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SYSTEM LIFE DATA ANALYSIS WITH DEPENDENT PARTIAL KNOWLEDGE ON THE EXACT CAUSE OF SYSTEM FAILURE

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Abstract: Because of cost and time factors the exact cause of system failure may be known only partially. For example, the cause is narrowed down to a component in a subsystem or a smaller set of components. This is called "masking" of the exact failure mode. Our paper focuses on reliability estimation when the masking probability is dependent on the particular cause of failure.

1. INTRODUCTION

In field data and prototype data of system life the exact cause may be unknown because of cost and time factors. The cause is narrowed down to a component in a subsystem or a smaller set of components, but is not precisely determined. For engineering examples, see Usher [1] , Miyakawa [2], Usher and Hodgson [3] , and Guess, Usher and Hodgson [4,5]. For biological examples compare Dinse [6,7]and Gross [8]. This has been called masking of the cause of failure.

References $1-5$ need the probability of masking with a certain subset of components to be independent of the cause of failure and discuss several situations where this is realistic. See our Section 2 for more technical details. References 1 and $3-5$ warn to check for this type of independence of masking.

This paper discusses analysis of such system and component life data when the masking probabilities are dependent upon the component in the masking set. We suggest a simple way of checking for this independence or dependence via subsampling, while minimizing the total cost of early prototype and later field life testing of systems. Section 2 sets up basic notation and the likelihood for this dependent data. Section 3 presents proportional dependent masking and illustrates how far off the maximum likelihood estimates (MLE's) of failure rates assuming independent masking can be from the correct MLE's under various degrees of dependent masking. Conclusions are stated in Section 4.

2. DEPENDENT MASKING AND LIKELIHOOD

Following the notation of Guess, Usher, and Hodgson [4,5] , we let T_i be the random life for the i^{th} system where $i=1,\ldots,n$ in a sample of n systems each consisting of J components in series. Let T_{ij} be the random life of the j^{th} component in the i^{th} system where $j=1,\ldots J$. Note that

$$T_i = \min(T_{i1}, \ldots, T_{iJ})$$

for $i=1,\ldots, n$. We assume that the T_{ij}'s are independent. (Aside: dependence among the components lifetimes could be modeled using a dependent multivariate distribution and a competing risk model.

See, for example, References 9 – 13.) For each fixed j the $T_{1j}, \ldots, T_{nj}$ would represent a sample of size n from component j's life distribution F_j.

We require the very mild condition that F_j has a density f_j indexed by a parameter vector $\underline{\theta}_j$. For each j, a different number of parameters in $\underline{\theta}_j$ is allowed if needed. Let $\overline{F}_j(t) = 1 - F_j(t)$ be the reliability of component j at time t. Let K_i be the index of the component causing the failure of system i. Due to the life distributions being continuous the cause of failure K_i is unique. Note that K_i is a random variable and that K_i may or may not be observed. That is, the component causing system failure may be masked. Before the sample is taken there is the <u>minimum</u> <u>random</u> <u>subset</u>, M_i, of components known to contain the true cause of failure of system i. In short, $K_i \in M_i$ and M_i is minimum.

After the sample data is obtained, we have $M_i = S_i \subset \{1,2,\ldots,J\}$ and $T_i = t_i$ where $i=1,\ldots,n$. Note that as t_i is the realized sample value of T_i, so is S_i the realized sample set of M_i. If $S_i = \{j\}$ then we know that $K_i = j$, and hence, the cause of failure is not masked. If, for example, $S_i = \{1,2\}$, we have that $K_i \in S_i$ but the true value of K_i is masked. The observed data here is

$$(t_1, S_1), \ldots, (t_n, S_n).$$

As derived in Guess, Usher, and Hodgson [4,5], the full likelihood for this data is

$$L_F = \prod_{i=1}^{n} \left\{ \sum_{j \in S_i} \left(f_j(t_i) \prod_{\substack{s=1 \\ s \neq j}}^{J} \overline{F}_s(t_i) \cdot P(M_i=S_i \mid T_i=t_i, K_i=j) \right) \right\}.$$

Note that the term $f_j(t_i) \prod_{\substack{s=1 \\ s \neq j}}^{J} \overline{F}_s(t_i)$ is from system i failing at time t_i due to cause j (i.e., due to component j).

The expression $P(M_i=S_i \mid T_i=t_i, K_i=j)$ represents the conditional probability that the observed minimum random subset is S_i given that system i failed at time t_i and the true cause was component j. For $S_i = \{j\}$, this expression is the conditional probability that the cause of failure is known. For S_i containing more than j, it yields the conditional probability of masking with the set S_i. It should also be noted for $J = 2$, the masking probabilities are similar to Dinse's [7] uncertainty rates. For $J > 2$, however, the masking probabilities generalize his uncertainty rates. Also, note that our assumption (2.1) relates to his excellent discussion of equal uncertainty rates in studying diseases.

For the observation (t_i, S_i) we sum over all possible failure causes j in S_i. The product is then over each of the observations to yield the full likelihood L_F.

For the industrial problems in References 1 – 5, they found their masking typically occurred due to constraints of time and the expense of failure analysis. Schedules often dictated that complete failure analysis (to determine the true cause of failure) be curtailed. In such settings they found that for j' fixed and in S_i that

$$P(M_i=S_i \mid T_i=t_i, K_i=j') = P(M_i=S_i \mid T_i=t_i, K_i=j) \quad \text{for all } j \in S_i. \tag{2.1}$$

We call these masking probabilities independent over the causes $j \in S_i$. When for any $j' \in S_i$

$$P(M_i=S_i \mid T_i=t_i, K_i=j') \neq P(M_i=S_i \mid T_i=t_i, K_i=j) \quad \text{for some } j \in S_i. \tag{2.1'}$$

the masking probabilities are dependent in S_i.

Note that the masking probabilities can be a function of time. We assume only that the masking probabilities conditional on time are not functions of the life distribution parameters. For future reference , we state this as

$$P(M_i=S_i \mid T_i=t_i,\ K_i=j) \text{ does not depend on the life distribution parameters.} \tag{2.2}$$

This is analogous to a censoring distribution not depending on the life distribution parameters. Compare Miller [14] and Lawless [15].

As a result of (2.1) and (2.2), the reduced or proper partial likelihood is

$$L_R = \prod_{i=1}^{n} \left\{ \sum_{j \epsilon S_i} \Big(f_j\,(t_i) \prod_{\substack{s=1 \\ s \neq j}}^{J} \bar{F}_s\,(t_i) \Big) \right\}.$$

Under (2.1) and (2.2), maximizing L_R with respect to the life parameters is equivalent to using L_F . This is similar to the usual derivation of a time censored (and not masked) data partial likelihood.

The full likelihood from Guess, Usher, and Hodgson [4,5] clarifies and extends Miyakawa's [2] likelihood. They point out that it is best to view his likelihood (as well as L_R above) as a partial likelihood that under appropriate conditions will yield good, consistent estimators. Under (2.1′) such a partial likelihood can yield inconsistent estimators. For proper statistical applications, they stress that it is important to be clearly aware of the effects of masking probabilities and checking for needed conditions. For further details on that and to easily allow for system lifetime censoring, see Guess, Usher, Hodgson [4,5]. In the next section we discuss (2.1′) further and a type of dependent masking.

3. PROPORTIONAL DEPENDENT MASKING AND EXAMPLES

A natural setting where (2.1′) would occur instead of (2.1) is for high heat generating components that sometimes sear a subsystem (module) black and a repair scheme of replacing the entire subsystem (module) whenever it is blackened. For example, suppose a module consist of two components 1 and 2 such that due to the size and heat generation of component 2 it is twice as likely at anytime to sear the entire module black as component 1. Here we have

$$2\,P(M_i=\{1,2\} \mid T_i=t_i,\ K_i=1) \;=\; P(M_i=\{1,2\} \mid T_i=t_i,\ K_i=2) \;\neq\; P(M_i=\{1,2\} \mid T_i=t_i,\ K_i=1)$$

and (2.1′) applies with it dependent in $S_i = \{1,2\}$ masking. More generally we have proportional probabilities for $j', j \,\epsilon\, S_i$ and $j' \neq j$ when

$$P(M_i=S_i \mid T_i=t_i,\ K_i=j') \;=\; c\,P(M_i=S_i \mid T_i=t_i,\ K_i=j) \quad , \tag{3.1}$$

where $c \geq 0$. The c is implicitly a function of j' and j . Note that for $c \neq 1$ there is the dependent masking over S_i of (2.1′) , while for $c = 1$ the special case of independent masking over S_i of (2.1) holds. For $c = 0$ and $P(M_i=S_i \mid T_i=t_i,\ K_i=j) > 0$ a very extreme form of dependent masking would occur.

Without loss of generality we can assume $c \leq 1$ by simply dividing by c and relabeling in (3.1). E. g., in the case of the two component module c was 2, but it is easy to divide and relabel to get the new c is $\frac{1}{2}$.

With (3.1) we illustrate the effects of various degrees of dependent masking on the MLE's of the failure rate for exponentially distributes components in a series of J components. For simplicity as in Usher and Guess [16] and Miyakawa [2], we let $J = 2$. Next set

$$\begin{aligned} p &= P(M_i = \{1,2\} \mid T_i = t_i, K_i = 1) \\ c\,p &= P(M_i = \{1,2\} \mid T_i = t_i, K_i = 2) \\ p_1 &= P(M_i = \{1\} \mid T_i = t_i, K_i = 1) \\ p_2 &= P(M_i = \{2\} \mid T_i = t_i, K_i = 2) \end{aligned}$$

and let

$$f_j(t) = \lambda_j e^{-\lambda_j t} \quad \text{and} \quad \overline{F}_j(t) = e^{-\lambda_j t}, \quad \text{for } t \geq 0, \text{ and } j = 1,2 .$$

Now we rewrite the full likelihood as

$$\begin{aligned} L_F = & \prod_{S_i=\{1\}} \left(\lambda_1 e^{-(\lambda_1 + \lambda_2) t_i} p_1 \right) \prod_{S_i=\{2\}} \left(\lambda_2 e^{-(\lambda_1 + \lambda_2) t_i} p_2 \right) \\ & \times \prod_{S_i=\{1,2\}} \left(\left(\lambda_1 p + \lambda_2 c\,p \right) e^{-(\lambda_1 + \lambda_2) t_i} \right) . \end{aligned}$$

Even though (2.1′) holds, because of (2.2) maximizing L_F for λ_1 and λ_2 is equivalent to maximizing the partial likelihood

$$\begin{aligned} L_{\mathcal{P}}(c) = & \prod_{S_i=\{1\}} \left(\lambda_1 e^{-(\lambda_1 + \lambda_2) t_i} \right) \prod_{S_i=\{2\}} \left(\lambda_2 e^{-(\lambda_1 + \lambda_2) t_i} \right) \\ & \times \prod_{S_i=\{1,2\}} \left(\left(\lambda_1 + \lambda_2 c \right) e^{-(\lambda_1 + \lambda_2) t_i} \right) . \end{aligned}$$

Note that $L_{\mathcal{P}}(c) = L_R$ for $c = 1$, but for $c \neq 1$ $L_{\mathcal{P}}(1) \neq L_R$. This enables us to compare how far off the MLE's of failure rates assuming independent masking over S_i can be from the correct MLE's under various degrees of dependent masking as indexed by c.

When $c = 1$ maximizing with respect to L_F, L_R, or $L_{\mathcal{P}}$ yield the same estimates. We stress for $c \neq 1$, however, that maximizing with respect to L_R is <u>not</u> equivalent to maximizing the L_F, but maximizing with respect to $L_{\mathcal{P}}$ will be equivalent to optimizing the L_F.

First we write the log likelihood equations below

$$\partial \log L_{\mathcal{P}}(c) \,/\, \partial \lambda_1 = n_1 / \lambda_1 + n_{12} / (\lambda_1 + \lambda_2 c) - \sum_{i=1}^{n} t_i \tag{3.2a}$$

$$\partial \log L_{\mathcal{P}}(c) \,/\, \partial \lambda_2 = n_2 / \lambda_2 + (n_{12} / (\lambda_1 + \lambda_2 c))\, c - \sum_{i=1}^{n} t_i \tag{3.2b}$$

where n_1, n_2 denotes the number of failures with $M_i = \{1\}$, $M_i = \{2\}$ respectively and n_{12} denotes the count of $M_i=\{1,2\}$, i.e., n_{12} is the number of masked failures here. For $c = 1$ they are equivalent to the special cases in Miyakawa [2] and Usher and Hodgson [3] .

Solving for (3.2 a and b), we find the MLE's under (2.1′) and (2.2) are

$$\hat{\lambda}_1 = \left\{ -cn_2 + (1-c)\,n_{12} + (1-2c)\,n_1 + \sqrt{\left(cn_2 - (1-c)\,n_{12} + n_1\right)^2 + 4\,(1-c)\,n_1\,n_{12}} \right\} / \,[2\,(1-c)\,T]$$

and

$$\hat{\lambda}_2 = \left\{ n_1 + (1-c)\,n_{12} + (2-c)\,n_2 - \sqrt{\left(c(n_2 + n_{12}) - (n_1 + n_{12})\right)^2 + 4\,c\,n_1\,n_2} \right\} / \,[2\,(1-c)\,T] \quad ,$$

where $T = \sum_{i=1}^{n} t_i$

is the total time on test. (For another quick check, note that these estimates for the case of $c = 1$ reduces to Miyakawa's special results.)

The relationships of these estimators to the dependency variable c are best understood graphically (for various values of n_1, n_2, and n_{12}) in Figures 1.1 – 1.4 and 2.1 – 2.4. Recall the case in (3.1) of $c = 1$ yields MLE's of the independent condition (2.1). Under (3.1) as $c \rightarrow 0$, then a strong dependency in S_i masking exists (i. e., (2.1′) holds strongly). For example, when the $c = 0$ extreme case occurs, then all the masking of $S_i = \{1, 2\}$ would be due to $K_i = 1$, never due to $K_i = 2$. For convenience we have rescaled the total time on test to be $T = 1.0$ units of time. This simply rescales the vertical axis in each Figure.

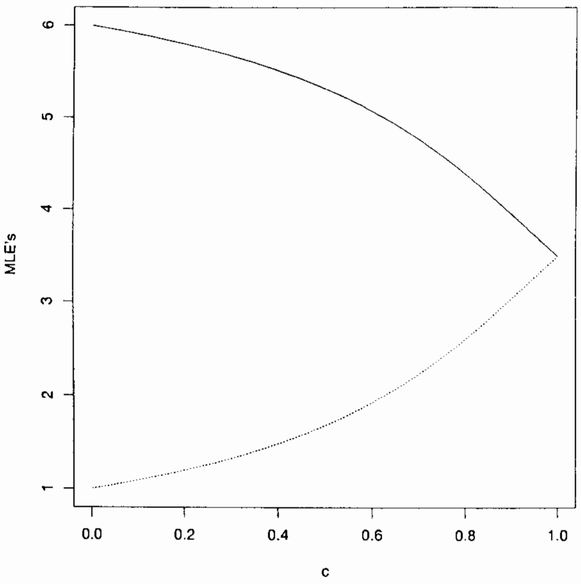

Figure 1.1 $(n_1, n_2, n_{12}) = (1, 1, 5)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

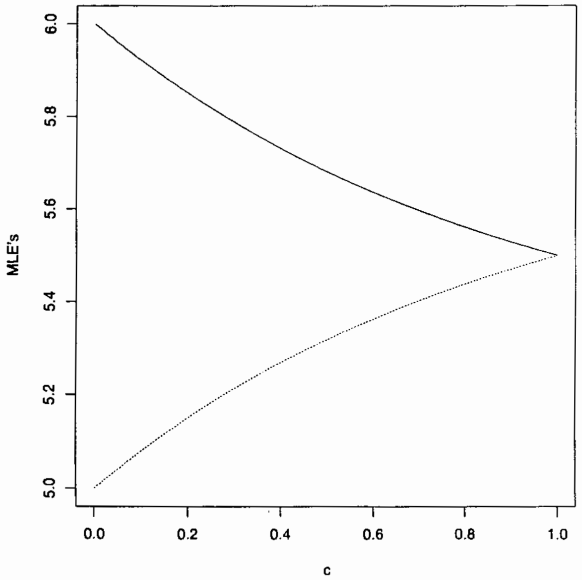

Figure 1.2 $(n_1, n_2, n_{12}) = (5, 5, 1)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

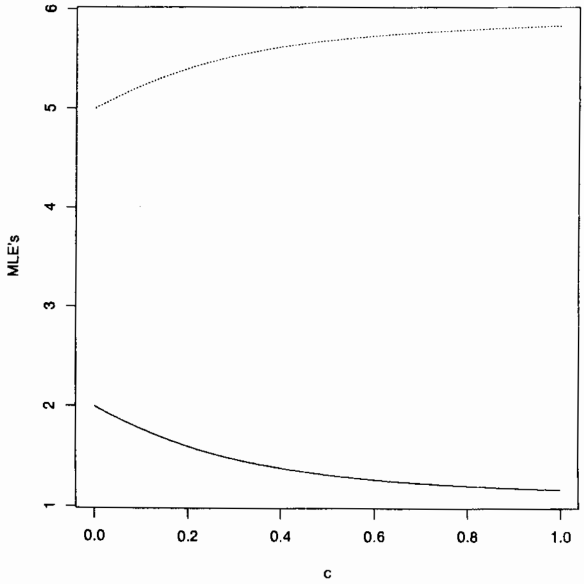

Figure 1.3 $(n_1, n_2, n_{12}) = (1, 5, 1)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

Note in particular for Figure 2.1 that the true MLE's of the failure rates actually flip as c varies from 1 down to 0. Intuitively this makes sense due to the larger number of masking $n_{12} = 9$ being allocated according to the dependency number c. This reemphasizes the warnings in References 1 and 3 – 5 about possible violations of (2.1).

In this paper we have introduced (3.1) and c to study when violation of (2.1) is minor and when it might be a significant problem. In Figures 1.4, 2.3, and 2.4 the MLE's are seen to change very little as a function of c. This robustness to c is also true for Figure 1.2 (we have deliberately blown up the vertical scale in Figure 1.2, since the MLE's vary only between 5 and 6). The MLE's are

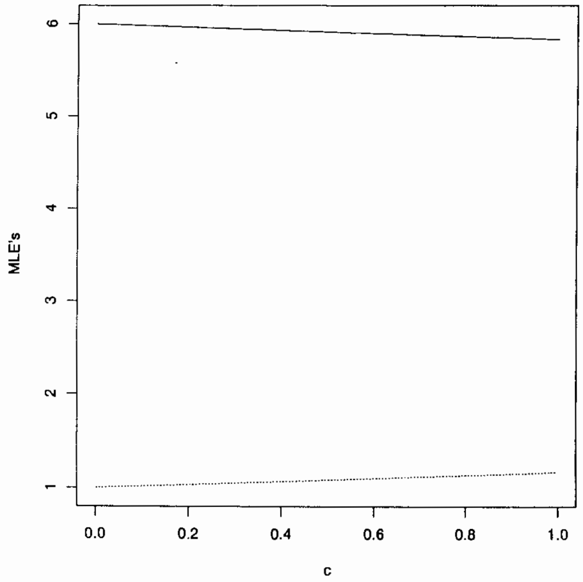

Figure 1.4 $(n_1, n_2, n_{12}) = (5, 1, 1)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

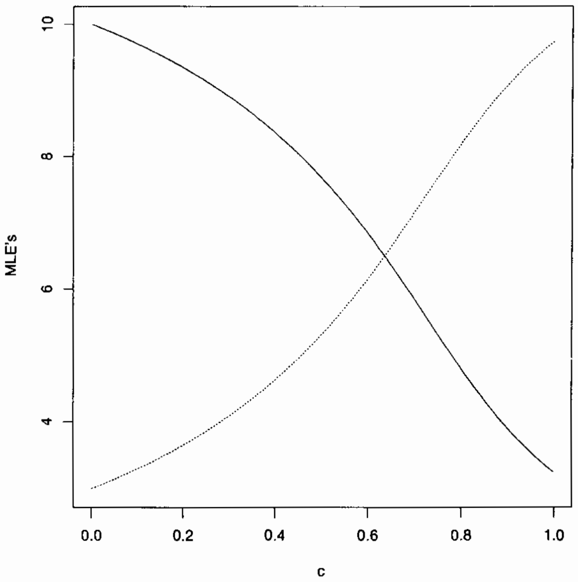

Figure 2.1 $(n_1, n_2, n_{12}) = (1, 3, 9)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

also relatively robust in Figure 1.3 although somewhat less robust in Figure 2.2. The Figure 2.1, however, clearly demonstrates the possible dangerous lack of robustness when dependence in S_i holds.

4. CONCLUSIONS

The graphs and discussion in Section 3 stress the importance for a user verification of (2.1) when masking is relatively large and not caused solely by scheduling and production constraints or else a sensitive analysis of c via (3.1). One way to check (2.1) and hold down costs is to take a

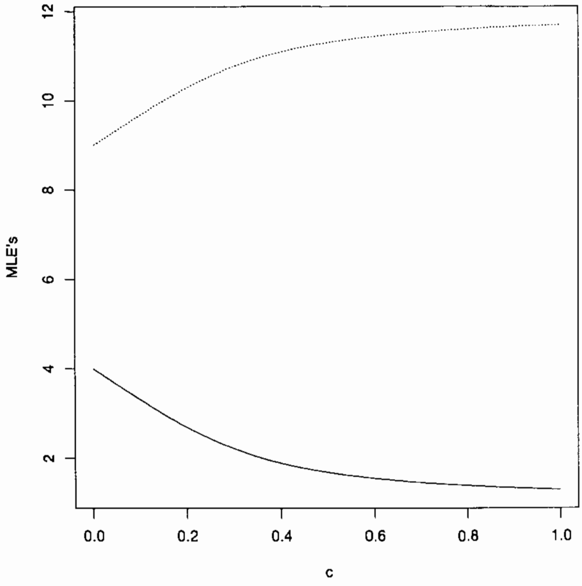

Figure 2.2 $(n_1, n_2, n_{12}) = (1, 9, 3)$ and $\widehat{\lambda}_1$, ____ ; $\widehat{\lambda}_2$, …

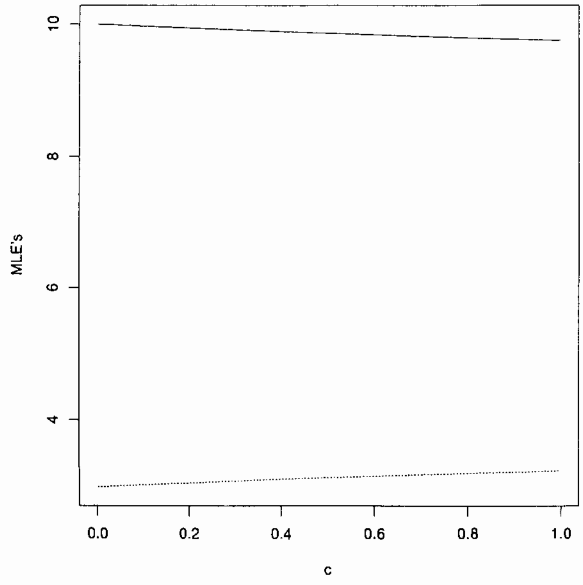

Figure 2.3 $(n_1, n_2, n_{12}) = (9, 3, 1)$ and $\widehat{\lambda}_1$, ____ ; $\widehat{\lambda}_2$, …

subsample of masked failures when the number of maskings is large relative to the number of observed unmasked failures.

When the number of maskings is small relative to the numbers of observed failures these MLE's will often be relatively robust. Using (3.1) and c provides a user a way of quantifying how much variability in the MLE's could arise due to unaccounted dependency in S_i masking.

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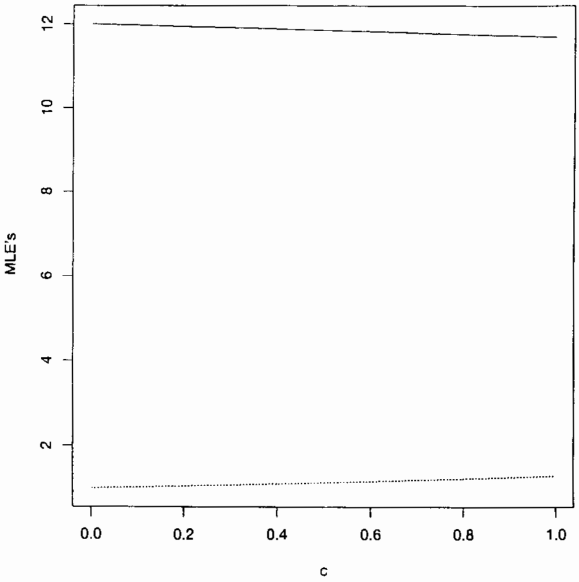

Figure 2.4 $(n_1, n_2, n_{12}) = (9, 1, 3)$ and $\hat{\lambda}_1$, ___ ; $\hat{\lambda}_2$, ...

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