Computational Statistics & Data Analysis 15 (1993) 147–157
North-Holland

Generating alias relationships for two-level Plackett and Burman designs

Dennis K.J. Lin
University of Tennessee, Knoxville, TN, USA

Norman R. Draper
University of Wisconsin, Madison, WI, USA

Received February 1991
Revised January 1992

Abstract: When the number of runs N in a Plackett and Burman design is a power of two, the design is a 2_{III}^{k-p} fractional factorial design, and the alias relationships are easily obtained. When N is a multiple of four but not a power of two, the alias relationships are extremely complicated. When only a few factors are expected to be relevant, and if all interactions involving three or more factors are tentatively assumed to be zero, knowledge of the alias relationships is valuable. It is then often possible to disentangle, either completely or partially, the main effects and two-factor interactions of those factors that appear to be of most importance in the initial analysis. In this article, we consider cases for $N \leq 100$ and explain how to sequentially construct the alias table for Plackett and Burman designs generated by cyclic generation and foldover, as well as the $N = 28$ run design generated by block permutation. (This excludes only the $N = 52$, 76 and 100 designs generated via block permutation and the $N = 92$ run design, which is of a special construction type.) Applications of these tables are briefly discussed.

Keywords: Alias table; Foldover; Interaction; Orthogonal array; Projection; Two-factor interactions.

1. Introduction

Consider a two-level factorial design with coded levels of $+1$ and -1 for each factor. Plackett and Burman (1946, pp. 323–324) provided orthogonal designs for N equal to a multiple of four where $N \leq 100$ and $N \neq 92$. There are three basic methods for constructing these Plackett and Burman designs:

Correspondence to: Norman R. Draper, Ph.D., University of Wisconsin, Department of Statistics, 1210 West Dayton Street, Madison, WI 53706-1393, USA.

Table 1
The 12-run Plackett and Burman design

Run	1	2	3	4	5	6	7	8	9	10	11	Observation
1	+	+	−	+	+	+	−	−	−	+	−	y_1
2	+	−	+	+	+	−	−	−	+	−	+	y_2
3	−	+	+	+	−	−	−	+	−	+	+	y_3
4	+	+	+	−	−	−	+	−	+	+	−	y_4
5	+	+	−	−	−	+	−	+	+	−	+	y_5
6	+	−	−	−	+	−	+	+	−	+	+	y_6
7	−	−	−	+	−	+	+	−	+	+	+	y_7
8	−	−	+	−	+	+	−	+	+	+	−	y_8
9	−	+	−	+	+	−	+	+	+	−	−	y_9
10	+	−	+	+	−	+	+	+	−	−	−	y_{10}
11	−	+	+	−	+	+	+	−	−	−	+	y_{11}
12	−	−	−	−	−	−	−	−	−	−	−	y_{12}

(1) *Cyclic Generation*. Take a (specific) row of $N-1$ plus and minus signs, provided by Plackett and Burman (1946, pp. 323–324). Construct $N-2$ further rows by cyclicly permuting the signs in the first row. Add a row of all minus signs. This gives N rows (= runs) of ± 1 levels for $N-1$ variables or factors (= columns). The $N=12$ case, shown in Table 1, is developed in this manner. So are the designs for $N=8$, 16, 20, 24, 32, 36, 44, 48, 60, 68, 72, 80, and 84.

(2) *Foldover*. A block of plus and minus signs which we denote by $\boldsymbol{D}$ is given. The design is obtained by writing down

$$\begin{bmatrix} \boldsymbol{u} & \boldsymbol{D} & \boldsymbol{D} \\ -\boldsymbol{u} & -\boldsymbol{D} & \boldsymbol{D} \end{bmatrix}, \tag{1.1}$$

where $\boldsymbol{u}$ denotes a unit column of all plus signs. Designs for $N=40$, 56, 64, 88, and 96 are obtained in this manner from those of 20, 28, 32, 44, and 48 runs, respectively. Note that this method can be applied for any N-run design when N is a multiple of eight and an $(\frac{1}{2}N)$-run design is available.

(3) *Block Permutation*. Several square blocks of plus and minus signs are given. Further rows are obtained by cyclic permutation of the blocks. A row of minus signs is then added. Designs for $N=28$, 52, 76 and 100 are of this type.

Note. The special case, $N=92$ was not given by Plackett and Burman (1946) but was discovered by Baumert, Golomb, and Hall (1962). The construction method used was different from those used above, being a modification of Williamson's (1944) method.

Plackett and Burman designs are often regarded as main-effect designs because, if we can assume (at least tentatively) that all interactions between the factors can be ignored, all main effects are estimable. In many circumstances, however, at least some two-factor interactions exist. Interpretation based upon the existence only of main-effects can then be misleading, and one needs to know the confounding pattern linking main-effects and two-factor interactions. This motivates our study below of the alias relationships.

When N is not only a multiple of four but is also a power of two, the Plackett and Burman designs are members of families of 2_{III}^{k-p} designs. The alias relationships in these cases can easily be obtained (see, for example, Box, Hunter and Hunter, 1978, p. 409; also Box and Draper, 1987, p. 154). Alias relationships for other Plackett and Burman designs are, however, much more complicated; see, for example, the comments of Margolin (1968, p. 571). In Section 2, we discuss methods which can be used to generate alias relationship tables for Plackett and Burman designs. Some additional remarks follow in Section 3.

2. Alias matrices

2.1. Estimation and confounding

If all two-factor or higher order interactions are 'negligible' (i.e., we assume tentatively that they are zero), the estimation of main-effects is carried out as follows. For factor $\boldsymbol{i}$, attach to the entries in the y column the signs in the i column and divide the total by the divisor $N/2$, which is the number of plus signs in column i (and in every factor column). For example, for the 12-run case of Table 1, the main-effect of factor **1** is

$$l_1 = (y_1 - y_2 + y_3 - y_4 - y_5 - y_6 + y_7 + y_8 + y_9 - y_{10} + y_{11} - y_{12})/6. \qquad (2.1)$$

The overall mean effect is obtained in a similar fashion using the column $\boldsymbol{u}$ of plusses and the divisor 12 (N, in general). Such estimated effects can also be seen from fitting the model $y_j = \beta_0 + \sum_{i=1}^{11} \beta_i x_{ij} + \epsilon_j$, $j = 1, 2, \ldots, 12$ via the standard least squares calculation $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y} = \frac{1}{12}\boldsymbol{X}'\boldsymbol{y}$, where $\boldsymbol{X}$ is a 12×12 matrix formed by a column of 1's adjoined to the block of ± 1's defined by Table 1, and $\boldsymbol{y}$ is a 12×1 column of y_i's. Note that the main-effect of factor $\boldsymbol{i}$, l_i, is double the value of $\hat{\beta}_i$, the $(i+1)$th element of $\hat{\boldsymbol{\beta}}$. This is because $\hat{\beta}_i$ measures a one unit effect, while l_i measures the two-unit effect from $x_i = -1$ to $x_i = 1$.

Suppose that some two-factor interactions are not zero but that all interactions of order three or more are zero. Then, the quantity (2.1) will estimate the main-effect of factor **1** plus a linear combination of certain two-factor interactions. The confounding patterns can be obtained using the standard alias (or bias) calculations suggested by Box and Wilson (1951). See, also, for example, Box and Draper (1987, p. 154).

Suppose we wish to fit the regression model $\mathrm{E}(\boldsymbol{y}) = \boldsymbol{X\beta}$ by least squares. Provided the model is correct, the estimator $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$ is unbiased, i.e., $\mathrm{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$. If the model is not correct, the estimator is biased; the extent of the bias depends not only on the postulated and the true models but also on the values of the x-variables which enter the regression calculations. If the correct model takes the form

$$\mathrm{E}(\boldsymbol{y}) = \boldsymbol{X\beta} + \boldsymbol{X}_1\boldsymbol{\beta}_1, \qquad (2.2)$$

and thus includes terms $\boldsymbol{X}_1\boldsymbol{\beta}_1$ which we did not consider in our estimation procedure, then it can be shown that

$$\mathrm{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta} + \boldsymbol{A}\boldsymbol{\beta}_1, \tag{2.3}$$

where $\boldsymbol{A} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{X}_1$ is called the *alias matrix* (or *bias matrix*). The individual equations in (2.3) are called the *alias relationships*.

In this article, we are particularly interested in the confounding patterns between main-effects and two-factor interactions, ignoring interactions of higher orders. Our $\boldsymbol{\beta}$ then consists of all the main effects, $\boldsymbol{\beta}_1$ consists of all the two-factor interactions, and the alias matrix $\boldsymbol{A}$ is an $k \times k(k-1)/2$ matrix, where k is the number of factors. We associate main effects $(\mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{k})$ with the rows of the $\boldsymbol{A}$ matrix, and interactions $(\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{3}), \ldots, (\boldsymbol{k}-\mathbf{1}, \boldsymbol{k})$ with the columns. It is convenient to display the transpose $\boldsymbol{A}'$ and call it an *alias table*.

Clearly, the size of $\boldsymbol{A}$ expands rapidly as k increases. Thus, while in principle (2.3) can be evaluated for any Plackett and Burman design, in practice the size of $\boldsymbol{A}$ can make these calculations physically difficult, and inhibit their being performed. It is, however, possible to construct the complete alias tables for all Plackett and Burman designs sequentially, by taking advantage of their special structures. We shall discuss this here for cases $N \le 96$, excepting $N = 52$ and 76 (and the $N = 92$ case which was not given by Plackett and Burman, 1946).

2.2. Alias tables for cyclic generation cases

For designs obtained by cyclic generation, because of the cyclic design structure, it can be shown that the cross-product of the (α, β)-interaction column with column γ is identical to the cross-product of the $(\alpha+1, \beta+1)$-interaction column with column $\gamma+1$. In symbols we can write this, for all $1 \le \alpha < \beta < k$, $1 \le \gamma < k$, as

$$(\alpha, \beta) \otimes \gamma = (\alpha+1, \beta+1) \otimes (\gamma+1),$$

and, when $\gamma = \boldsymbol{k}$, we recycle back to $\gamma + 1 = \mathbf{1}$, that is,

$$(\alpha, \beta) \otimes \boldsymbol{k} = (\alpha+1, \beta+1) \otimes \mathbf{1}.$$

The consequences of this are as follows: If the ith block $\boldsymbol{A}_i$ in the alias table, for $i = 1, 2, \ldots, k-1$, is regarded as partitioned into the form

$$\begin{array}{c|cccc} \boldsymbol{A}_i & \mathbf{1}\ \mathbf{2} \cdots \boldsymbol{i}-1 & \boldsymbol{i} & \boldsymbol{i}+1 \cdots \boldsymbol{k}-1 & \boldsymbol{k} \\ \hline (\boldsymbol{i}, \boldsymbol{i}+\mathbf{1}) & & 0 & & \\ (\boldsymbol{i}, \boldsymbol{i}+\mathbf{2}) & \left[\quad \boldsymbol{G} \quad\right] & 0 & \left[\quad \boldsymbol{H} \quad\right] & \left[\ \boldsymbol{f}\ \right] \\ \vdots & & \vdots & & \\ (\boldsymbol{i}, \boldsymbol{k}-\mathbf{1}) & & 0 & & \\ (\boldsymbol{i}, \boldsymbol{k}) & [\quad \boldsymbol{g}' \quad] & 0 & [\quad \boldsymbol{f}' \quad] & 0 \end{array}, \tag{2.4}$$

then the next block, A_{i+1}, can be obtained by first dropping the last row and then moving the last column to the front, namely,

$$\begin{array}{c|c} A_i+1 & \mathbf{1} \quad \mathbf{2}\cdots \boldsymbol{i-1}\ \boldsymbol{i} \quad \boldsymbol{i+1} \quad \boldsymbol{i+2}\cdots \boldsymbol{k} \\ \hline \begin{array}{c} (\boldsymbol{i+1},\boldsymbol{i+1}) \\ (\boldsymbol{i+1},\boldsymbol{i+3}) \\ \vdots \\ (\boldsymbol{i+1},\boldsymbol{k}) \end{array} & \begin{bmatrix} \\ \boldsymbol{f} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{G} \\ \\ \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array} \begin{bmatrix} \\ \boldsymbol{H} \\ \\ \end{bmatrix} \end{array}. \qquad (2.5)$$

This idea can be repeated block by block for $i = 1, 2, \ldots, (k-1)$. Thus, one can obtain the complete alias table by knowledge of only the first block, A_1. This manipulation can be easily done in a computer. The first block can be generated each time of use, or permanently stored in a file.

Example. For the 12-run design shown in Table 1, the A_1 block is shown in Table 2. The makeup of A_1 in the notation of (2.4) is such that:

(a) $\boldsymbol{G}$ and $\boldsymbol{g}'$ do not exist;

(b) $\boldsymbol{f}' = (-\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$;

(c) $\boldsymbol{H}$ is what is left when the bottom row $\boldsymbol{f}'$, the last column $\boldsymbol{f}$ and the zero at their intersection are all deleted from A_1 in Table 2.

The blocks $A_2, A_3, \ldots, A_{10}$ are constructed sequentially using the rule given above, and are shown separated and marked in the complete alias table in Table 3.

Similar A_1 tables for $N = 20$, 24, and 36 are given in Lin and Draper (1991). In the general case:

(a) For $i = 1$, $\boldsymbol{G}$ and $\boldsymbol{g}'$ do not exist because the column of zeros occupies the first column;

(b) The diagonal elements in $\boldsymbol{H}$, which is a square symmetric matrix of dimension $(k - i - 1)$, are always zeros, because the $(\boldsymbol{i}, \boldsymbol{j})$ interaction is orthogonal to both factors $\boldsymbol{i}$ and $\boldsymbol{j}$;

Table 2
The first block, A_1, of the A' matrix for the 12-run Plackett and Burman design of Table 1

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
(1, 2)	0	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3
(1, 3)	0	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3
(1, 4)	0	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3
(1, 5)	0	1/3	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3
(1, 6)	0	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3
(1, 7)	0	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3	1/3
(1, 8)	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3
(1, 9)	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0	1/3	−1/3
(1, 10)	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3
(1, 11)	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0

Table 3
The complete alias table for the 12-run Plackett and Burman design

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
(1, 2)	0	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3
(1, 3)	0	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3
(1, 4)	0	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3
(1, 5)	0	1/3	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3
(1, 6)	0	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3
(1, 7)	0	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3	1/3
(1, 8)	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3
(1, 9)	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0	1/3	−1/3
(1, 10)	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3
(1, 11)	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0
(2, 3)	−1/3	0	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3
(2, 4)	1/3	0	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3
(2, 5)	1/3	0	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3
(2, 6)	−1/3	0	1/3	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3
(2, 7)	−1/3	0	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3
(2, 8)	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3
(2, 9)	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3
(2, 10)	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0	1/3
(2, 11)	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	−1/3	0
(3, 4)	−1/3	−1/3	0	0	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3
(3, 5)	−1/3	1/3	0	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3	−1/3
(3, 6)	1/3	1/3	0	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3
(3, 7)	−1/3	−1/3	0	1/3	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3
(3, 8)	1/3	−1/3	0	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3
(3, 9)	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3
(3, 10)	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3
(3, 11)	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0
(4, 5)	−1/3	−1/3	−1/3	0	0	−1/3	1/3	1/3	−1/3	−1/3	1/3
(4, 6)	−1/3	−1/3	1/3	0	−1/3	0	−1/3	−1/3	1/3	−1/3	1/3
(4, 7)	−1/3	1/3	1/3	0	1/3	−1/3	0	−1/3	−1/3	−1/3	−1/3
(4, 8)	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3	0	1/3	1/3	−1/3
(4, 9)	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3
(4, 10)	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3
(4, 11)	1/3	−1/3	−1/3	0	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0
(5, 6)	1/3	−1/3	−1/3	−1/3	0	0	−1/3	1/3	1/3	−1/3	−1/3
(5, 7)	1/3	−1/3	−1/3	1/3	0	−1/3	0	−1/3	−1/3	1/3	−1/3
(5, 8)	−1/3	−1/3	1/3	1/3	0	1/3	−1/3	0	−1/3	−1/3	−1/3
(5, 9)	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3	0	1/3	1/3
(5, 10)	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3	1/3	0	−1/3
(5, 11)	−1/3	1/3	−1/3	1/3	0	−1/3	−1/3	−1/3	1/3	−1/3	0
(6, 7)	−1/3	1/3	−1/3	−1/3	−1/3	0	0	−1/3	1/3	1/3	−1/3
(6, 8)	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3	0	−1/3	−1/3	1/3
(6, 9)	−1/3	−1/3	−1/3	1/3	1/3	0	1/3	−1/3	0	−1/3	−1/3
(6, 10)	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3	0	1/3
(6, 11)	−1/3	−1/3	−1/3	1/3	−1/3	0	−1/3	1/3	−1/3	1/3	0

Table 3 (continued)

	1	2	3	4	5	6	7	8	9	10	11
(7, 8)	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0	0	−1/3	1/3	1/3
(7, 9)	1/3	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3	0	−1/3	−1/3
(7, 10)	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0	1/3	−1/3	0	−1/3
(7, 11)	1/3	1/3	−1/3	−1/3	−1/3	−1/3	0	1/3	−1/3	−1/3	0
(8, 9)	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0	0	−1/3	1/3
(8, 10)	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3	0	−1/3
(8, 11)	−1/3	−1/3	−1/3	−1/3	−1/3	1/3	1/3	0	1/3	−1/3	0
(9, 10)	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0	0	−1/3
(9, 11)	−1/3	−1/3	1/3	−1/3	1/3	−1/3	−1/3	1/3	0	−1/3	0
(10, 11)	−1/3	1/3	1/3	−1/3	−1/3	1/3	−1/3	−1/3	−1/3	0	0

(c) This method can also be applied to other cyclic designs such as those in John (1987).

2.3. Alias tables for foldover cases

The designs obtained via foldover are all generated via (1.1) from the cyclicly generated designs just discussed. Because we already have the alias tables for these cyclic designs, we can construct the alias tables for the folded designs in the following manner.

We first recall that each row of the alias table has an (i, j) label, where now $i = 1, 2, \ldots, 2k$, and $j = (i+1), (i+2), \ldots, (2k+1)$ looped within each i. Here, k is the number of columns in $\boldsymbol{D}$ in (1.1), and so the folded design has $(2k+1)$ columns (which we shall renumber 1 through $2k+1$) and $2k+2$ rows. The rows of the alias table can then be specified as follows (we use the notation $\boldsymbol{A}'$ for the complete alias table for design $\boldsymbol{D}$, $\boldsymbol{A}_i$ for its submatrix in block i as given above; $\mathbf{0}$ denotes a column vector of zeros, $\boldsymbol{O}$ denotes a matrix of zeros, and $\boldsymbol{I}$ is an identity matrix, all of appropriate sizes):

(a) Rows of the alias table for $i = 1$, $j = 2, 3, \ldots, (2k+1)$

$$\begin{bmatrix} \mathbf{0} & \boldsymbol{O} & \boldsymbol{I} \\ \mathbf{0} & \boldsymbol{I} & \boldsymbol{O} \end{bmatrix}.$$

We can see this by looking at (1.1). Consider the row of $\boldsymbol{A}$ which is specified by a product of the first column of (1.1) with any of the columns $2, 3, \ldots, (k+1)$ of (1.1). Because of the double negative signs in (1.1), any such column product will provide a column identical to the corresponding column in the $(k+2), (k+3), \ldots, (2k+1)$ columns of (1.1), respectively. Thus the summed triple product of column, 1, j, $(j+k)$ of (1.1), divided by N, will leave a unit entry in $\boldsymbol{A}'$. Otherwise the $\boldsymbol{A}'$ entry will be zero. We can now reverse the roles of columns $[2, 3, \ldots, (k+1)]$ and $[(k+2), (k+3), \ldots, (2k+1)]$ and employ a similar argument.

(b) Rows of the alias table for $i = 2, 3, \ldots, k$, $j = 3, 4, \ldots, (k+1)$

$$[\mathbf{0}\ \boldsymbol{O}\ \boldsymbol{A}_{i-1}].$$

(Because of the renumbering of factors, we write $\boldsymbol{A}_{i-1}$ and not $\boldsymbol{A}_i$ here.) The above result can be argued as follows. Any column product $(\boldsymbol{i}, \boldsymbol{j})$ of the form defined produces two identical half-column portions. When this is converted into a triple product using any of the columns $1, 2, 3, \ldots, (k+1)$ of (1.1), we obtain the result of zero. When converted into a triple product using any of the columns $(k+2), (k+3), \ldots, (2k+1)$, an element of $\boldsymbol{A}_{i-1}$ emerges. An identical argument (with different columns) accounts for (d) below, and a similar argument occurs for (c) below, but with a sign reversed.

(c) Rows of the alias table for $i = 2, 3, \ldots, (k+1)$, $j = (k+2), (k+3), \ldots, (2k+1)$

$$[\mathbf{0}\ \boldsymbol{B}_{i-1}\ \boldsymbol{O}], \qquad \text{where } \boldsymbol{B}_1 = \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \boldsymbol{A}_1 \end{bmatrix},$$

and $\boldsymbol{B}_i$ (for $i = 2, 3, \ldots, (k+1)$) is obtained from $\boldsymbol{B}_{i-1}$ by: (1) moving the last row of $\boldsymbol{B}_i$ to the first row; (2) moving the last column of $\boldsymbol{B}_i$ to the second column, i.e., after the column of zeros.

(d) Rows of the alias table for $i = (k+2), (k+3), \ldots, 2k$, $j = (k+3), (k+4), \ldots, (2k+1)$

$$[\mathbf{0}\ \boldsymbol{O}\ \boldsymbol{A}'].$$

Such an alias table for the 24-run Plackett and Burman design is available in Lin and Draper (1991). (The 24-run design results could also be obtained via cyclic permutation.)

2.4. Alias tables for block permutation cases

The block permutation cases are not as simple as the cyclic generation and foldover cases. Nevertheless, there are interesting patterns to be found. We briefly illustrate with the 28-run case. Very massive alias matrices arise for the larger cases of 52, 76, and 100 runs. The 28-run, 27-column design matrix can be written in the format

$$\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{a} \\ \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} \\ -\boldsymbol{u}' & -\boldsymbol{u}' & -\boldsymbol{u}' \end{bmatrix},$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are the 9×9 matrices given by Plackett and Burman (1946, p. 323) and each $\boldsymbol{u}'$ is a row vector of nine 1's. The alias table then takes the form

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{L} & \boldsymbol{M} \\ \boldsymbol{U} & \boldsymbol{V} & \boldsymbol{W} \\ \boldsymbol{V}^* & \boldsymbol{W}^* & \boldsymbol{U}^* \\ \boldsymbol{M} & \boldsymbol{K} & \boldsymbol{L} \\ \boldsymbol{W} & \boldsymbol{U} & \boldsymbol{V} \\ \boldsymbol{L} & \boldsymbol{M} & \boldsymbol{K} \end{bmatrix},$$

where U^*, V^*, W^* are obtained via certain permutations from U, V, and W respectively. For their specific forms, see Lin and Draper (1991). The sizes of these matrices are 36×9 for K, L, M and 81×9 for U, V, W (similarly, for the 52-run, 76-run and 100-run cases, we would split the columns of their alias tables into 5, 13 and 9 blocks of columns, respectively).

3. Additional remarks

The knowledge of the details of alias relationships for Plackett and Burman designs is especially useful for those designs that are not 2_{III}^{k-p} type designs (the 2_{III}^{k-p} designs have a relatively simple alias pattern, with coefficients 0 or ± 1). When Plackett and Burman designs are used in factor screening situations, and the estimates of $N-1$ combinations of effects are obtained, it is typically true that relatively few (usually 25% or fewer) of the estimates are judged as 'real'. It would then be customary to say, at least tentatively, that these larger contrasts represent the effective factors and that the other factors can be ignored in the initial assessment of the results. While this is not the only possible assessment, it is often a basically reasonable one.

Suppose, for example, that analysis of the results from the 12-run Plackett and Burman design of Table 1 indicated that the contrasts associated with columns 1, 2, 3, and 4 were large compared with the local error measurement, and the remaining contrasts were small. We might then assume tentatively that only factors **1–4** were relevant, and regard the design as one projected down from the full 11 dimensions to the four dimensions of interest. By 'deleting' all two-factor interactions involving factors **5–11** in Table 3, we see that the eleven contrasts estimate combinations of the four main effects and the six two factor interactions arising between the four factors. This enables us to backsolve for these main effects and interactions, achieving in 12 runs what would otherwise require the 16 runs of a 2^4 design to obtain. This enhances the value of the Plackett and Burman designs as exploratory tools. The following remarks note some additional connections of this paper to works of other authors.

In the cases where N is a power of two, each row in the alias table contains one -1 or one $+1$ and all other elements are 0. Reading the alias table row by row, provides the so-called *interaction table* which has been used in assigning factors to the columns. (See Taguchi, 1987, pp. 1129–1143.)

Hedayat, Raktoe and Federer (1974) suggested using $[\text{trace}(A'A)]^{1/2}$ where $A=(X'X)^{-1}X'X_1$ is the alias matrix defined in (2.3), as a design criterion to measure the bias of a design. If we work with a Hadamard matrix as X, so that $\bar{y}$ is estimated in addition to the $(N-1)$ contrasts, then $X'X=NI=XX'$, and this criterion reduces to the form

$$\text{trace}(A'A)^{1/2}=\text{trace}\left(N^{-1}X_1'X_1\right)^{1/2}=\left[d(d-1)/2\right]^{1/2},$$

where d is the number of factors tentatively retained after analysis, as discussed in the first paragraph of this section, where $d = 4$. The obvious conclusion from this is that the bias criterion $[\text{trace}(A'A)]^{1/2}$ depends only on the number of factors retained, and not on which specific factors they are.

If we add a column of 1's to a Plackett and Burman design, we obtain a Hadamard matrix $\boldsymbol{H}$ which satisfies $\boldsymbol{H}'\boldsymbol{H} = N\boldsymbol{I}$. For $N = 12$, $\boldsymbol{H}$ is unique, but for higher N this is not true. For example, there are five non-equivalent classes (one cyclic) for $N = 16$ (Hall, 1961), and three (one cyclic) for $N = 20$ (Hall, 1965). It can be shown that non-equivalent Hadamard matrices provide different alias relationships. If all factors are considered to be equally important, cyclic structure might be considered preferable because it achieves a complete balance in the alias relationships.

Acknowledgements

N.R. Draper gratefully acknowledges partial support from the National Science Foundation under Grant DMS-8900426, and from the Wisconsin Alumni Research Foundation through the University of Wisconsin Graduate School Research Committee. D.K.J. Lin was partially supported by a Faculty Research Fellowship through the College of Business Administration, The University of Tennessee.

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