Journal of Statistical Planning and Inference 28 (1991) 95-98
North-Holland

On the identity relationships of 2^{k-p} designs

Jiahua Chen

Department of Statistics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

Dennis K.J. Lin

Department of Statistics, University of Tennessee, 331 Stokely Management Center, Knoxville, TN 37996, USA

Received 6 December 1989; revised manuscript received 7 May 1990
Recommended by A.S. Hedayat

Abstract: Draper and Mitchell (1970) gave an example which showed that two 2^{k-p} designs with the same word length pattern can be different. The difference was detected via a 'letter pattern comparison' test. These authors went on to conjecture that two designs with equivalent letter pattern matrices are indeed equivalent designs. We show this conjecture is not true by giving an example of two different designs with identical letter pattern matrices. Thus, more effort is needed to construct a complete set of 2^{k-p} designs.

AMS Subject Classification: 62K15.

Key words and phrases: Letter pattern comparison; minimum aberration; moment; word length pattern.

1. Introduction

Consider any two-level fractional factorial design; we follow the Box and Hunter (1961) notation here. The numbers $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}$, attached to the factors, are called letters. A product (in common sense) of any possible subset of these letters is called a *word*. For any 2^{k-p} design, we can choose a (non-unique) set of p words, which generate the design. The p selected *generators* determine the design. Let $\mathbf{I}$ be the *identity* defined so that, for all words $\mathbf{W}$, $\mathbf{IW} = \mathbf{WI} = \mathbf{W}$ and $\mathbf{W}^2 = \mathbf{I}$. This enables us to write the *product* $\mathbf{UW}$ of two words $\mathbf{U}$ and $\mathbf{W}$ in a minimally reduced form. The set of distinct words formed by all possible products involving the p generators gives the *defining relation*, which contains 2^p terms, including the identity term $\mathbf{I}$. The words in the defining relation correspond to those products of the design columns that are the same as the identity $\mathbf{I}$. A design is uniquely determined by the defining relation and vice versa.

Designs can appear in different ways. Two designs are said to be *equivalent* if one can be obtained from the other via sign changes in the columns, rearrangement of runs, and rearrangement of columns. Testing whether two given defining relation are equivalent seems not to have been resolved in the literature. Burton and Connor (1957) develop a necessary and sufficient condition on the existence of an identity relationship. No method is given to distinguish two designs with the same *word length pattern* defined as the vector $\boldsymbol{\gamma}=(\gamma_1,\gamma_2,\ldots,\gamma_k)$, where γ_t is the number of words of length t in its defining relation. If word length patterns are different, two designs are necessarily different (see also John, 1966). Draper and Mitchell (1970) have found that designs are not uniquely determined by word length pattern. The following examples of two different 2^{12-3} designs with the same word length patterns, $\boldsymbol{\gamma}=(0,0,0,0,0,4,0,3,0,0,0,0)$, can be derived from their 1968 paper (designs 3.4 and 3.5 on p. 248).

(1) $\mathbf{I}=\mathbf{12367\overline{10}}=\mathbf{12389\overline{11}}=\mathbf{6789\overline{10}\,\overline{11}}=\mathbf{14569\overline{12}}=\mathbf{234579\overline{10}\,\overline{12}}=\mathbf{234568\overline{11}\,\overline{12}}=\mathbf{14578\overline{10}\,\overline{11}\,\overline{12}}$,

(2) $\mathbf{I}=\mathbf{13469\overline{10}}=\mathbf{13578\overline{11}}=\mathbf{456789\overline{10}\,\overline{11}}=\mathbf{1234568\overline{12}}=\mathbf{2589\overline{10}\,\overline{12}}=\mathbf{2467\overline{11}\,\overline{12}}=\mathbf{12379\overline{10}\,\overline{11}\,\overline{12}}$.

(The overbar is used to distinguish between, e.g., 'fifteen' and 'one-five'.) Note that the frequencies of the letters appearing in the length-eight words are different. Because such frequencies are invariant under relabeling, designs (1) and (2) must be different. This sort of check was called a *letter pattern comparison* by Draper and Mitchell (1970, p. 877).

In general, to make a 'letter pattern comparison' we first examine the defining relation of a design and count the number, a_{ij}, of words of length j in which letter i appears. Then we form the $k\times k$ letter pattern matrix $\mathbf{A}=\{a_{ij}\}$ for each design, and declare two letter patterns (for design $\mathbf{D}$ and for design $\mathbf{D}^*$) to be equivalent if and only if $\mathbf{A}=\mathbf{P}(\mathbf{A}^*)$, where $\mathbf{P}(\mathbf{A}^*)$ is some permutation of rows of $\mathbf{A}^*$ and where $\mathbf{A}$ and $\mathbf{A}^*$ are the letter pattern matrices corresponding to $\mathbf{D}$ and $\mathbf{D}^*$. Note that the word length pattern, $\boldsymbol{\gamma}$, is just

$$\left(\sum_{i=1}^{k} a_{i1},\ldots,j^{-1}\sum_{i=1}^{k} a_{ij},\ldots,k^{-1}\sum_{i=1}^{k} a_{ik}\right),$$

so that two designs having equivalent letter pattern matrices necessarily have equivalent word length pattern vectors.

Draper and Mitchell (1970, p. 878) conjectured that two designs with equivalent letter pattern matrices are indeed equivalent designs. We show, however, that identical letter pattern matrices *can* be obtained from two different specific 2^{k-p} designs.

2. Example and discussion

Table 1 shows two specific 2_{VII}^{31-16} design. Factors $\mathbf{1},\mathbf{2},\ldots,\mathbf{\overline{15}}$ define a 2^{15} design

Table 1
Generators of two 2_{VII}^{31-16} designs (use factors $\mathbf{1}, \mathbf{2}, \ldots, \overline{\mathbf{15}}$ as basic factors)

Design (a)	Design (b)
$\mathbf{\overline{16} = 1234567}$	$\mathbf{\overline{16} = 149\,\overline{10}\,\overline{14}\,\overline{15}}$
$\mathbf{\overline{17} = 123489\overline{10}}$	$\mathbf{\overline{17} = 12459\,\overline{11}\,\overline{14}}$
$\mathbf{\overline{18} = 125689\overline{11}}$	$\mathbf{\overline{18} = 2356\,\overline{10}\,\overline{12}\,\overline{15}}$
$\mathbf{\overline{19} = 13578\,\overline{10}\,\overline{11}}$	$\mathbf{\overline{19} = 13679\,\overline{10}\,\overline{11}\,\overline{13}\,\overline{14}\,\overline{15}}$
$\mathbf{\overline{20} = 14679\,\overline{10}\,\overline{11}}$	$\mathbf{\overline{20} = 12789\,\overline{11}\,\overline{12}}$
$\mathbf{\overline{21} = 1234\,\overline{12}\,\overline{13}\,\overline{14}}$	$\mathbf{\overline{21} = 2389\,\overline{10}\,\overline{12}\,\overline{13}}$
$\mathbf{\overline{22} = 1256\,\overline{12}\,\overline{13}\,\overline{15}}$	$\mathbf{\overline{22} = 349\,\overline{10}\,\overline{11}\,\overline{13}\,\overline{14}}$
$\mathbf{\overline{23} = 1357\,\overline{12}\,\overline{13}\,\overline{15}}$	$\mathbf{\overline{23} = 45\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{14}\,\overline{15}}$
$\mathbf{\overline{24} = 1467\,\overline{13}\,\overline{14}\,\overline{15}}$	$\mathbf{\overline{24} = 14569\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}\,\overline{14}}$
$\mathbf{\overline{25} = 23456789\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}\,\overline{14}\,\overline{15}}$	$\mathbf{\overline{25} = 2567\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}\,\overline{14}\,\overline{15}}$
$\mathbf{\overline{26} = 134567\,\overline{10}\,\overline{11}\,\overline{14}\,\overline{15}}$	$\mathbf{\overline{26} = 1346789\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}}$
$\mathbf{\overline{27} = 1245679\,\overline{11}\,\overline{13}\,\overline{15}}$	$\mathbf{\overline{27} = 245789\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}\,\overline{14}}$
$\mathbf{\overline{28} = 1235678\,\overline{11}\,\overline{12}\,\overline{15}}$	$\mathbf{\overline{28} = 35689\,\overline{10}\,\overline{11}\,\overline{12}\,\overline{13}\,\overline{14}\,\overline{15}}$
$\mathbf{\overline{29} = 1234679\,\overline{10}\,\overline{13}\,\overline{14}}$	$\mathbf{\overline{29} = 167\,\overline{11}\,\overline{12}\,\overline{13}}$
$\mathbf{\overline{30} = 1234578\,\overline{10}\,\overline{12}\,\overline{14}}$	$\mathbf{\overline{30} = 278\,\overline{12}\,\overline{13}\,\overline{14}}$
$\mathbf{\overline{31} = 12345689\,\overline{12}\,\overline{13}}$	$\mathbf{\overline{31} = 389\,\overline{13}\,\overline{14}\,\overline{15}}$

and factors $\overline{\mathbf{16}}, \overline{\mathbf{17}}, \ldots, \overline{\mathbf{31}}$ are associated with the generators. Both designs of Table 1 have the same letter pattern matrix, with rows (0, 0, 0, 0, 0, 0, 35, 120, 0, 0, 1848, 3360, 0, 0, 8835, 9429, 0, 0, 5320, 3360, 0, 0, 345, 120, 0, 0, 0, 0, 0, 0, 1), and thus the same word length pattern (0, 0, 0, 0, 0, 0, 155, 465, 0, 0, 5208, 8680, 0, 0, 18259, 18259, 0, 0, 8680, 5208, 0, 0, 465, 155, 0, 0, 0, 0, 0, 0, 1).

The 155 words of length seven and the 465 words of length eight in the two defining relations are given in Lin and Chen (1989, Appendix A). All $\binom{31}{4} = 31\,465$ four-factor interactions in design (a) are confounded with either three-factor interactions or other four-factor interactions, but in design (b), 9765 (out of 31 465) four-factor interactions are *not* confounded with any other three- and four-factor interactions, e.g. **1234**. They are confounded with five-factor or higher order interactions. This proves that design (a) and (b) are *not* identical, even though they have identical letter pattern matrices. If we assume all five-factor and higher order interactions are negligible, we can estimate all main effects, two- and three-factor interactions *only*, in design (a), but in addition, we can estimate another 9765 four-factor interactions in design (b).

Aberration criteria essentially use word length patterns in their comparisons. For a criterion that minimizes the number of words in the defining relation that are of minimum length, see Fries and Hunter (1980). For a criterion that maximizes odd moments and minimizes even moments, see Franklin (1984). Computer searches based on a letter pattern comparison test (see Draper and Mitchell, 1970; Fries and Hunter, 1980) always discard designs that have the same letter pattern matrix as a design already given. As we see from this example, the discarded design(s) may be different and perhaps better than the design retained. Although the letter pattern comparison test is a more sensitive test than the word length pattern test, it is not in a one-to-one correspondence with a design, and so is not a sufficient criterion for determining uniqueness.

Much attention has been focused in finding all intrinsically different 2^{k-p} designs (given the run size), especially recently, when the two-level fractional factorial designs have proved to be an important and useful tool in Quality Engineering (see, e.g., Box, Bisgaard and Fung, 1988). Letter pattern methods successfully found more designs than word length pattern methods. However, it is unknown whether those results are *indeed* all the possibilities if letter pattern and design do *not* correspond one-to-one. The existence of our counterexample implies that more research work needs to be done.

Note: Fractional factorial designs are known to be equivalent to *linear error-correcting codes* (Robillard, 1968). The concept of word length pattern is also investigated in the coding literature, and is called *weight distribution*. It is known that two different codes can share the same weight distribution (see, for example, Belekemp, 1968, *Algebraic Coding Theory*, p. 394). These two examples were modified from examples in the coding literature, in which letter pattern comparisons are not used.

Acknowledgements

We thank one referee for his thorough review of the manuscript that greatly improved the article.

References

Burton, R.C. and W.S. Connor (1957). On the identity relationship for fractional replicates of the 2^n series. *Ann. Math. Statist.* **28**, 762–767.

Box, G.E.P., S. Bisgaard and C.A. Fung (1988). An explanation and critique of Taguchi's contributions to quality engineering. Technical Report No. 28, Center for Quality and Productivity Improvement, University of Wisconsin-Madison.

Box, G.E.P. and J.S. Hunter (1961). The 2^{k-p} fractional factorial designs. *Technometrics* **3**, 311–351, 449–458.

Draper, N.R. and T.J. Mitchell (1968). Construction of the set of 256-run designs of resolution ≥5 and the set of even 512-run designs of resolution ≥6 with special reference to the unique saturated designs. *Ann. Math. Statist.* **39**, 246–255.

Draper, N.R. and T.J. Mitchell (1970). Construction of a set of 512-run designs of resolution ≥5 and a set of even 1024-run designs of resolution ≥6. *Ann. Math. Statist.* **41**, 876–887.

Franklin, M.F. (1984). Construction tables of minimum aberration p^{n-m} designs. *Technometrics* **26**, 225–232.

Fries, A. and W.G. Hunter (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22**, 601–608.

Lin, D.K.J. and J.H. Chen (1989). On the identity relationships of 2^{k-p} designs. Technical Report No. 7, Department of Statistics, University of Toronto.

John, P.W.M. (1966). On identity relationships for 2^{n-r} designs having words of equal length. *Ann. Math. Statist.* **37**, 1842–1843.

Robillard, P. (1968). Combinatorial problems in the theory of factorial designs and error correcting codes. Inst. Statist. Mimeo Ser. 594, Univ. of North Carolina, Chapel Hill.