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Bayesian Inference for Masked System Lifetime Data

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SUMMARY

Estimating component and system reliabilities frequently requires using data from the system level. Because of cost and time constraints, however, the exact cause of system failure may be unknown. Instead, it may only be ascertained that the cause of system failure is due to a component in a subset of components. This paper develops methods for analysing such masked data from a Bayesian perspective. This work was motivated by a data set on a system unit of a particular type of IBM PS/2 computer. This data set is discussed and our methods are applied to it.

Keywords: Censoring; Masked data; Posterior probability; Reliability

1. Introduction

Failure data from systems are often analysed as a means of estimating the reliability of the systems' components. Such component estimates are useful since they reflect the reliability of components in actual degradation that occurs within the system manufacturing, assembly or transportation process. Once obtained, these component reliability estimates can then be used to predict the reliability of new systems better. When a complex system fails, however, one or more modules each containing components are usually replaced to bring the system back to an operating state. The analysis of life data on complex systems constructed from many components is thus often complicated because the exact cause of failure is not precisely known. Rather the failure cause may only be isolated down to some subset of components. Such

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data are said to be *masked* since the cause of failure is masked from view. Masked data are becoming more widespread owing to the modular nature of today's complex systems.

For example, the data in Table 1 result from a test of the system unit (no monitor or keyboard) of a particular type of IBM PS/2 computer, where the times to failure have been suitably transformed. The system is viewed as being made up of three components in a series, i.e.

(a) the planar (motherboard),
(b) the direct access storage device (disc drives) and
(c) the power supply.

682 systems were tested. Among them, eight failed and 674 were right censored at various times. The cause of failure is indicated below, specifying the set of components known to have caused system failure. For example {1} indicates that component 1 is known to have caused failure, whereas {1, 3} indicates that the cause of failure has been isolated to components 1 and 3.

Depending on the exact cause of failure, the technicians are sometimes unable to determine the exact cause. Using only basic diagnostic equipment they can sometimes find the true cause, whereas at other times they can only narrow the cause down to one of several possibilities. For example, when the cause of failure is listed as {1, 3}, this indicates that it was only possible to isolate the cause down to either the planar board (component 1) or the power supply (component 3). The engineer is forced to work with the masked data to estimate the reliability of the various components within the system. With the estimates in hand, an evaluation can still be made of the areas within the system where improvement efforts are likely to yield the best results. Additional engineering examples are given in Usher and Hodgson (1988) and Miyakawa (1984).

Previous work on the analysis of masked data has focused on point estimation; for example, see Guess *et al.* (1991), Usher and Guess (1989) and Usher and Hodgson (1988), and references therein. In this paper, we discuss a Bayesian analy-

TABLE 1
IBM PS/2 computer failure data†

Time to failure (h)	*Cause of failure*
1	{1}
1	{1}
1	{1, 3}
1	{1, 2, 3}
16	{3}
17	{2, 3}
21	{2}
222	{2}

†348 unfailed systems were removed at 67 h; 246 unfailed systems were removed at 200 h; 26 unfailed systems were removed at 800 h; 54 unfailed systems were removed at 4000 h.

sis, obtaining the relevant posterior distributions on which we base our inferences. Section 2 briefly discusses the general Bayesian approach under the assumption of exponentially distributed lifetimes. The analysis of the data described above is discussed in Section 3. For a $J = 2$ components system, closed form expressions for the Bayesian approach can be explicitly derived and are given in Section 4. Some concluding remarks are presented in Section 5.

2. General Bayesian Approach

The field of reliability is well suited to the use of Bayesian methods. As components for a system are developed, knowledge about performance is gained. A practitioner would probably like to model that previous experience and to incorporate it into the analysis. We shall illustrate the general Bayesian approach assuming exponentially distributed component lifetimes. The method can be extended to other distributions, although the computations may be somewhat complicated.

Let T_i be the random life for the ith system where $i = 1, \ldots, n$ in a sample of n systems each consisting of J components in a series. Let T_{ij}s be the random life of the jth component in the ith system where $j = 1, \ldots, J$. Note that $T_i = \min(T_{i1}, \ldots, T_{iJ})$ for $i = 1, \ldots, n$. We assume that the T_{ij} are independent. The dependence among the lifetimes of the components could be modelled by using a dependent multivariate distribution and a competing risk model. (See, for example, Barlow and Proschan (1981) and Basu and Klein (1982).) For each fixed j, the $T_{1j}, \ldots, T_{nj}$ would represent a sample of size n from component j's life distribution F_j. We require the very mild condition that F_j has a density f_j indexed by a parameter vector θ_j. For each j, a different number of parameters in θ_j is allowed if needed. Let $\bar{F}_j(t) = 1 - F_j(t)$ be the reliability of component j at time t. Let K_i be the index of the component causing the failure of system i. Since the life distribution is continuous, the cause of failure K_i is unique. Note that K_i is a random variable and that K_i may or may not be observed. Before the sample is taken, there is the *minimum random subset* M_i of components known to contain the true cause of failure of system i.

After the sample data have been obtained, we have $M_i = S_i \subset \{1, 2, \ldots, J\}$ and $T_i = t_i$ where $i = 1, \ldots, n$. As t_i is the realized sample value of T_i, so S_i is the realized sample set of M_i. The observed data here are the points $(t_1, S_1), \ldots, (t_n, S_n)$. The full likelihood for these data is given by

$$L_{\mathrm{F}} = \prod_{i=1}^{n} \left[\sum_{j \in S_i} \left\{ f_j(t_i) \prod_{s=1,\, s \neq j}^{J} \bar{F}_s(t_i) P(M_i = S_i \mid T_i = t_i,\, K_i = j) \right\} \right]. \tag{1}$$

The term $f_j(t_i) \Pi_{s=1, s \neq j} \bar{F}_s(t_i)$ arises from system i failing at time t_i, caused by j (component j), and $P(M_i = S_i \mid T_i = t_i, K_i = j)$ denotes the conditional probability that the observed minimum random subset is S_i, given that system i failed at time t_i and the true cause was component j. The product in equation (1) is then taken over each of the observations to yield the full likelihood L_{F}.

For industrial problems, the masking typically occurs because of constraints of time and the expense of failure analysis. Schedules often dictate that complete failure analysis (to determine the true cause of failure) be curtailed. In such settings it is reasonable to assume, for fixed $j' \in S_i$, that

$$P(M_i = S_i \mid T_i = t_i,\ K_i = j') = P(M_i = S_i \mid T_i = t_i,\ K_i = j) \qquad \text{for all } j \in S_i. \quad (2)$$

Assumption (2) states that the masking probabilities are independent of the causes $j \in S_i$, or in other words that the probability that a particular subset of components is masked is the same regardless of which of the components in the subset is actually the cause of failure. Furthermore, if $P(M_i = S_i \mid T_i = t_i, K_i = j)$ does not depend on the life distribution parameters, the reduced likelihood is

$$L_R = \prod_{i=1}^{n} \left[\sum_{j \in S_i} \left\{ f_j(t_i) \prod_{s=1,\, s \neq j}^{J} \bar{F}_s(t_i) \right\} \right]. \quad (3)$$

Maximizing L_R with respect to the life parameters is now equivalent to maximizing L_F. This is similar to the usual derivation of a time-censored (and not masked) data partial likelihood. The *dependence* case where assumption (2) is violated is currently under study and will be addressed in a separate paper. See Guttman *et al.* (1994).

In many situations, such as the IBM PS/2 example, the system life data are right censored. The reduced likelihood (3) can be generalized (Guess *et al.*, 1991) to

$$L_R = \prod_{i=1}^{n} \left[\left\{ \sum_{j \in S_i} f_j(t_i) \prod_{s=1,\, s \neq j}^{J} \bar{F}_s(t_i) \right\}^{\alpha_i} \left\{ \prod_{s=1}^{J} \bar{F}_s(t_i) \right\}^{1-\alpha_i} \right] \quad (4)$$

where

$$\alpha_i = \begin{cases} 1 & \text{if system } i \text{ is uncensored,} \\ 0 & \text{if system } i \text{ is censored.} \end{cases}$$

Assuming that each component life has an exponential density with failure rate λ_j, $j = 1, \ldots, J$, we find, since $\bar{F}_j(t) = \exp(-\lambda_j t)$, that

$$L_R = \prod_{i=1}^{n} \sum_{j \in S_i} \left\{ \lambda_j \exp(-\lambda_j t_i) \prod_{s=1,\, s \neq j}^{J} \exp(-\lambda_s t_i) \right\} = \exp(-T\delta) \prod_{i=1}^{n} \sum_{j \in S_i} \lambda_j, \quad (5)$$

where $T = \Sigma_{i=1}^{n} t_i$ and $\delta = \Sigma_{s=1}^{J} \lambda_s$. If we use the usual invariant improper prior, $\pi(\lambda_1, \ldots, \lambda_J) \propto 1/\Pi_{s=1}^{J} \lambda_s$, for example, the posterior of $\lambda_1, \ldots, \lambda_J$ is then $p(\lambda_1, \ldots, \lambda_J \mid \text{data}) \propto L_R/\Pi_{s=1}^{J} \lambda_s$ from which Bayesian inferences follow. The above prior is appropriate when there is only vague prior information available on the λ_i. It provides a useful comparison point with frequency-based inference and a base-line for examining the influence of informative priors. This prior is used throughout this paper. We note, however, that strong prior information could frequently be effectively modelled by gamma priors. In this case the formulae derived below would require some simple modification.

For any specific system G, the failure time of component j, as mentioned above, is T_{G_j}. The reliability of system G at time t is

$$P(T_G \geqslant t) = P(T_{G_j} \geqslant t, \text{ for all } j = 1, \ldots, J) = \exp\left(-\sum_{j=1}^{J} \lambda_j t \right)$$

$$= \exp(-\delta t) = R(\delta;\ t). \quad (6)$$

Consequently, inference on δ, or equivalently, for fixed t, $R(\delta; t)$, is of interest. In addition, we frequently want to carry out inference on the individual component reliabilities. The reliability of component j at time t is $R_j = R_j(\lambda_j,\ t) = \bar{F}_j(t) =$

$\exp(-\lambda_j t)$. Posterior distributions for the component reliabilities can be obtained by transformation of variables. For right-censored exponentially distributed system life data, we obtain from equation (4) that

$$L_R = \exp(-T\delta) \prod_{i=1}^{n} \left(\sum_{j \in S_i} \lambda_j \right)^{\alpha_i}. \tag{7}$$

We now return to the data described in Section 1.

3. Analysis of IBM PS/2 Data

For the data in Table 1 described in Section 1, it follows from equation (7) that $L_R = \exp\{-T(\lambda_1 + \lambda_2 + \lambda_3)\}\lambda_1^2\lambda_2^2\lambda_3^1(\lambda_1 + \lambda_2)^0(\lambda_1 + \lambda_3)^1(\lambda_2 + \lambda_3)^1(\lambda_1 + \lambda_2 + \lambda_3)^1$. Using the usual non-informative prior, $p(\lambda_1, \lambda_2, \lambda_3) \propto 1/\lambda_1\lambda_2\lambda_3$, gives the posterior of $(\lambda_1, \lambda_2, \lambda_3)$ as

$$p(\lambda_1, \lambda_2, \lambda_3 \,|\, \text{data}) = K \exp\{-(\lambda_1 + \lambda_2 + \lambda_3)T\} H(\lambda_1, \lambda_2, \lambda_3) \tag{8}$$

where

$$H(\lambda_1, \lambda_2, \lambda_3) = \lambda_1\lambda_2(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)$$

and $K^{-1} = 70/T^8$ by integrating out equation (8) term by term. Note that the total time on test $T = \Sigma_{i=1}^{n} t_i = 309\,596$. The marginal posterior for each λ_i can be obtained via term-by-term integration of equation (8):

$$\left.\begin{aligned} p(\lambda_1 \,|\, \text{data}) &= \frac{T^2}{70} \exp(-\lambda_1 T)\lambda_1(3\lambda_1^2 T^2 + 16\lambda_1 T + 20); \\ p(\lambda_2 \,|\, \text{data}) &= \frac{T^2}{70} \exp(-\lambda_2 T)\lambda_2(3\lambda_2^2 T^2 + 16\lambda_2 T + 20); \\ p(\lambda_3 \,|\, \text{data}) &= \frac{T}{70} \exp(-\lambda_3 T)(\lambda_3^3 T^3 + 8\lambda_3^2 T^2 + 24\lambda_3 T + 24). \end{aligned}\right\} \tag{9}$$

These marginal posteriors are plotted in Fig. 1. Note that λ_1 and λ_2 are identically distributed. Moreover,

$$E(\lambda_1 \,|\, \text{data}) = E(\lambda_2 \,|\, \text{data}) = \frac{104}{35T} = 9.5978 \times 10^{-6},$$

$$\operatorname{var}(\lambda_1 \,|\, \text{data}) = \operatorname{var}(\lambda_2 \,|\, \text{data}) = \frac{4304}{1225T^2} = 3.6656 \times 10^{-11},$$

$$E(\lambda_3 \,|\, \text{data}) = \frac{72}{35T} = 6.6446 \times 10^{-6},$$

$$\operatorname{var}(\lambda_3 \,|\, \text{data}) = \frac{3636}{1225T^2} = 3.0967 \times 10^{-11}.$$

The corresponding standard errors are large, compared with the estimates. We may use the marginals in equations (9) to find posterior probability limits for the λ_is

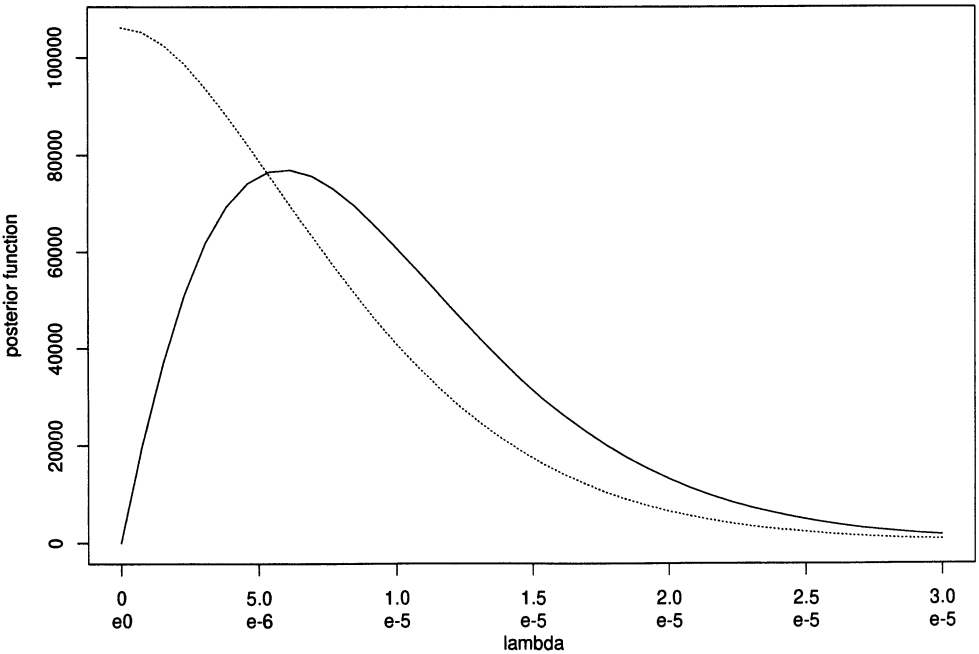

Fig. 1. Marginal posterior for each λ: ———, λ_1, λ_2; · · · · ·, λ_3

numerically in the usual way. However, we note that there is an interesting connection of the densities in equations (9) with χ^2-distributions, so that existing routines for the cumulative density functions (CDFs) of χ^2-distributions may be utilized. To sketch this connection, we first note that, in general, if λ has the distribution

$$f(\lambda \mid m; T) = \frac{T^{m+1}}{\Gamma(m+1)} \lambda^m \exp(-T\lambda),$$

when $m \geqslant 0$, $\lambda \geqslant 0$ and zero otherwise, then it is easy to show that $W = 2T\lambda \sim \chi^2_{2(m+1)}$. Hence, from equations (9), we have

$$\begin{aligned} P(\lambda_1 \leqslant a) &= \frac{3\times 6}{70}\int_0^a f(\lambda_1 \mid 3;\ T)\,\mathrm{d}\lambda_1 + \frac{2\times 16}{70}\int_0^a f(\lambda_1 \mid 2;\ T)\,\mathrm{d}\lambda_1 + \frac{20}{70}\int_0^a f(\lambda_1 \mid 1; T)\,\mathrm{d}\lambda_1 \\ &= \frac{9}{35} P(\chi_8^2 \leqslant 2Ta) + \frac{16}{35} P(\chi_6^2 \leqslant 2Ta) + \frac{10}{35} P(\chi_4^2 \leqslant 2Ta), \end{aligned}$$

$$P(\lambda_2 \leqslant a) = \frac{9}{35} P(\chi_8^2 \leqslant 2Ta) + \frac{16}{35} P(\chi_6^2 \leqslant 2Ta) + \frac{10}{35} P(\chi_4^2 \leqslant 2Ta),$$

$$P(\lambda_3 \leqslant a) = \frac{3}{35} P(\chi_8^2 \leqslant 2Ta) + \frac{8}{35} P(\chi_6^2 \leqslant 2Ta) + \frac{12}{35} P(\chi_4^2 \leqslant 2Ta) + \frac{12}{35} P(\chi_2^2 \leqslant 2Ta)$$

and it is easy to see that iteration with respect to a using readily available standard χ^2 CDF routines will produce percentage points of the CDFs for the λ_is, so that Bayesian probability intervals may be constructed. This procedure was used to find 95% intervals, which are

$$\left(\frac{0.8673}{2T}, \frac{15.0901}{2T}\right) = (1.4007 \times 10^{-6}, 2.4371 \times 10^{-5}),$$
$$\left(\frac{0.1459}{2T}, \frac{12.8629}{2T}\right) = (2.3559 \times 10^{-7}, 2.0774 \times 10^{-5}) \quad (10)$$

for λ_1 (and λ_2) and λ_3 respectively. An examination of the posterior means of the λ_is, the probability intervals and the graphs of the marginal posteriors in Fig. 1 indicates that the failure rate for the power supply (component 3) is smaller than the failure rates for the other two components.

For inference on a system's lifetime, paralleling the work in Section 2, we have that $R = \exp(-\delta t) = R(\delta; t)$, with $\delta = \lambda_1 + \lambda_2 + \lambda_3$, and R is a decreasing function of δ, so that inference on R follows from inference on δ. Now, to obtain the posterior for δ, we use the transformation of variables

$$\delta = \lambda_1 + \lambda_2 + \lambda_3, \qquad \gamma = \lambda_1 + \lambda_3, \qquad \tau = \lambda_2 + \lambda_3$$

in equation (8), and by integrating out γ and τ find

$$p(\delta \mid \text{data}) = \iint p(\delta, \gamma, \tau \mid \text{data})\, \mathrm{d}\gamma\, \mathrm{d}\tau = \frac{T^8}{\Gamma(8)} \delta^7 \exp(-T\delta), \quad (11)$$

which, interestingly, is to say that, conditional on the data,

$$2T\delta = 2T(\lambda_1 + \lambda_2 + \lambda_3) \sim \chi^2_{16}.$$

Hence, a $1 - \alpha$ two-sided probability interval for δ is

$$[a_1, a_2] = [\chi^2_{16, 1-\alpha/2}/2T, \ \chi^2_{16, \alpha/2}/2T],$$

so that a $1 - \alpha$ two-sided probability interval for R is

$$[\exp(-a_2 t), \ \exp(-a_1 t)].$$

One-sided intervals for R are similarly constructed. We provide a graph of lower $1 - \alpha$ probability bounds, say R_L, for R for various α and t, in Fig. 2, for use in a probability interval for R of the form $[R_L, 1]$.

From equation (11), for fixed t ($t > 0$), letting $R = \exp(-\delta t)$, we have, for $0 < R < 1$,

$$p(R \mid \text{data}, t) = \frac{T^8}{\Gamma(8)} \left\{\frac{1}{t}(-\log R)\right\}^7 (tR)^{-1} R^{T/t} = \frac{T^8}{7!} \frac{(-\log R)^7}{t^8} R^{T/t-1}. \quad (12)$$

Using equation (12), we find that

$$E(R \mid \text{data}, t) = (1 + t/T)^{-8}. \quad (13)$$

We graph equation (13) in the same figure as the lower bounds for R (Fig. 2). As expected, this figure well illustrates the fall-off of the system reliability as a function of t. For this particular example, high or moderate system reliability (e.g. $R > 0.8$) can be obtained for a system lifetime, which is less than about 2500, with a high posterior confidence. Posterior 95% probability intervals for the component reliabilities at any given time can be obtained by applying equations (10) to $R_j = \exp(-\lambda_j t)$, $j = 1, 2, 3$.

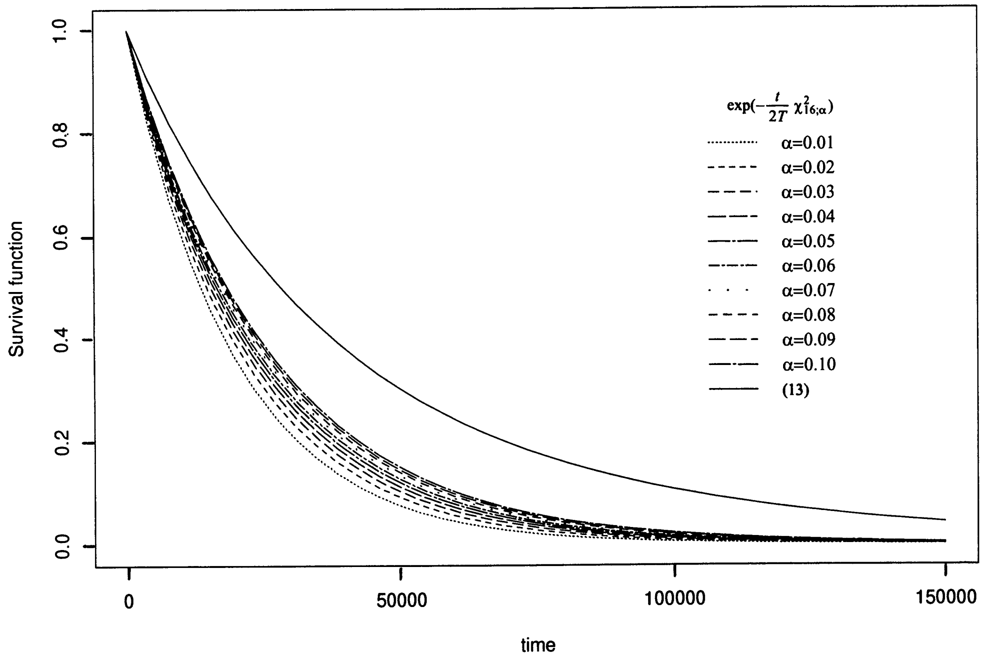

Fig. 2. Survival functions for various significance levels

4. $J = 2$ Components System

The special case of a $J = 2$ components system where closed form expressions for Bayesian analysis can be explicitly derived is now presented. For larger systems ($J > 2$), writing down the posteriors in general becomes difficult as a consequence of the large number of potential S_i. However, any particular data set can be handled as was illustrated in Section 3. Consider a two-component system, $J = 2$ with n systems put on test. Let n_1 and n_2 be the numbers of system failures for which the cause of failure is known to be components 1 and 2 respectively, while n_{12} denotes the number of failed systems where the cause is not directly known. Then,

$$\begin{aligned} p(\lambda_1, \lambda_2 \mid \text{data}) &= K \exp\{-T(\lambda_1 + \lambda_2)\} \lambda_1^{n_1 - 1} \lambda_2^{n_2 - 1} (\lambda_1 + \lambda_2)^{n_{12}} \\ &= K \exp\{-T(\lambda_1 + \lambda_2)\} \left\{ \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \lambda_1^{n_1 + j - 1} \lambda_2^{n_2 + n_{12} - j - 1} \right\} \end{aligned} \tag{14}$$

where we have obtained

$$K^{-1} = T^{-n^*} \sum_{j=0}^{n_{12}} \left\{ \binom{n_{12}}{j} \Gamma(n_1 + j) \Gamma(n_2 + n_{12} - j) \right\}$$

by integration, $n^* = n_1 + n_2 + n_{12}$. Such a posterior is plotted in Fig. 3, for $T = 209.8$, $n_1 = 14$, $n_2 = 12$ and $n_{12} = 4$ as an example. The scales for λ_1 and λ_2 range from 0 to 0.15 in Fig. 3.

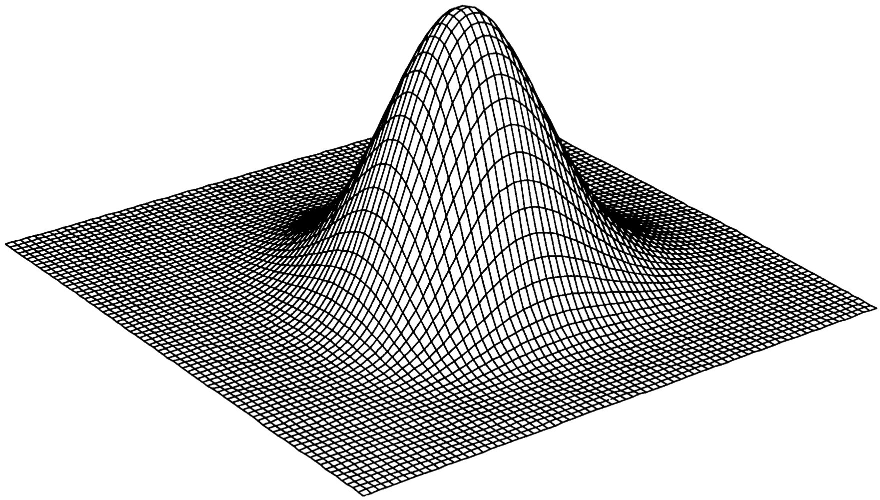

Fig. 3. Posterior for the $J = 2$ components system: $(n_1, n_2, n_{12}, T) = (14, 12, 4, 209.8)$; x-y-range (0, 0.15)

Consequently, the marginal distributions of λ_1 and λ_2 may be obtained by integration in equation (14) to be

$$\begin{aligned} p(\lambda_1 \mid \text{data}) &= K \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \lambda_1^{n_1+j-1} \exp(-\lambda_1 T) \frac{\Gamma(n_2 + n_{12} - j)}{T^{n_2+n_{12}-j}}, \\ p(\lambda_2 \mid \text{data}) &= K \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \lambda_2^{n_2+n_{12}-j-1} \exp(-\lambda_2 T) \frac{\Gamma(n_1 + j)}{T^{n_1+j}}. \end{aligned} \tag{15}$$

A plot of these marginal posteriors is given in Fig. 4 for the situation used in Fig. 3. It is clear from Fig. 4 that λ_1 tends to be larger than λ_2. Marginal posteriors for the component reliabilities $R_j = \exp(-\lambda_j t)$, $j = 1, 2$, can be readily derived from equations (15). We may further deduce that

$$\begin{aligned} E(\lambda_1 \mid \text{data}) &= K \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\Gamma(n_1 + j + 1)\,\Gamma(n_2 + n_{12} - j)}{T^{n^*+1}}, \\ E(\lambda_2 \mid \text{data}) &= K \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\Gamma(n_1 + j)\,\Gamma(n_2 + n_{12} - j + 1)}{T^{n^*+1}}. \end{aligned} \tag{16}$$

These expectations may, of course, be used as point estimators for the respective λ_is ($i = 1, 2$).

The maximum likelihood estimates (MLEs) are (see, for example, Usher and Hodgson (1988))

$$\begin{aligned} \lambda_1 &= \frac{n_1}{T} \frac{n_1 + n_2 + n_{12}}{n_1 + n_2}, \\ \lambda_2 &= \frac{n_2}{T} \frac{n_1 + n_2 + n_{12}}{n_1 + n_2}. \end{aligned} \tag{17}$$

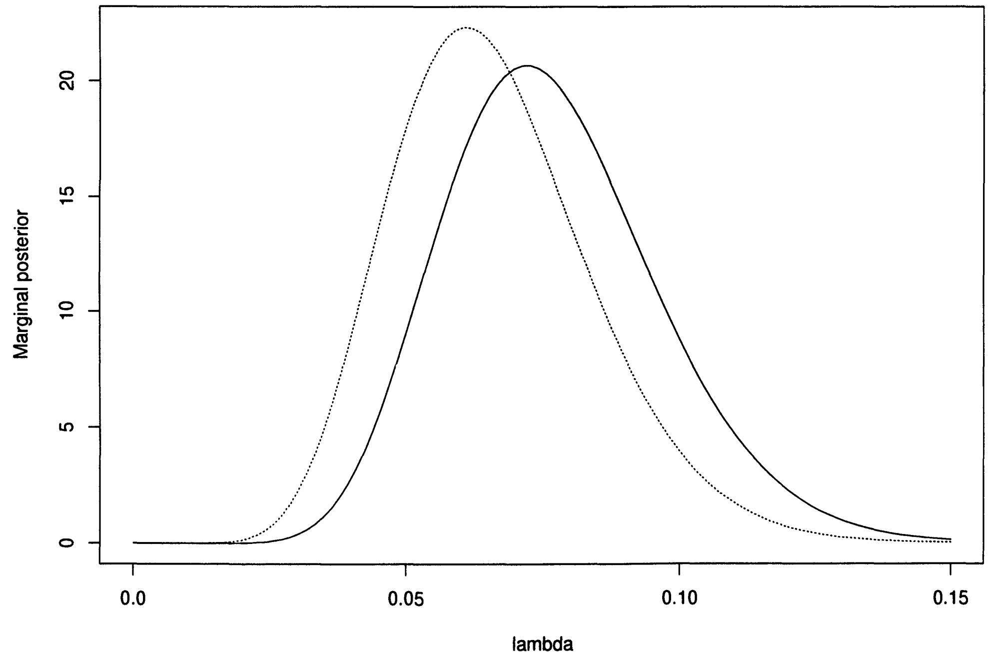

Fig. 4. Marginal distributions for λ_1 (———) and λ_2 (······): $(n_1, n_2, n_{12}, T) = (14, 12, 4, 209.8)$

It is interesting that the MLEs are identical with the Bayes estimates given in equations (16). (See the detailed derivation in Reiser *et al.* (1993).) We have not been able to prove this equivalence for $J > 2$. In other words, if not much prior information is available, the Bayes estimates based on improper non-informative priors are as good as the MLEs. In addition, we may utilize equations (14) and/or (15) to provide Bayesian probability regions or intervals for (λ_1, λ_2). This can require substantial computation as was illustrated in Section 3 for the PS/2 data.

We turn now to inference on the system reliability R, which in this case is, from equation (6),

$$R(\delta, t) = \exp(-\delta t) \tag{18}$$

where $\delta = \lambda_1 + \lambda_2$. We proceed by first determining the posterior of δ, using the transformation

$$\delta = \lambda_1 + \lambda_2, \qquad \gamma = \lambda_2$$

in equation (14). Note that $0 < \gamma < \delta$, and the absolute value of the Jacobian is 1. Hence,

$$p(\delta, \gamma \mid \text{data}) = K \exp(-\delta T) \left\{ \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} (\delta - \gamma)^{n_1+j-1} \gamma^{n_2+n_{12}-j-1} \right\}$$

and the posterior of δ is now

$$p(\delta \mid \text{data}) = K \exp(-\delta T) \sum_{j=0}^{n_{12}} \left\{ \binom{n_{12}}{j} \int_0^{\delta} (\delta - \gamma)^{n_1+j-1} \gamma^{n_2+n_{12}-j-1} \, \mathrm{d}\gamma \right\}.$$

Consequently, it is easy to see that (see Reiser *et al.* (1993) for details)

$$p(\delta \mid \text{data}) = K^* \exp(-\delta T)\delta^{n^*-1}, \tag{19}$$

where, normalizing directly, $K^* = T^{n^*}/\Gamma(n^*)$. It follows that the posterior of δ, given the data, is such that

$$2T\delta \sim \chi^2_{2n^*}.$$

Thus, it follows that

$$E(\delta \mid \text{data}) = n^*/T,$$

which may be obtained directly from equation (19), and a two-sided $1-\alpha$ posterior probability interval for δ is given by

$$\left[\frac{1}{2T}\chi^2_{2n^*;\alpha/2}, \frac{1}{2T}\chi^2_{2n^*;1-\alpha/2}\right]. \tag{20}$$

Using expressions (18) and (20), it is easy to see that a $1-\alpha$ posterior probability interval for R, the system reliability at time t, is

$$\left[\exp\left(-\frac{t}{2T}\chi^2_{2n^*;\alpha/2}\right), \exp\left(-\frac{t}{2T}\chi^2_{2n^*;1-\alpha/2}\right)\right].$$

This analysis also holds for censored data. If there is no censoring, $n^* = n_1 + n_2 + n_{12} = n$, whereas for censoring $n^* < n$, since for $\alpha_i = 0$ neither the cause nor the time of failure is observed.

5. Conclusions

We have presented a Bayesian approach for estimating the reliability of components through the analysis of masked system life data. Under the assumptions of independent masking and exponentially distributed component life, we have shown that masked system life data can be effectively used in the estimation process. Such an analysis is often needed in industrial settings because of the prevalence of masked data. Although our analysis used non-informative priors, it is readily seen that the use of an informative gamma prior for the various λ_is would not lead to much further complication. The case of dependence masking (i.e. when assumption (2) does not hold) that does lead to further complication is the subject of another paper. See Guttman *et al.* (1994).

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