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CHARACTERIZING PROJECTED DESIGNS: REPEAT AND MIRROR-IMAGE RUNS

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ABSTRACT

Two-level designs are useful to examine a large number of factors in an efficient manner. It is typically anticipated that only a few factors will be identified as important ones. The results can then be reanalyzed using a projection of the original design, projected into the space of the factors that matter. An interesting question is how many intrinsically different type of projections are possible from an initial given design. We examine this question here for the Plackett and Burman screening series with N = 12, 20 and 24 runs and pro-

jected dimensions $k \leq 5$. As a characterization criterion, we look at the number of repeat and mirror-image runs in the projections. The idea can be applied to any two-level design projected into fewer dimensions.

1. INTRODUCTION

The purpose of screening designs is to examine a large number of factors in relatively few runs. It is anticipated that only a few of the factors will be important and that, once these factors are identified, the analysis can be refocused and redone with a view to planning what to examine next.

Because screening experiments are conducted at an early stage of experimentation, use of more than two levels of each factor would be unusual and often wasteful. There is little point in trying to assess quadratic effects of factors that may not even affect the response linearly. The two-level Plackett and Burman (1946) designs form a general class of two-level orthogonal arrays for which N is a multiple of four, for all $N \leq 100$ (except 92). They are *saturated* in the sense that they examine $N-1$ factors in N runs. For other values of N, see Seberry (1978).

Whenever N is a power of two, the Plackett and Burman designs are always 2_R^{k-p} fractional factorials, with resolution R = III (see Box and Hunter, 1961). The 2_{III}^{k-p} series provides some extremely popular screening designs. They permit estimation of the main effects of all the factors being explored, assuming that all interactions between the factors can, at least tentatively, be ignored. Moreover, once the factors of importance have been tentatively identified, the initial design can then be projected into the dimensions of those factors. This leads to another 2_R^{k-p} design, usually of higher resolution than the first, depending on specifically which factors have been eliminated. It is easy to determine what type the reduced design is, merely by deletion of the ignored factors in the defining relation.

When N is *not* a power of two (but only a multiple of four) the Plackett and Burman designs are still useful screening designs, filling the gaps between N = 8, 16, 32, ... and again function as main effect designs if interactions are ignored. Otherwise, their alias structures are very intricate, and their projected designs are difficult to characterize.

2. SELECTING COLUMNS FROM PLACKETT AND BURMAN DESIGNS

When $k \leq N-1$ columns are chosen, we might ask *how many designs there are for given k and N*. We would like to be able to distinguish designs that are intrinsically different and those that are obtainable from others via sign changes in the columns, rearrangement of rows (points), and rearrangement of columns (re-naming of variables). A complete tabulation of all such designs would be very extensive, however. A more limited, but nevertheless useful display for the experimenter, can be made by distinguishing subsets of designs through their *repeat and mirror-image patterns*, as will now be explained.

Table 1. *Repeat and mirror-image patterns for columns (1,2,3,4,5) from the 24-run Plackett and Burman design*

Run No.	1	2	3	4	5	Repeat & Mirror
1	+	+	+	+	+	Mirror of #24
2	+	+	+	+	–	Mirror of #20
3	+	+	+	–	+	
4	+	+	–	+	–	Mirror of #15
5	+	–	+	–	+	Mirror of #16
6	–	+	–	+	+	Mirror of #17
7	+	–	+	+	–	
8	–	+	+	–	–	Mirror of #10 & identical to #12
9	+	+	–	–	+	Mirror of #11 & identical to #13
10	+	–	–	+	+	Mirror of #8 & #12
11	–	–	+	+	–	Mirror of #9 & #13
12	–	+	+	–	–	Mirror of #10 & identical to #8
13	+	+	–	–	+	Mirror of #11 & identical to #9
14	+	–	–	+	–	
15	–	–	+	–	+	Mirror of #4
16	–	+	–	+	–	Mirror of #5
17	+	–	+	–	–	Mirror of #6
18	–	+	–	–	–	
19	+	–	–	–	–	Mirror of #23
20	–	–	–	–	+	Mirror of #2
21	–	–	–	+	+	
22	–	–	+	+	+	
23	–	+	+	+	+	Mirror of #19
24	–	–	–	–	–	Mirror of #1

For a specific design, we can determine how many runs belong to mirror-image pairs (i.e., all signs of one run opposite to the signs of the other run), how many runs occur in repeat pairs, how many runs occur as mirror-images to a repeat-pair, how many runs occur in triples, how many singles there are (i.e., neither repeats nor mirror-images involved), and so on. For example, if we choose columns (1, 2, 3, 4, 5) from a 24-run Plackett and Burman design, we have (see Table 1) 6 mirror-image pairs (12 runs), 2 repeat pairs (4 runs) and moreover those 2 repeat pairs are also mirror-images of two other runs, one per pair (2 runs), no triples, and 6 singles (6 runs). Note that these numbers of runs add to 24, as they should. A much simpler example is given in Draper (1985): If we choose any 5 columns from a 12-run Plackett and Burman design, we have only 2 possible repeat and mirror-image patterns; one with a repeat pair and ten singles, and one with a mirror-image pair and ten singles. We note that:

1. The repeat and mirror-image pattern characterization is invariant not only under changes in column order, but also under the switching of signs in any set of columns. Obviously, identical designs give the same pattern. However, it is not known whether or not the same pattern implies identical designs.

2. Mirror-image runs provide points at opposite extremes to each other in the predictor variable space, each point being farther from its mirror image than from the remaining points. This is the essential concept of *foldover*.

3. Repeat runs provide information on pure error. If we wanted to use a projected design by itself as an investigating tool, and if reduction in the total number of runs were critical for reasons of expense or time, however, some of the repeat runs could be eliminated, particularly if it were known that the error variation were small. (Conversely, if the error were large, repeat runs become more desirable.) Note that when repeat runs are eliminated, the orthogonality is lost, causing correlation among the estimates. (A reverse approach is dis-

cussed by Taguchi (1987) who counts twice some runs performed only once! This trick of *partially supplementing designs* can be used to "restore orthogonality" for a quick, neat, but approximate analysis.)

4. If the projected design by itself were used to estimate main effects and two factor interactions, the mirror-image runs would not provide mirror-image levels in terms of the two factor interactions, but "repeat levels" instead. This further de-aliases the main effects from the two factor interactions, and so is desirable.

3. RESULTS FOR N = 12, 20 AND 24

The three cases presented here cover about the practical range of factors for screening designs. The 24 run design can handle 23, 22, 21 and 20 factors; the 20 run design can handle 19, 18, 17 and 16; next in line comes the 16 run 2_{III}^{k-p} design, and then the 12 run design, followed by the 8 and 4 run 2_{III}^{k-p} designs, to cover the complete range. All through this range of factors, orthogonal columns are preserved for the main effect columns. (The details for the 28-run case have also been evaluated but occupy 18 pages of text. For that reason, they are omitted.)

We first describe the specific way in which the designs we studied were generated. Plackett and Burman (1946) provide the following rows of signs:

$N = 12$: + + − + + + − − − + −

$N = 20$: + + − − + + + + − + − + − − − − + + −

$N = 24$: + + + + + − + − + + − − + + − − + − + − − − −

Designs are generated by cyclic permutation, row by row, cycling to the left; that is, we remove the leftmost sign, place it on the extreme right and move all signs one place back to the left. (Thus, for example, the second row of signs for the 12–run design is + – + + + – – – + – +.) If cycling is done in any other manner, e.g., cycling right as in Plackett and Burman (1946), or writing Plackett and Burman columns in row as in Box, Hunter and Hunter (1978), the columns would have to re-labeled. Apart from that, essentially identical results would be obtained. Thus these three designs represent all the designs of this type, without loss of generality.

Because of the cyclic structures of the base Plackett and Burman designs, we always have $(x_1, x_2, \ldots, x_k) = (x_1+i, x_2+i, \ldots, x_k+i) \bmod n$, for any i. This means that, for any combination of column numbers, we can always find another combination with $x_1=1$ that gives the same design. Letting x_1 always be the first column implies that there are only $\binom{N-2}{k-1}$, rather than $\binom{N-1}{k}$, possible choices for the remaining columns. This considerably reduces the amount of computation needed.

We have examined various choices of $k-1$ columns from the $N-2$ possible, and have categorized the resulting designs by repeat and mirror-image patterns and the frequencies with which the various patterns occurred. Tables 2, 3, and 4, show the various results for $2 \le k \le 5$ and $N = 12$, 20 and 24, respectively. We restrict $k \le 5$ on the basis that, in general, it would be unusual to have more than five active factors when using these designs to

Table 2. *All possible repeat & mirror-image patterns for choosing k columns from the 12-run Plackett and Burman design*

k	Design number	Columns chosen	Pattern	Frequency
2	2.1	1,2	$\begin{bmatrix}3\\3\end{bmatrix}^2$	10
3	3.1	1,2,3	$\begin{bmatrix}2\\1\end{bmatrix}^4$	45
4	4.1	1,2,3,4	$\begin{bmatrix}2\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}^3 \begin{bmatrix}1\\0\end{bmatrix}^4$	120
5	5.1	1,2,3,4,5	$\begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}^{10}$	180
	5.2	1,2,3,5,8	$\begin{bmatrix}2\\0\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}^{10}$	30

screen factors. Extended tables for $k \leq (N/2)-1$ are, however, available from the authors. Note that designs for k and $N-k-1$ are complementary and can be derived one from the other.

Table 2 exhibits the patterns for the 12-run case. For $k = 3$, there is just one pattern which could be obtained via the choice of columns 1, 2, and 3, and which would produce four repeat pairs, each pair having a single run as mirror-image. When the design is generated as described previously, runs 1 and 5 form a repeat pair, and run 8 is the mirror-image of both runs 1 and 5. Three other such sets exist. For $k = 4$, the design formed by columns 1, 2, 3, 4 has a repeat pair (runs 2 and 10) and three mirror-image pairs (runs 1 and 8, 3 and 6, 4 and 7) but the repeats and mirror-images are otherwise not interconnected. For $k = 5$, there are two distinct patterns, one with a mirror-image pair (design 5.1) and one with a repeat pair (design 5.2).

The pattern notation of Table 2 summarizes these connections. For example, for k=3, $\begin{bmatrix}2\\1\end{bmatrix}^4$ means that a pair of repeats is mirror-image to another run and this pattern occurs four times over. For k=4, $\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}^4$ means that there is an otherwise unconnected pair of repeats, three mirror image pairs and four singles (a single has no repeats and no mirror images). Design 5.1 has one mirror image pair and 10 singles, while design 5.2 has a pair of repeats and 10 singles. As a check, note that the cross product $\sum$ (sum of elements within square brackets) $\times$ (power of the bracket) $= N$, where the sum is taken over all brackets. For example, for Design 4.1, we have $(2+0)\times 1+(1+1)\times 3+(1+0)\times 4=12=N$. The last column of Table 2 shows the frequencies for the various patterns; these add up to $\binom{N-2}{k-1}$ as already mentioned. If for $k = 12$, one of the five factor designs were to be used as an experimental design, the choice reduces to whether a repeat run is required or not. (For a first order model both choices have the same D-criterion.) On the other hand, if 11 design factors were assigned to the Plackett and Burman columns at random, the frequencies show that the two specific five factor projected designs will occur in a 180:30 or 6:1 ratio; in only one-seventh of the projections would a projection with a repeat run occur. Tables 3 and 4 cover the 20-run and 24-run cases respectively.

Table 3. *All possible repeat and mirror-image patterns for choosing k columns from the 20-run Plackett and Burman design*

k	Design number	Columns chosen	Pattern	Frequency
2	2.1	1,2	$\begin{bmatrix}5\\5\end{bmatrix}^2$	18
3	3.1	1,2,3	$\begin{bmatrix}3\\2\end{bmatrix}^4$	144
	3.2	1,3,6	$\begin{bmatrix}4\\1\end{bmatrix}^4$	9
4	4.1	1,2,3,4	$\begin{bmatrix}2\\1\end{bmatrix}^4\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^3$	576
	4.2	1,2,3,6	$\begin{bmatrix}2\\0\end{bmatrix}^3\begin{bmatrix}2\\1\end{bmatrix}^3\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$	192
	4.3	1,2,3,16	$\begin{bmatrix}2\\2\end{bmatrix}^3\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^4$	48
5	5.1	1,2,3,4,5	$\begin{bmatrix}1\\1\end{bmatrix}^5\begin{bmatrix}1\\0\end{bmatrix}^{10}$	495
	5.2	1,2,3,4,6	$\begin{bmatrix}2\\0\end{bmatrix}^3\begin{bmatrix}1\\1\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}^{10}$	360
	5.3	1,2,3,4,9	$\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}^{10}$	810
	5.4	1,2,3,6,16	$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}^3\begin{bmatrix}1\\1\end{bmatrix}^3\begin{bmatrix}1\\0\end{bmatrix}^5$	270
	5.5	1,2,3,4,14	$\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}^{10}$	360
	5.6	1,2,3,4,15	$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}^5$	180
	5.7	1,2,3,5,12	$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}^5\begin{bmatrix}1\\0\end{bmatrix}^5$	405
	5.8	1,2,3,6,9	$\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}^5\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^5$	45
	5.9	1,2,3,6,10	$\begin{bmatrix}2\\0\end{bmatrix}^4\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^{10}$	135

Table 4. *All possible repeat and mirror-image patterns for choosing k columns from the 24-run Plackett and Burman design*

k	Design number	Columns chosen	Pattern	Frequency
2	2.1	1,2	$\begin{bmatrix}6\\6\end{bmatrix}^2$	22
3	3.1	1,2,3	$\begin{bmatrix}3\\3\end{bmatrix}^4$	132
	3.2	1,3,6	$\begin{bmatrix}4\\2\end{bmatrix}^4$	99
4	4.1	1,2,3,4	$\begin{bmatrix}2\\2\end{bmatrix}^4\begin{bmatrix}1\\1\end{bmatrix}^4$	132
	4.2	1,2,3,5	$\begin{bmatrix}2\\1\end{bmatrix}^8$	528
	4.3	1,2,3,6	$\begin{bmatrix}3\\1\end{bmatrix}^2\begin{bmatrix}2\\2\end{bmatrix}^2\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^2$	528
	4.4	1,2,3,8	$\begin{bmatrix}3\\0\end{bmatrix}^2\begin{bmatrix}2\\1\end{bmatrix}^6$	352
5	5.1	1,2,3,4,11	$\begin{bmatrix}2\\1\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^6\begin{bmatrix}1\\0\end{bmatrix}^6$	495
	5.2	1,2,3,4,7	$\begin{bmatrix}2\\1\end{bmatrix}^2\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}^6$	2,310
	5.3	1,2,3,4,8	$\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^6\begin{bmatrix}1\\0\end{bmatrix}^8$	1,320
	5.4	1,2,3,4,9	$\begin{bmatrix}2\\1\end{bmatrix}^2\begin{bmatrix}2\\0\end{bmatrix}^4\begin{bmatrix}1\\1\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}^6$	1,100
	5.5	1,2,3,4,14	$\begin{bmatrix}1\\1\end{bmatrix}^8\begin{bmatrix}1\\0\end{bmatrix}^8$	220
	5.6	1,2,3,5,15	$\begin{bmatrix}2\\0\end{bmatrix}^4\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}^8$	1,485
	5.7	1,2,3,6,9	$\begin{bmatrix}3\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^6\begin{bmatrix}1\\0\end{bmatrix}^6$	165
	5.8	1,2,4,5,10	$\begin{bmatrix}3\\0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}^2\begin{bmatrix}1\\1\end{bmatrix}^4\begin{bmatrix}1\\0\end{bmatrix}^6$	110
	5.9	1,2,6,7,9	$\begin{bmatrix}2\\0\end{bmatrix}^6\begin{bmatrix}1\\1\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}^8$	110

Certain general statements can be made. For any Plackett and Burman design projected into $k = 2$ dimensions, we obtain a 2^2 full factorial, $N/4$ times over. It follows that every run has $N/4$ repeats (including itself in the count) and $N/4$ mirror-image runs. For all projections into $k = 3$ dimensions, we can make the following general characterization. The N points that arise after projection into the three-dimensional space of (x_u, x_v, x_w), where $u \neq v \neq w$ are any of 1, 2, ..., k, will always consist of a combination of

(a) a 2^{3-1} design with $x_u x_v x_w = 1$, occurring f_1 times (say), and

(b) a 2^{3-1} design with $x_u x_v x_w = -1$, occurring f_{-1} times (say).

Although the values of f_1 and f_{-1} will depend on the particular choice of u, v, and w, they will always satisfy $f_1 + f_{-1} = N/4$. Specifically, for $N = 12$; 20; and 24, the combinations (f_1, f_{-1}) = (2,1); (3,2) or (4,1); and (3,3) or (4,2) can all arise, as the tables indicate. (These numbers are not affected by the choice of the particular Plackett and Burman design selected.) For $k \geq 4$, further study is needed.

4. THE USEFULNESS OF THESE PROJECTION RESULTS

The Plackett and Burman screening designs are most frequently performed in situations where only a few of the many factors investigated are expected to be useful for future study. Such a situation has been called *effect sparsity* by Box and Meyer (1986). Thus, once the response results have enabled us to find the "large effects", often via a normal plot, it becomes appropriate to project the design used into the smaller number of dimensions identified with these large effects. Knowledge of the geometry of all possible projections is useful in several ways.

1. If no prior knowledge of which predictor variables *might* be important is available, the variables would typically be randomly assigned to the columns. Tables 2-4 show the various frequencies of all the projections into k=2-5 dimensions that are possible, and how often they occur. We see, for example, that a "desirable" five-factor projection like 5.5 in Table 4, which is a three-quarters fraction of a 2^5 (John, 1962), will occur only rarely (probability 220/7315=0.0301), about 3% of the time, when a 24-run design is used.

2. If prior knowledge of which predictor variables *might* be important does exist, however, we can use it to allocate variables to columns in an advantageous way. Continuing the example in (1), if five variables were thought likely to have real effects, they could be allocated to columns 1, 2, 3, 4, and 14 (see 5.5 in table 4) and the desirable projection would then be attained if the prior information was substantiated by the data. Similarly, suppose we decide to screen up to 23 variables in 24 runs, but believe that *four* specific variables are likely to be important. Entering Table 4 with k=4 shows that the projection 4.2, obtained from columns 1, 2, 3, and 5, consists of a full 2^4 plus a 2^{4-1} design. Thus the four variables of interest may sensibly allocated to columns 1, 2, 3, and 5. Other choices shown in Table 4 have projections whose points are more irregularly deployed in the four-dimensional projected space.

3. Because all the possible projections are known from Tables 2-4, the experimenter can decide upon additional follow-up runs to achieve whatever pattern appears desirable. Completion, in the projected spaces of k=2-5 factors, of full and 2^{k-p} fractional factors when N=12 has been discussed by Lin and Draper (1992). Similar extensions can be made using the results in Tables 3 and 4.

Note that, even when all columns of the design are not used initially, the unused columns can provide estimates of errors. The same is true of the columns that do not belong in the chosen projected space. Projections of the Plackett and Burman designs are useful in other (non-screening) connections also:

4. They can be used as the "cube" portions of *small composite designs* (see, Draper, 1985; Box and Draper, 1987; and Draper and Lin, 1990). The pairwise orthogonality of the columns of the **X** matrix, where the model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ is to be fitted to the response vector **y**, is an important feature of these designs in the first order case. When axial points are added to the Plackett and Burman points to form a second-order composite design, pairwise orthogonality of any main effect column with any two-factor interaction column in the new **X** is preserved. The pure quadratic columns of **X** are, of course, not orthogonal to one another, nor to 1, the column of 1's, arising from the intercept in the model, and the two-factor interactions are not mutually orthogonal, though they are orthogonal to the pure quadratic columns.

5. A different connection is found in the area of mixture experiments. If there are q mixture ingredients, $(q-1)$ columns of a Plackett and Burman design can be used to allocate two levels of each of $(q-1)$ ingredients. The q-th ingredient is then assigned levels to make the ingredients sum to unity, or some other required value. (A variation of this method is needed when there are additional constraints on the ingredients.) See, for example, Snee and Marquardt (1974, 1976) and Piepel (1990). An indirect application of our results to Piepel (1990) is discussed in next section.

5. AN INDIRECT APPLICATION TO MIXTURES DESIGNS

We see from Table 2 that only one essentially different choice of four columns from the 12-run Plackett and Burman design is possible. Because the full design is unique in the same manner, it follows that only one essentially different choice of seven columns is possible. Piepel (1990) chooses five 12 run designs, $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$, say. By rearranging column order and changing signs, we can establish that Piepel's five choices are initially equivalent to one another as follows (apart from a rearrangement of run order):

$\mathbf{P}_1$: Plackett and Burman columns (1,2,3,4,5,6,7).

$\mathbf{P}_2$: The changes (–8,–1,10,–5,–6,11,–3) make the design equivalent to $\mathbf{P}_1$.

$\mathbf{P}_3$: The columns (5,6,7,8,9,10,11) form a cyclic permutation of $\mathbf{P}_1$.

$\mathbf{P}_4$: The changes (–10,–11,–6,2,–1,–3,9) make the design equivalent to $\mathbf{P}_1$.

$\mathbf{P}_5$: The reordering (7,4,10,2,3,8,9) makes the design equivalent to $\mathbf{P}_1$.

In Piepel's specific application to mixtures, $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_5$ provide *different* mixture designs, however, because of (i) the allocation of the high and low levels to the + and -- signs, and (ii) the "choice of levels back-correction" needed when the initial allocation of the last mixture level (or the last two, or last three; see Piepel, 1990, Table 3) causes the mixture total not to add to unity at first. Nevertheless, it is clear that any variance criteria used to assess the various Piepel first order mixtures designs cannot vary much, because the initial design before mixture level allocation are all essentially the same. This is evident in the calculations presented in Piepel (1990, Table 4), and a similar flatness can be observed in Piepel (1990, Tables 6,7).

6. REMARKS

1. Another possibility for characterizing projected designs is by a criterion of design optimality, such as D-optimality. The *d-value* is defined as $d = |\mathbf{X}'\mathbf{X}|^{1/p}/N$, where p is the number of parameters in $\boldsymbol{\beta}$, so that $\mathbf{X}$ is $N \times p$. Note that, because of the complete orthogonality of Plackett and Burman designs, the first order matrix $\mathbf{X}'\mathbf{X}$,where $\mathbf{X} = (1, x_1, x_2, \cdots, x_p)$, is diagonal with all diagonal elements equal to N for a general N-run case, and therefore has the determinant $|\mathbf{X}'\mathbf{X}| = N^{(p+1)}$. This implies that, for first-order fitting, the d-value will not be affected by the specific choice of k columns. The experimenter can thus choose a design using Tables 2-4, knowing that all are D-optimal and so equally good from *that* point of view.

2. If we add a column of +1's to a Plackett and Burman design, we obtain a square N by N matrix (**H**, say) with orthogonal columns such that $\mathbf{H}'\mathbf{H} = N\mathbf{I}$. Such a matrix is, of course, a Hadamard matrix. For a lucid discussion, see Hedayat and Wallis (1978). Hadamard matrices of order higher than 12 are not unique. Hadamard matrices not equivalent to the Plackett and Burman designs will produce some patterns different from those in this paper. This can be seen by performing computations on the five non-equivalent Hadamard matrices of order 16, and three (namely two in addition to the design discussed already) of order 20, given in Hall (1961, 1965). For related details of the 16-run case, see Lin and Draper (1991) and Sun and Wu (1993). A complete search for N=20 shows that no projections other than those in Table 3 arise when k=2, 3, and 4. For N=20 and k=5, all the projections of Table 3 occur for all designs and there is, in addition, a new projection of the type

$$\begin{bmatrix}2\\0\end{bmatrix}^6\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}^5$$

that does not arise from the Plackett and Burman design. For N=24, the number of non-equivalent Hadamard matrices is not known.

3. Hadamard matrices can be characterized using $\chi(a)$, the *quadratic character* modulo $N-1$. The orthogonality of Plackett and Burman designs as main effects designs is related to the fact that for any i,j=1, . . . ,$N-1$

$$\sum_{x=1}^{N-1} \chi(x+i)\,\chi(x+j) = -1,$$

(usually stated in terms of the *Legendre symbol*, see, e.g., Hedayat and Wallis, 1978). This sum is directly related to the (i,j)th element of the $\mathbf{X}'\mathbf{X}$ matrix for the main effects model. To investigate questions relating to second-order designs, one could look at similar sums

$$\sum_{x=1}^{N-1} \chi(x+i)\,\chi(x+j)\,\chi(x+l) \quad \text{and} \quad \sum_{x=1}^{N-1} \chi(x+i)\,\chi(x+j)\,\chi(x+l)\,\chi(x+m)\ ,$$

which are related to the elements of $\mathbf{X}_M'\mathbf{X}_I$ and $\mathbf{X}_I'\mathbf{X}_I$, respectively, for $i,j,l,m=1,\ldots,N-1$, where $\mathbf{X}_1$ and $\mathbf{X}_M$ are the portion of the $\mathbf{X}$ matrix corresponding to main effects and two-factor interactions, respectively.

4. Vijayan (1976) has shown that any square N by N Hadamard matrix with $N-d$ columns can be extended to the full matrix essentially uniquely only for $d\leq 3$. When only d zero-sum orthogonal columns are given, knowledge of the patterns given here will guide one to an identification of the full design.

5. Srivastava and Anderson (1970) also select columns from some Plackett and Burman designs. Their work and orientation are different however, in several ways. They make their selections from balanced incomplete block designs, some of which are equivalent to Plackett and Burman designs. They also use foldover, and do not study all the possibilities that could arise.

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