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# MAKING FULL USE OF TAGUCHI'S ORTHOGONAL ARRAYS

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## SUMMARY

Taguchi[1] has provided 18 orthogonal arrays which have been widely touted as useful frameworks for planning experiments. Thirteen of these are 'saturated designs', that is, they are appropriate for investigating $(N - 1)$ factors in $N$ runs, thus using the full capacity of the design. Here, the other five 'non-saturated' designs are discussed. By creating additional, orthogonal columns which provide estimates of interaction effects, we can essentially wring out some additional information over and above that suggested by Taguchi, without additional cost. In particular, if only the linear effect is of interest for any specific factor, one can accommodate more factors than the number suggested by Taguchi. An example is given for illustration.

KEY WORDS Design of experiments Interaction Main-effects plan Unsaturated design Supplementary orthogonal column

## INTRODUCTION

The *scientific approach* to quality improvement is becoming more widespread in industrial practice. The application of statistical methods, in particular the design of experiments, has had considerable impact. The ideas of a very successful leading quality consultant in Japan, Dr. Genichi Taguchi, have been adopted by many American companies in both manufacturing and scientific contexts. The experimental designs employed by Taguchi,[1] known as *orthogonal arrays*, are essentially *fractional factorials*. These fractional factorial designs have been used for many years, certainly since Yates[2] wrote about them in 1935. About 20 years later, large compilations of two-level, three-level and mixed-level (two- and three-level) designs were made available by the U.S. National Bureau of Standards.[3–5]

In applying Taguchi's (or any other) designs, certain basic questions need to be addressed:

1. What response (output) variable(s) $y$ should be recorded?
2. Which input factors, $x_i$ might affect $y$, and how might they interact?
3. Over what region in the input space should experiments be performed?
4. How many levels are needed for each factor?

These questions have long been treated as 'given by professional knowledge', but typically need very careful consideration. Once this is done, the choice of the proper design and the column assignments is fairly routine.

Taguchi[1] has offered 18 useful designs which can be classified into three groups based upon their practical characteristics; namely, two-level, $p$-level $(p \geq 3)$, and mixed-level designs. The two-level designs in group I, including $L_4(2^3)$, $L_8(2^7)$, $L_{12}(2^{11})$, $L_{16}(2^{15})$, $L_{32}(2^{31})$ and $L_{64}(2^{63})$, are essentially Plackett and Burman[6] type designs. Note that Plackett and Burman[6] also provided $L_{20}(2^{19})$, $L_{24}(2^{23})$, $L_{28}(2^{27})$ ,..., etc., namely orthogonal arrays of $L_{4t}(2^{4t-1})$, for $t \leq 25$ (excepting $t = 23$). Designs in group II, including $L_9(3^4)$, $L_{16}(4^5)$, $L_{25}(5^6)$, $L_{27}(3^{13})$, $L_{64}(4^{21})$ and $L_{81}(3^{40})$, are the so-called orthogonal arrays of strength two.[7, 8] Designs in groups I and II are *saturated* in the sense that each degree of freedom is used to estimate one main effect, and all degrees of freedom are used. Designs in group III, including $L_{18}(2^1 \times 3^7)$, $L'_{32}(2^1 \times 4^9)$, $L'_{36}(2^3 \times 3^{13})$, $L_{36}(2^{11} \times 3^{12})$, $L_{50}(2^1 \times 5^{11})$ and $L_{54}(2^1 \times 3^{25})$, can be constructed by the methods of references 9 and 10. Among these six orthogonal arrays in group III, note that only the $L_{36}(2^{11} \times 3^{12})$ array is saturated; the other five arrays are not.

The unused or unaccommodated columns in orthogonal arrays do not typically estimate 'error', but interactions (or linear combinations of interactions). If separate estimation of the standard deviation is possible, as is true for most of Taguchi's experiments in which replicates are usually made, estimates from unaccommodated columns provide excellent additional information. Treating these estimates as error causes a too-large estimated variance, which provides reduced capability for detecting significant factors. The unaccommodated columns have no impact at the experimentation stage, of course.

### *Example*

Suppose four two-level factors $(A, B, C, D)$ are under study. Estimates of the two-factor interactions $A \times B$ and $A \times C$ are also required. An $L_8$ array is thus employed. Through the aid of an interaction

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Table I. Simulated data using the $L_8$ array

| Run | 1 $A$ | 2 $B$ | 3 $AB$ | 4 $C$ | 5 $AC$ | 6 e | 7 $D$ | Observations $y_i$ | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | 33·09 | 31·45 |
| 2 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | 21·13 | 19·26 |
| 3 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | 13·59 | 13·55 |
| 4 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | 18·68 | 24·69 |
| 5 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | 12·95 | 16·39 |
| 6 | 1 | −1 | 1 | 1 | −1 | 1 | −1 | 5·61 | 3·74 |
| 7 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | 3·43 | 2·42 |
| 8 | 1 | 1 | −1 | 1 | −1 | −1 | 1 | 14·40 | 15·20 |

table (see, for example, Reference 1, p. 1129), columns 1, 2, 4 and 7 are assigned to factors $A$, $B$, $C$ and $D$, respectively. Two replicates are made for each experiment. The results are given in Table I. In this case, column 3 is for the $A \times B$ interaction and column 5 is for $A \times C$. Note that column 6 is unused, and thus the symbol $e$ is attached to it.

Table II shows the estimated effects for $A$, $B$, $C$, $D$, $A \times B$ and $A \times C$ effects, as well as the analysis of variance (ANOVA) table. Factor $A$ appears to be the only significant effect. The overall $R^2$ is 62·9 per cent and the mean square error (MSE) is 52·43 for the estimated error variance of $\sigma^2$.

However, if we take the unaccommodated column $e$ into account, the estimated effect for $e$ is 5·25 and corresponding sum of squares is 442·23. Because of the property of orthogonal columns, the sum of squares and the estimated effects remain the same for all other factors. Accordingly, the sum of squares for error becomes 29·63 (so that the mean square error then is 3·70, a much smaller estimated value for $\sigma^2$). The $R^2$ increases to 97·7 per cent (compared to $R^2$ = 69·9 per cent originally). Besides the main effect $A$, the effects $B$, $A \times B$ and $e$ are now identified as significant effects. What does the column $e$ stand for? From the interaction table, we see that column 6 is for the $A \times D$ (= $B \times C$) interaction. Although this interaction was not of interest initially, we see that use of the $e$ column as 'error' leads to faulty conclusions. If the estimated effect of any unaccommodated column is indeed non-significant, then it can be treated as 'error'. Otherwise, one should identify which interaction effect(s) it estimates.

For Group I and II arrays, the confounding patterns are well-understood (for example, References 11 and 12).

## THE SUPPLEMENTARY ORTHOGONAL COLUMNS FOR UNSATURATED CASES

Among the 18 arrays suggested by Taguchi, five arrays are not saturated. Table III shows that there are 2, 3, 6, 4 and 2 unused degrees of freedom in $L_{18}$, $L'_{32}$, $L'_{36}$, $L'_{50}$ and $L_{54}$ respectively.

The corresponding supplementary orthogonal columns are given in Tables IV, V, VI, and VII for $L_{18}$, $L'_{36}$, $L_{50}$ and $L_{54}$ respectively. These columns are apparently new. For the case $L_{32}$, see Reference 9. We can use these columns to (a) locate interaction effects, and/or (b) accommodate new factors.

### (*a*) *Using the supplementary orthogonal columns as interaction columns*

The supplementary orthogonal columns represent certain types of interaction. Specifically, the two supplementary orthogonal columns $a$ and $b$ represent the interaction between factors **1** and **2** (**1** $\times$ **2** interaction) in both the $L_{18}$ and $L_{54}$ arrays. Factor **1** is two-level and factor **2** is three-level; thus the **1** $\times$ **2** interaction carries two degrees of freedom, represented by the two columns in Table IV. For the $L'_{36}$ array, the supplementary columns $a$ and $b$ represent the **1** $\times$ **4** interaction; $c$ and $d$ represent the **2** $\times$ **4** interaction; and $e$ and $f$ represent the **3** $\times$ **4** interactions (each with two degrees of freedom). In fact, these interactions are also confounded with other two-factor interactions not listed.

Table II. The ANOVA table for the data in Table I

| Parameters | Estimated effects | Degree of freedom | Sum of squares | Mean squares | $F$ ratio* | $F$ ratio† |
|---|---|---|---|---|---|---|
| $A$ | 6·332 | 1 | 641·52 | 641·52 | 12·24 | 173·38 |
| $B$ | 2·355 | 1 | 88·70 | 88·70 | 1·69 | 23·97 |
| $AB$ | 1·949 | 1 | 60·76 | 60·76 | 1·16 | 16·42 |
| $C$ | 0·261 | 1 | 1·09 | 1·09 | 0·02 | 0·29 |
| $AC$ | 0·731 | 1 | 8·55 | 8·55 | 0·16 | 2·31 |
| $D$ | −0·210 | 1 | 0·71 | 0·71 | 0·01 | 0·19 |
| Error | | 9 | 471·86 | 52·43 | | |
| $e$ | 5·250 | 1 | 442·23 | 442·23 | | 119·52 |
| Pure error | | 8 | 29·63 | 3·70 | | |
| Total | | 15 | 1273·18 | | | |

*using $s^2$ = 52·34 †using $s^2$ = 3·70

Table III. Unsaturated orthogonal arrays

| Design | No. of rows | No. of factors | Max no. of columns at these levels | | | | Unused degrees of freedom |
|---|---|---|---|---|---|---|---|
| | | | 2 | 3 | 4 | 5 | |
| $L_{18}(2^1 \times 3^7)$ | 18 | 8 | 1 | 7 | — | — | 2 |
| $L'_{32}(2^1 \times 4^9)$ | 32 | 10 | 1 | — | 9 | — | 3 |
| $L'_{36}(2^3 \times 3^{13})$ | 36 | 16 | 3 | 13 | — | — | 6 |
| $L_{50}(2^1 \times 5^{11})$ | 50 | 12 | 1 | — | — | 11 | 4 |
| $L_{54}(2^1 \times 3^{25})$ | 54 | 26 | 1 | 25 | — | — | 2 |

Table IV. The supplementary orthogonal columns for $L_{18}$

| Run | *a* | *b* |
|---|---|---|
| 1 | 0 | −2/3 |
| 2 | 0 | −2/3 |
| 3 | 0 | −2/3 |
| 4 | 1 | 1/3 |
| 5 | 1 | 1/3 |
| 6 | 1 | 1/3 |
| 7 | −1 | 1/3 |
| 8 | −1 | 1/3 |
| 9 | −1 | 1/3 |
| 10 | 0 | 2/3 |
| 11 | 0 | 2/3 |
| 12 | 0 | 2/3 |
| 13 | −1 | −1/3 |
| 14 | −1 | −1/3 |
| 15 | −1 | −1/3 |
| 16 | 1 | −1/3 |
| 17 | 1 | −1/3 |
| 18 | 1 | −1/3 |

Table V. The supplementary orthogonal columns for $L'_{36}$

| Run | *a* | *b* | *c* | *d* | *e* | *f* |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1/3 | 0 | 1/3 |
| 2 | 1 | 1 | 1 | 1/3 | 0 | 1/3 |
| 3 | 1 | 1 | 1 | 1/3 | 0 | 1/3 |
| 4 | 1 | −1 | −1 | 1/3 | 0 | −1/3 |
| 5 | 1 | −1 | −1 | 1/3 | 0 | −1/3 |
| 6 | 1 | −1 | −1 | 1/3 | 0 | −1/3 |
| 7 | −1 | 1 | −1 | −1/3 | −1/3 | 0 |
| 8 | −1 | 1 | −1 | −1/3 | −1/3 | 0 |
| 9 | −1 | 1 | −1 | −1/3 | −1/3 | 0 |
| 10 | −1 | −1 | 1 | −1/3 | 1/3 | 0 |
| 11 | −1 | −1 | 1 | −1/3 | 1/3 | 0 |
| 12 | −1 | −1 | 1 | −1/3 | 1/3 | 0 |
| 13 | 0 | 0 | 0 | −2/3 | 0 | −2/3 |
| 14 | 0 | 0 | 0 | −2/3 | 0 | −2/3 |
| 15 | 0 | 0 | 0 | −2/3 | 0 | −2/3 |
| 16 | 0 | 0 | 0 | −2/3 | 0 | 2/3 |
| 17 | 0 | 0 | 0 | −2/3 | 0 | 2/3 |
| 18 | 0 | 0 | 0 | −2/3 | 0 | 2/3 |
| 19 | 0 | 0 | 0 | 2/3 | 2/3 | 0 |
| 20 | 0 | 0 | 0 | 2/3 | 2/3 | 0 |
| 21 | 0 | 0 | 0 | 2/3 | 2/3 | 0 |
| 22 | 0 | 0 | 0 | 2/3 | −2/3 | 0 |
| 23 | 0 | 0 | 0 | 2/3 | −2/3 | 0 |
| 24 | 0 | 0 | 0 | 2/3 | −2/3 | 0 |
| 25 | −1 | −1 | −1 | 1/3 | 0 | 1/3 |
| 26 | −1 | −1 | −1 | 1/3 | 0 | 1/3 |
| 27 | −1 | −1 | −1 | 1/3 | 0 | 1/3 |
| 28 | −1 | 1 | 1 | 1/3 | 0 | −1/3 |
| 29 | −1 | 1 | 1 | 1/3 | 0 | −1/3 |
| 30 | −1 | 1 | 1 | 1/3 | 0 | −1/3 |
| 31 | 1 | −1 | 1 | −1/3 | −1/3 | 0 |
| 32 | 1 | −1 | 1 | −1/3 | −1/3 | 0 |
| 33 | 1 | −1 | 1 | −1/3 | −1/3 | 0 |
| 34 | 1 | 1 | −1 | −1/3 | 1/3 | 0 |
| 35 | 1 | 1 | −1 | −1/3 | 1/3 | 0 |
| 36 | 1 | 1 | −1 | −1/3 | 1/3 | 0 |

### (*b*) *Using the supplementary orthogonal columns to accommodate new factors*

If the assumption of 'no interactions' is made, the extra columns can be used to accommodate new factors. In Table IV, for example, for any quantitative three-level factor, if only the linear effect is expected to be non-zero, then column *a* can be used to accommodate a new factor. Thus, the original $L_{18}(2^1 \times 3^7)$ can be extended to $L_{18}(2^1 \times 3^7 \times \mathbf{3}^1)$, when the quadratic effect of the added column is negligible (it is partially confounded with the column **2** of the original $L_{18}$ array). Note that only a linear effect in under consideration for the new factor, namely, the difference between level 1 and level 2 is the same as that between level 2 and level 3.

Similarly, we can accommodate three, two, and one more factors in the original $L'_{36}$, $L_{50}$ and $L_{54}$ arrays, respectively (e.g. columns *a*, *b*, *c* in Table V; columns *a*, *c* in Table VI; and column *a* in Table VII). *These factors must be quantitative factors, and only their linear effects are of interest.* It was argued by Taguchi (Reference 1, chapter 12) that the linear effect usually dominates the factor effect (from quadratic effect, cubic effect,. . .etc). The so-called 'main effect principle' is based upon the empirical observation that linear main effects are more important than higher order effects. Other columns in Tables IV–VII remain to be interaction columns; their non-equal occurrence property prohibits the use of assigning new factors. (If the equal occurrence property is not a concern, then all columns in Tables IV–VII can be used to accommodate new factors.)

Table VI. The supplementary orthogonal columns for $L_{50}$

| Run | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| 1 | 0 | 2 | 0 | −6 |
| 2 | 0 | 2 | 0 | −6 |
| 3 | 0 | 2 | 0 | −6 |
| 4 | 0 | 2 | 0 | −6 |
| 5 | 0 | 2 | 0 | −6 |
| 6 | 2 | −2 | 1 | −1 |
| 7 | 2 | −2 | 1 | −1 |
| 8 | 2 | −2 | 1 | −1 |
| 9 | 2 | −2 | 1 | −1 |
| 10 | 2 | −2 | 1 | −1 |
| 11 | 1 | 1 | −2 | 4 |
| 12 | 1 | 1 | −2 | 4 |
| 13 | 1 | 1 | −2 | 4 |
| 14 | 1 | 1 | −2 | 4 |
| 15 | 1 | 1 | −2 | 4 |
| 16 | −1 | 1 | 2 | 4 |
| 17 | −1 | 1 | 2 | 4 |
| 18 | −1 | 1 | 2 | 4 |
| 19 | −1 | 1 | 2 | 4 |
| 20 | −1 | 1 | 2 | 4 |
| 21 | −2 | −2 | −1 | −1 |
| 22 | −2 | −2 | −1 | −1 |
| 23 | −2 | −2 | −1 | −1 |
| 24 | −2 | −2 | −1 | −1 |
| 25 | −2 | −2 | −1 | −1 |
| 26 | 0 | −2 | 0 | 6 |
| 27 | 0 | −2 | 0 | 6 |
| 28 | 0 | −2 | 0 | 6 |
| 29 | 0 | −2 | 0 | 6 |
| 30 | 0 | −2 | 0 | 6 |
| 31 | −2 | 2 | −1 | 1 |
| 32 | −2 | 2 | −1 | 1 |
| 33 | −2 | 2 | −1 | 1 |
| 34 | −2 | 2 | −1 | 1 |
| 35 | −2 | 2 | −1 | 1 |
| 36 | −1 | −1 | 2 | −4 |
| 37 | −1 | −1 | 2 | −4 |
| 38 | −1 | −1 | 2 | −4 |
| 39 | −1 | −1 | 2 | −4 |
| 40 | −1 | −1 | 2 | −4 |
| 41 | 1 | −1 | −2 | −4 |
| 42 | 1 | −1 | −2 | −4 |
| 43 | 1 | −1 | −2 | −4 |
| 44 | 1 | −1 | −2 | −4 |
| 45 | 1 | −1 | −2 | −4 |
| 46 | 2 | 2 | 1 | 1 |
| 47 | 2 | 2 | 1 | 1 |
| 48 | 2 | 2 | 1 | 1 |
| 49 | 2 | 2 | 1 | 1 |
| 50 | 2 | 2 | 1 | 1 |

Table VII. The supplementary orthogonal columns for $L_{54}$

| Run | $a$ | $b$ |
|---|---|---|
| 1 | 0 | −2/3 |
| 2 | 0 | −2/3 |
| 3 | 0 | −2/3 |
| 4 | 0 | −2/3 |
| 5 | 0 | −2/3 |
| 6 | 0 | −2/3 |
| 7 | 0 | −2/3 |
| 8 | 0 | −2/3 |
| 9 | 0 | −2/3 |
| 10 | 1 | 1/3 |
| 11 | 1 | 1/3 |
| 12 | 1 | 1/3 |
| 13 | 1 | 1/3 |
| 14 | 1 | 1/3 |
| 15 | 1 | 1/3 |
| 16 | 1 | 1/3 |
| 17 | 1 | 1/3 |
| 18 | 1 | 1/3 |
| 19 | 1 | 1/3 |
| 20 | 1 | 1/3 |
| 21 | 1 | 1/3 |
| 22 | 1 | 1/3 |
| 23 | 1 | 1/3 |
| 24 | 1 | 1/3 |
| 25 | 1 | 1/3 |
| 26 | 1 | 1/3 |
| 27 | 1 | 1/3 |
| 28 | 0 | 2/3 |
| 29 | 0 | 2/3 |
| 30 | 0 | 2/3 |
| 31 | 0 | 2/3 |
| 32 | 0 | 2/3 |
| 33 | 0 | 2/3 |
| 34 | 0 | 2/3 |
| 35 | 0 | 2/3 |
| 36 | 0 | 2/3 |
| 37 | −1 | −1/3 |
| 38 | −1 | −1/3 |
| 39 | −1 | −1/3 |
| 40 | −1 | −1/3 |
| 41 | −1 | −1/3 |
| 42 | −1 | −1/3 |
| 43 | −1 | −1/3 |
| 44 | −1 | −1/3 |
| 45 | −1 | −1/3 |
| 46 | −1 | −1/3 |
| 47 | −1 | −1/3 |
| 48 | −1 | −1/3 |
| 49 | −1 | −1/3 |
| 50 | −1 | −1/3 |
| 51 | −1 | −1/3 |
| 52 | −1 | −1/3 |
| 53 | −1 | −1/3 |
| 54 | −1 | −1/3 |

*Note*

The ideas are also applicable to other designs that are orthogonal arrays, not only the Taguchi designs mentioned above.

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