 & 0.9286 & 0.9286 \end{bmatrix} \rightarrow \begin{bmatrix} 0.2371 & -0.2123 & 0.2662 \\ 0.0000 & 0.1332 & -0.5834 \\ -0.4054 & 0.1296 & 0.5676 \\ 0.6803 & -0.2549 & -0.3196 \\ 0.2466 & 0.7452 & 0.3589 \\ -0.2636 & -0.8843 & 0.0000 \\ -0.8362 & 0.4527 & -0.2180 \end{bmatrix}$$

Similarly, the nine-run optimal outer array points in a ball are:

$$\begin{bmatrix} 0.0556 & 0.3889 & 0.7222 \\ 0.1667 & 0.8333 & 0.5000 \\ 0.2778 & 0.2778 & 0.2778 \\ 0.3889 & 0.7222 & 0.0556 \\ 0.5000 & 0.1667 & 0.8333 \\ 0.6111 & 0.6111 & 0.6111 \\ 0.7222 & 0.0556 & 0.3889 \\ 0.8333 & 0.5000 & 0.1667 \\ 0.9444 & 0.9444 & 0.9444 \end{bmatrix} \rightarrow \begin{bmatrix} 0.0848 & -0.0646 & -0.3664 \\ -0.3669 & -0.4102 & 0.0000 \\ 0.2900 & -0.1015 & 0.5756 \\ -0.3244 & 0.6144 & 0.2236 \\ 0.5291 & 0.2958 & -0.5123 \\ -0.1886 & -0.6338 & -0.5318 \\ 0.7975 & -0.3149 & 0.2642 \\ 0.0000 & 0.4705 & 0.8150 \\ -0.8721 & 0.4224 & -0.1537 \end{bmatrix}$$

Designing Outer Array Points Over a Spherical Surface

The quality of a ball bearing is sometimes judged by the uniformity and evenness of hardness over the spherical surface of the bearing. Outer array points over the sphere can be obtained from the following transformation from $[0,1]^2$ to the sphere $\left\{(x,y,z) : \sqrt{x^2+y^2+z^2} = 1\right\}$:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} (1-2c_1) \\ \sqrt{2c_1(1-c_1)}\ \cos(2\pi c_2) \\ \sqrt{2c_1(1-c_1)}\ \sin(2\pi c_2) \end{bmatrix}.$$

Comparison with Alternative Methods

Grid Mapping

A k-dimensional grid is the Cartesian product of k one-dimensional grids. It has been shown that the product of two optimal one-dimensional grids is no longer optimal in the 2-dimensional space. A full factorial design is a grid, and an orthogonal array is simply a subset of the full factorial design. For integral approximation, optimal representative points are more accurate than points selected by an orthogonal array of the same size. Shaw (1988) showed that the 2-dimensional grid $\{((2i-1)/32, (2j-1)/32);\ i,j = 1,\ldots,16\}$ has an error size of five times the size of the 256 optimal representative points. Moreover, the optimal representative points are much more flexible with respect to the run size selection.

Latin Hypercube Sampling

Another possibility for designing outer array points is by using Latin hypercube sampling (see, for example, McKay, Beckman, and Conover (1979)). Latin hypercube sampling takes a random subset of size n from the product of k stratified random samples, where each stratified sample of size n is taken for every univariate component of the noise variable. Latin hypercube sampling has the desirable "space-filling" property. Because the one-dimensional random sample is drawn from an equal-distance partition for every univariate component, Latin hypercube sampling points have n different values for each coordinate. These properties are also shared by the optimal representative points method.

Note that optimal representative points are deterministic, but Latin hypercube sampling points are random. In other words, the optimal representative point method is much easier to use in planning and execution. Also, no optimal property can be proven for a given Latin hypercube sampling because it is a realization of the random process. The optimal representative points are known to be optimal in integral approximation. Although some asymptotical results have been derived for Latin hypercube sampling, finite sample situations are more likely to be found in practice.

Concluding Remarks

The choice of design mainly depends upon the objective of the experiment. Classical designs such as fractional factorials will continue to be used in many circumstances. However, if the purpose of experiments is to evaluate the overall performance, as in the case of outer array designs, the optimal representative points approach is shown to be superior to other plans. This is simply because the empirical cumulative density function (cdf) is closer to the the true cdf in the optimal representative points approach, resulting in more efficient estimation for the performance measurement.

Appendix. Table for Optimal Representative Points, Uniform Distribution Assumed

TABLE A1. The Optimal Representative Points in $[0,1]^k$ Assuming a Uniform Distribution for $n = 16, \ldots, 25$ (all values need to be multiplied by 10^{-4})

Number of Runs, n	Number of Factors, k											
	1	2	3	4	5	6	7	8	9	10	11	12
16	313	5938	9063	8438	2188	3438	5313	2813	9688	1563	7813	6563
	938	1563	7813	6563	4688	7188	313	5938	9063	3438	5313	2813
	1563	7813	6563	4688	7188	313	5938	9063	8438	5313	2813	9688
	2188	3438	5313	2813	9688	4063	938	1563	7813	7188	313	5938
	2813	9688	4063	938	1563	7813	6563	4688	7188	9063	8438	2188
	3438	5313	2813	9688	4063	938	1563	7813	6563	313	5938	9063
	4063	938	1563	7813	6563	4688	7188	313	5938	2188	3438	5313
	4688	7188	313	5938	9063	8438	2188	3438	5313	4063	938	1563
	5313	2813	9688	4063	938	1563	7813	6563	4688	5938	9063	8438
	5938	9063	8438	2188	3438	5313	2813	9688	4063	7813	6563	4688
	6563	4688	7188	313	5938	9063	8438	2188	3438	9688	4063	938
	7188	313	5938	9063	8438	2188	3438	5313	2813	938	1563	7813
	7813	6563	4688	7188	313	5938	9063	8438	2188	2813	9688	4063
	8438	2188	3438	5313	2813	9688	4063	938	1563	4688	7188	313
	9063	8438	2188	3438	5313	2813	9688	4063	938	6563	4688	7188
	9688	4063	938	1563	7813	6563	4688	7188	313	8438	2188	3438
17	294	5588	8529	7941	2059	3235	5000	2647	9118	1471	7353	6176
	882	1471	7353	6176	4412	6765	294	5588	8529	3235	5000	2647
	1471	7353	6176	4412	6765	294	5588	8529	7941	5000	2647	9118
	2059	3235	5000	2647	9118	3824	882	1471	7353	6765	294	5588
	2647	9118	3824	882	1471	7353	6176	4412	6765	8529	7941	2059
	3235	5000	2647	9118	3824	882	1471	7353	6176	294	5588	8529
	3824	882	1471	7353	6176	4412	6765	294	5588	2059	3235	5000
	4412	6765	294	5588	8529	7941	2059	3235	5000	3824	882	1471
	5000	2647	9118	3824	882	1471	7353	6176	4412	5588	8529	7941
	5588	8529	7941	2059	3235	5000	2647	9118	3824	7353	6176	4412
	6176	4412	6765	294	5588	8529	7941	2059	3235	9118	3824	882
	6765	294	5588	8529	7941	2059	3235	5000	2647	882	1471	7353
	7353	6176	4412	6765	294	5588	8529	7941	2059	2647	9118	3824
	7941	2059	3235	5000	2647	9118	3824	882	1471	4412	6765	294
	8529	7941	2059	3235	5000	2647	9118	3824	882	6176	4412	6765
	9118	3824	882	1471	7353	6176	4412	6765	294	7941	2059	3235
	9706	9706	9706	9706	9706	9706	9706	9706	9706	9706	9706	9706

TABLE A1. — *Continued*

n	1	2	3	4	5	6	7	8	9	10	11	12
18	278	4167	3056	7500	9167	5278	3611	1389	1944	9722	833	8611
	833	8611	6389	4722	8056	278	7500	3056	4167	9167	1944	6944
	1389	2500	9722	1944	6944	5833	833	4722	6389	8611	3056	5278
	1944	6944	2500	9722	5833	833	4722	6389	8611	8056	4167	3611
	2500	833	5833	6944	4722	6389	8611	8056	278	7500	5278	1944
	3056	5278	9167	4167	3611	1389	1944	9722	2500	6944	6389	278
	3611	9722	1944	1389	2500	6944	5833	833	4722	6389	7500	9167
	4167	3611	5278	9167	1389	1944	9722	2500	6944	5833	8611	7500
	4722	8056	8611	6389	278	7500	3056	4167	9167	5278	9722	5833
	5278	1944	1389	3611	9722	2500	6944	5833	833	4722	278	4167
	5833	6389	4722	833	8611	8056	278	7500	3056	4167	1389	2500
	6389	278	8056	8611	7500	3056	4167	9167	5278	3611	2500	833
	6944	4722	833	5833	6389	8611	8056	278	7500	3056	3611	9722
	7500	9167	4167	3056	5278	3611	1389	1944	9722	2500	4722	8056
	8056	3056	7500	278	4167	9167	5278	3611	1389	1944	5833	6389
	8611	7500	278	8056	3056	4167	9167	5278	3611	1389	6944	4722
	9167	1389	3611	5278	1944	9722	2500	6944	5833	833	8056	3056
	9722	5833	6944	2500	833	4722	6389	8611	8056	278	9167	1389
19	263	3947	2895	7105	8684	5000	3421	1316	1842	9211	789	8158
	789	8158	6053	4474	7632	263	7105	2895	3947	8684	1842	6579
	1316	2368	9211	1842	6579	5526	789	4474	6053	8158	2895	5000
	1842	6579	2368	9211	5526	789	4474	6053	8158	7632	3947	3421
	2368	789	5526	6579	4474	6053	8158	7632	263	7105	5000	1842
	2895	5000	8684	3947	3421	1316	1842	9211	2368	6579	6053	263
	3421	9211	1842	1316	2368	6579	5526	789	4474	6053	7105	8684
	3947	3421	5000	8684	1316	1842	9211	2368	6579	5526	8158	7105
	4474	7632	8158	6053	263	7105	2895	3947	8684	5000	9211	5526
	5000	1842	1316	3421	9211	2368	6579	5526	789	4474	263	3947
	5526	6053	4474	789	8158	7632	263	7105	2895	3947	1316	2368
	6053	263	7632	8158	7105	2895	3947	8684	5000	3421	2368	789
	6579	4474	789	5526	6053	8158	7632	263	7105	2895	3421	9211
	7105	8684	3947	2895	5000	3421	1316	1842	9211	2368	4474	7632
	7632	2895	7105	263	3947	8684	5000	3421	1316	1842	5526	6053
	8158	7105	263	7632	2895	3947	8684	5000	3421	1316	6579	4474
	8684	1316	3421	5000	1842	9211	2368	6579	5526	789	7632	2895
	9211	5526	6579	2368	789	4474	6053	8158	7632	263	8684	1316
	9737	9737	9737	9737	9737	9737	9737	9737	9737	9737	9737	9737
20	250	2250	8250	750	1750	3750	4750	5250	6250	7750	9250	9750
	750	4750	6250	1750	3750	7750	9750	250	2250	5250	8250	9250
	1250	7250	4250	2750	5750	1250	4250	5750	8750	2750	7250	8750
	1750	9750	2250	3750	7750	5250	9250	750	4750	250	6250	8250
	2250	1750	250	4750	9750	9250	3750	6250	750	8250	5250	7750
	2750	4250	8750	5750	1250	2750	8750	1250	7250	5750	4250	7250
	3250	6750	6750	6750	3250	6750	3250	6750	3250	3250	3250	6750
	3750	9250	4750	7750	5250	250	8250	1750	9750	750	2250	6250
	4250	1250	2750	8750	7250	4250	2750	7250	5750	8750	1250	5750
	4750	3750	750	9750	9250	8250	7750	2250	1750	6250	250	5250
	5250	6250	9250	250	750	1750	2250	7750	8250	3750	9750	4750
	5750	8750	7250	1250	2750	5750	7250	2750	4250	1250	8750	4250
	6250	750	5250	2250	4750	9750	1750	8250	250	9250	7750	3750
	6750	3250	3250	3250	6750	3250	6750	3250	6750	6750	6750	3250
	7250	5750	1250	4250	8750	7250	1250	8750	2750	4250	5750	2750
	7750	8250	9750	5250	250	750	6250	3750	9250	1750	4750	2250
	8250	250	7750	6250	2250	4750	750	9250	5250	9750	3750	1750
	8750	2750	5750	7250	4250	8750	5750	4250	1250	7250	2750	1250
	9250	5250	3750	8250	6250	2250	250	9750	7750	4750	1750	750
	9750	7750	1750	9250	8250	6250	5250	4750	3750	2250	750	250

TABLE A1. — *Continued*

n	1	2	3	4	5	6	7	8	9	10	11	12
21	238	2143	7857	714	1667	3571	4524	5000	5952	7381	8810	9286
	714	4524	5952	1667	3571	7381	9286	238	2143	5000	7857	8810
	1190	6905	4048	2619	5476	1190	4048	5476	8333	2619	6905	8333
	1667	9286	2143	3571	7381	5000	8810	714	4524	238	5952	7857
	2143	1667	238	4524	9286	8810	3571	5952	714	7857	5000	7381
	2619	4048	8333	5476	1190	2619	8333	1190	6905	5476	4048	6905
	3095	6429	6429	6429	3095	6429	3095	6429	3095	3095	3095	6429
	3571	8810	4524	7381	5000	238	7857	1667	9286	714	2143	5952
	4048	1190	2619	8333	6905	4048	2619	6905	5476	8333	1190	5476
	4524	3571	714	9286	8810	7857	7381	2143	1667	5952	238	5000
	5000	5952	8810	238	714	1667	2143	7381	7857	3571	9286	4524
	5476	8333	6905	1190	2619	5476	6905	2619	4048	1190	8333	4048
	5952	714	5000	2143	4524	9286	1667	7857	238	8810	7381	3571
	6429	3095	3095	3095	6429	3095	6429	3095	6429	6429	6429	3095
	6905	5476	1190	4048	8333	6905	1190	8333	2619	4048	5476	2619
	7381	7857	9286	5000	238	714	5952	3571	8810	1667	4524	2143
	7857	238	7381	5952	2143	4524	714	8810	5000	9286	3571	1667
	8333	2619	5476	6905	4048	8333	5476	4048	1190	6905	2619	1190
	8810	5000	3571	7857	5952	2143	238	9286	7381	4524	1667	714
	9286	7381	1667	8810	7857	5952	5000	4524	3571	2143	714	238
	9762	9762	9762	9762	9762	9762	9762	9762	9762	9762	9762	9762
22	227	7500	5682	6136	3409	9318	5227	6591	7500	1591	2955	1136
	682	4773	1136	2045	7045	8409	227	2955	4773	3409	6136	2500
	1136	2045	7045	8409	227	7500	5682	9773	2045	5227	9318	3864
	1591	9773	2500	4318	3864	6591	682	6136	9773	7045	2045	5227
	2045	7045	8409	227	7500	5682	6136	2500	7045	8864	5227	6591
	2500	4318	3864	6591	682	4773	1136	9318	4318	227	8409	7955
	2955	1591	9773	2500	4318	3864	6591	5682	1591	2045	1136	9318
	3409	9318	5227	8864	7955	2955	1591	2045	9318	3864	4318	227
	3864	6591	682	4773	1136	2045	7045	8864	6591	5682	7500	1591
	4318	3864	6591	682	4773	1136	2045	5227	3864	7500	227	2955
	4773	1136	2045	7045	8409	227	7500	1591	1136	9318	3409	4318
	5227	8864	7955	2955	1591	9773	2500	8409	8864	682	6591	5682
	5682	6136	3409	9318	5227	8864	7955	4773	6136	2500	9773	7045
	6136	3409	9318	5227	8864	7955	2955	1136	3409	4318	2500	8409
	6591	682	4773	1136	2045	7045	8409	7955	682	6136	5682	9773
	7045	8409	227	7500	5682	6136	3409	4318	8409	7955	8864	682
	7500	5682	6136	3409	9318	5227	8864	682	5682	9773	1591	2045
	7955	2955	1591	9773	2500	4318	3864	7500	2955	1136	4773	3409
	8409	227	7500	5682	6136	3409	9318	3864	227	2955	7955	4773
	8864	7955	2955	1591	9773	2500	4318	227	7955	4773	682	6136
	9318	5227	8864	7955	2955	1591	9773	7045	5227	6591	3864	7500
	9773	2500	4318	3864	6591	682	4773	3409	2500	8409	7045	8864
23	217	7174	5435	5870	3261	8913	5000	6304	7174	1522	2826	1087
	652	4565	1087	1957	6739	8043	217	2826	4565	3261	5870	2391
	1087	1957	6739	8043	217	7174	5435	9348	1957	5000	8913	3696
	1522	9348	2391	4130	3696	6304	652	5870	9348	6739	1957	5000
	1957	6739	8043	217	7174	5435	5870	2391	6739	8478	5000	6304
	2391	4130	3696	6304	652	4565	1087	8913	4130	217	8043	7609
	2826	1522	9348	2391	4130	3696	6304	5435	1522	1957	1087	8913
	3261	8913	5000	8478	7609	2826	1522	1957	8913	3696	4130	217
	3696	6304	652	4565	1087	1957	6739	8478	6304	5435	7174	1522
	4130	3696	6304	652	4565	1087	1957	5000	3696	7174	217	2826
	4565	1087	1957	6739	8043	217	7174	1522	1087	8913	3261	4130
	5000	8478	7609	2826	1522	9348	2391	8043	8478	652	6304	5435
	5435	5870	3261	8913	5000	8478	7609	4565	5870	2391	9348	6739
	5870	3261	8913	5000	8478	7609	2826	1087	3261	4130	2391	8043
	6304	652	4565	1087	1957	6739	8043	7609	652	5870	5435	9348

TABLE A1. — *Continued*

n	1	2	3	4	5	6	7	8	9	10	11	12
23	6739	8043	217	7174	5435	5870	3261	4130	8043	7609	8478	652
	7174	5435	5870	3261	8913	5000	8478	652	5435	9348	1522	1957
	7609	2826	1522	9348	2391	4130	3696	7174	2826	1087	4565	3261
	8043	217	7174	5435	5870	3261	8913	3696	217	2826	7609	4565
	8478	7609	2826	1522	9348	2391	4130	217	7609	4565	652	5870
	8913	5000	8478	7609	2826	1522	9348	6739	5000	6304	3696	7174
	9348	2391	4130	3696	6304	652	4565	3261	2391	8043	6739	8478
	9783	9783	9783	9783	9783	9783	9783	9783	9783	9783	9783	9783
24	208	4375	8542	2292	6458	1458	9792	3542	625	4792	5625	7292
	625	8958	6875	4792	2708	3125	9375	7292	1458	9792	1042	4375
	1042	3125	5208	7292	9375	4792	8958	625	2292	4375	6875	1458
	1458	7708	3542	9792	5625	6458	8542	4375	3125	9375	2292	8958
	1875	1875	1875	1875	1875	8125	8125	8125	3958	3958	8125	6042
	2292	6458	208	4375	8542	9792	7708	1458	4792	8958	3542	3125
	2708	625	8958	6875	4792	1042	7292	5208	5625	3542	9375	208
	3125	5208	7292	9375	1042	2708	6875	8958	6458	8542	4792	7708
	3542	9792	5625	1458	7708	4375	6458	2292	7292	3125	208	4792
	3958	3958	3958	3958	3958	6042	6042	6042	8125	8125	6042	1875
	4375	8542	2292	6458	208	7708	5625	9792	8958	2708	1458	9375
	4792	2708	625	8958	6875	9375	5208	3125	9792	7708	7292	6458
	5208	7292	9375	1042	3125	625	4792	6875	208	2292	2708	3542
	5625	1458	7708	3542	9792	2292	4375	208	1042	7292	8542	625
	6042	6042	6042	6042	6042	3958	3958	3958	1875	1875	3958	8125
	6458	208	4375	8542	2292	5625	3542	7708	2708	6875	9792	5208
	6875	4792	2708	625	8958	7292	3125	1042	3542	1458	5208	2292
	7292	9375	1042	3125	5208	8958	2708	4792	4375	6458	625	9792
	7708	3542	9792	5625	1458	208	2292	8542	5208	1042	6458	6875
	8125	8125	8125	8125	8125	1875	1875	1875	6042	6042	1875	3958
	8542	2292	6458	208	4375	3542	1458	5625	6875	625	7708	1042
	8958	6875	4792	2708	625	5208	1042	9375	7708	5625	3125	8542
	9375	1042	3125	5208	7292	6875	625	2708	8542	208	8958	5625
	9792	5625	1458	7708	3542	8542	208	6458	9375	5208	4375	2708
25	200	4200	8200	2200	6200	1400	9400	3400	600	4600	5400	7000
	600	8600	6600	4600	2600	3000	9000	7000	1400	9400	1000	4200
	1000	3000	5000	7000	9000	4600	8600	600	2200	4200	6600	1400
	1400	7400	3400	9400	5400	6200	8200	4200	3000	9000	2200	8600
	1800	1800	1800	1800	1800	7800	7800	7800	3800	3800	7800	5800
	2200	6200	200	4200	8200	9400	7400	1400	4600	8600	3400	3000
	2600	600	8600	6600	4600	1000	7000	5000	5400	3400	9000	200
	3000	5000	7000	9000	1000	2600	6600	8600	6200	8200	4600	7400
	3400	9400	5400	1400	7400	4200	6200	2200	7000	3000	200	4600
	3800	3800	3800	3800	3800	5800	5800	5800	7800	7800	5800	1800
	4200	8200	2200	6200	200	7400	5400	9400	8600	2600	1400	9000
	4600	2600	600	8600	6600	9000	5000	3000	9400	7400	7000	6200
	5000	7000	9000	1000	3000	600	4600	6600	200	2200	2600	3400
	5400	1400	7400	3400	9400	2200	4200	200	1000	7000	8200	600
	5800	5800	5800	5800	5800	3800	3800	3800	1800	1800	3800	7800
	6200	200	4200	8200	2200	5400	3400	7400	2600	6600	9400	5000
	6600	4600	2600	600	8600	7000	3000	1000	3400	1400	5000	2200
	7000	9000	1000	3000	5000	8600	2600	4600	4200	6200	600	9400
	7400	3400	9400	5400	1400	200	2200	8200	5000	1000	6200	6600
	7800	7800	7800	7800	7800	1800	1800	1800	5800	5800	1800	3800
	8200	2200	6200	200	4200	3400	1400	5400	6600	600	7400	1000
	8600	6600	4600	2600	600	5000	1000	9000	7400	5400	3000	8200
	9000	1000	3000	5000	7000	6600	600	2600	8200	200	8600	5400
	9400	5400	1400	7400	3400	8200	200	6200	9000	5000	4200	2600
	9800	9800	9800	9800	9800	9800	9800	9800	9800	9800	9800	9800

Acknowledgments

We thank the past editor and two anonymous referees for making helpful suggestions that led to important improvements in the article. Dennis Lin gratefully acknowledges partial support from the National Science Foundation under Grant DMS-9204007, from the Professional Development Award, the University of Tennessee, and from a visiting Scientist Fellowship, IBM Watson Research Center. Some of Dennis Lin's work was completed while he was a visitor with the Institute of Statistical Sciences, Academia Sinica, Taiwan, ROC.

References

ANDREWS, D. F.; BICKEL, P. J.; HAMPEL, F. R.; HUBER, P. J.; ROGERS, W. H.; and TUKEY, J. W. (1972). *Robust Estimation of Location, Survey and Advances.* Princeton University Press, Princeton, NJ.

BOX, G. E. P. and JONES, S. P. (1992). "Split Plot Designs for Robust Product Experimentation". *Journal of Applied Statistics* 19, pp. 3-26.

HUA, L. K. and WANG, Y. (1981). *Applications of Number Theory to Numerical Analysis.* Springer-Verlag, Berlin.

KACKAR, R. N. and SHOEMAKER, A. C. (1986). "Robust Design: A Cost-Effective Method for Improving Manufacturing Process". *AT&T Technical Journal* 65, pp. 39-50.

KOROBOV, N. M. (1959). "Computation Of Multiple Integrals By The Method Of Optimal Coefficients". *Moscow State University Series Mathematics, Astronomy, Physics, and Chemistry* 4, pp. 19-25.

MCKAY, M. D.; BECKMAN, R. J.; and CONOVER, W. J. (1979). "A Comparison of Three Methods for Selecting Values of Input Variables in Analysis of Output From a Computer Code". *Technometrics* 21, pp. 239-245.

PHADKE, M. S. (1989). *Quality Engineering Using Robust Design.* Prentice Hall, Englewood Cliffs, NJ.

SHAW, J. E. H. (1988). "A Quasirandom Approach to Integration in Bayesian Statistics". *The Annals of Statistics* 16, pp. 859-914.

TAGUCHI, G. (1986). *Introduction to Quality Engineering.* UNIPUB/Kraus International Publications, White Plains, NY.

TAGUCHI, G. (1987). *System of Experimental Design.* UNIPUB/Kraus International Publications, White Plains, NY.

WANG, Y. and FANG, K. T. (1981). "Number Theoretic Methods In Applied Statistics (II)". *Chinese Annals of Mathematics* B 11, pp. 76-86.

WANG, Y. J.; FANG, K. T.; and LIN, D. K. J. (1992). "Optimal Representative Points Via Quantizer". Working Report No. 272, College of Business Administration, University of Tennessee-Knoxville, Knoxville, TN.

YATES, F. (1935). "Complex Experiments". *Journal of the Royal Statistical Society* B 2, pp. 181-223.

Key Words: *Fractional Factorial Designs, Orthogonal Array, Performance Measures.*