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Connections Between Two-Level Designs of Resolutions III* and V

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Resolution III* designs are resolution III designs in which no two-factor interactions are confounded with one another. They arise naturally in the search for small composite designs and were first suggested by Hartley (1959). New results given here link resolution III* designs to resolution V designs, enabling passage between the two types of designs to be made. This means that previously untabulated resolution III* designs can be derived directly from known resolution V designs. For example, a Hartley-type composite design can be based on a $2^{7-2}_{III^*}$ design (derived from a 2^{6-1}_{V} design); the "usual" composite design is based on a 2^{7-1}_{V} design, double the number of factorial runs. The resolution III* designs can also be usefully modified for specific industrial applications (Box and Jones 1989). Moreover, the maximum number of factors it is possible to accommodate in a resolution III* design of a given size can be obtained directly from the known maximum for resolution V designs of the same size.

KEY WORDS: Composite design; Conversion multiplier; Word-length pattern.

1. INTRODUCTION

A 2^k factorial design is one in which k variables or factors, labeled $(\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k})$, are each allocated two levels, conventionally ± 1 in coded coordinates, and every possible combination of the $\pm$ signs is run, typically in a randomized or randomized block order. A fractional two-level design is one that employs only a fraction of the 2^k runs. Many such designs use a 2^{-p} fraction of the whole 2^k runs and so have been designated 2^{k-p} fractional factorials. Strictly speaking, however, any selection of the 2^k runs forms a fractional design, but not necessarily a 2^{k-p} fraction. Two-level factorial and fractional factorial designs have been used for many years, certainly since Yates (1935). A large compilation of 2^{k-p} designs was made available by the National Bureau of Standards (1957), for example. Alternatives to the classical methods of formation and analysis were given by Box and Hunter (1961); see also Box, Hunter, and Hunter (1978) and Box and Draper (1987). We follow the Box and Hunter (1961) notation and development in this article. The numbers $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{k}$, attached to the factors, are called *letters*. A product of any subset of these variables, or *letters*, is called a *word*. Associated with every 2^{k-p} design is a set of p words, $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_p$, called *generators*. For $p > 1$, a set of generators is not unique, and the same design may be described via different sets of generators. Let I be the *identity*, defined so that, for all words $\mathbf{W}$, $\mathbf{IW} = \mathbf{WI} = \mathbf{W}$ and $\mathbf{W}^2 = \mathbf{I}$. This enables us to write the *product* $\mathbf{UW}$ of two words, $\mathbf{U}$ and $\mathbf{W}$, in a minimally reduced form. The set of distinct words formed by all possible products involving the p generators gives the *defining relation*, which contains 2^p terms including the identity term $\mathbf{I}$.

An important characteristic of a 2^{k-p} design is its resolution, a concept recognized by Bose (1947) and Rao (1947) and defined by Box and Hunter (1961), as follows: "A design of resolution R is one in which no p factor effect is confounded with any other effect containing less than $R - p$ factors. . . . In general, the *resolution* of a two-level fractional design is *the length of the shortest word in the defining relation*" (p. 319). Therefore, a resolution V design permits the estimation of all main effects and two-factor interactions when higher-order interaction effects are negligible. Every word in its defining relation contains five or more letters. One way to characterize a 2^{k-p}_R (two to the k minus p, resolution R) fractional factorial design is by its *word-length pattern*. Suppose that a design D of resolution R has γ_t words of length t in the defining relation of D, where $t = R, R + 1, \ldots, k$. The vector $\boldsymbol{\gamma} = (\gamma_R, \ldots, \gamma_k)$ will be called the *word-length pattern* of D. If the word-length patterns are different, the designs are necessarily different.

This article discusses a special class of designs that has a resolution we call III*, the value of which was first pointed out by Hartley (1959) in connection with the formation of reduced designs of the composite type (Box and Hunter 1957). A link between the little-known resolution III* designs and the well-known resolution V designs makes resolution III* designs easy to obtain. We discuss resolution III* designs and provide some specific examples without derivation in Section 2. Theorem 1 in Section 3 provides a way to derive resolution V designs from resolution III* designs, and the reverse derivation, which enables us to obtain resolution III* from those of resolution V, is given in Section 4. In Section 5, we give extensions of some of our results to designs of resolution "odd-star"—that is, V*, VII*, and so on—and also mention work by Box and Jones (1989) that makes practical use of III* designs.

2. RESOLUTION III* DESIGNS

We define a 2^{k-p} fractional factorial design to be of resolution III* if it is of resolution III *and* its defining relation contains no four-letter word. Thus the word-length pattern of a resolution III* design is $(\gamma_3, 0, \gamma_5, \ldots, \gamma_k)$, where $\gamma_3 \neq 0$. The value of such a design was first shown by Hartley (1959). In presenting their useful *composite* designs, Box and Hunter (1957, p. 227) combined a 2^{k-p} design with a set of $2k$ axial points $(\pm\alpha, 0, \ldots, 0)$, $(0, \pm\alpha, \ldots, 0), \ldots, (0, 0, \ldots, \pm\alpha)$ to estimate the coefficients of a second-order polynomial model. They recommended that the 2^{k-p} design be of resolution V or higher because then two-factor interactions would nowhere be confounded with two-factor interactions. Hartley pointed out that 2^{k-p} designs of resolution III* could also be used. Even though some main effects would then be confounded with two-factor interactions in the factorial portion of the design, the coefficients thus aliased could essentially be immediately de-aliased using the additional main-effect information derived from the pairs of axial points. Hartley (1959, pp. 613–615) offered two examples, a saturated $2^{4-1}_{\mathrm{III}^*}$ generated by $\mathbf{I} = \mathbf{123}$ when $\boldsymbol{\gamma} = (1, 0)$ and a saturated $2^{6-2}_{\mathrm{III}^*}$ with defining relation $\mathbf{I} = \mathbf{123} = \mathbf{456} = \mathbf{123456}$ (any two of the three words shown can be used as generators) for which $\boldsymbol{\gamma} = (2, 0, 0, 1)$. Of the 15 two-factor interactions in the latter, 6 are confounded with main effects and the remaining 9 are confounded with four-factor interactions. No two-factor interaction is confounded with any other two-factor interaction, however. Hartley also showed that there is no $2^{5-2}_{\mathrm{III}^*}$ design.

Westlake (1965, p. 325) provided a table indicating the existence of resolution III* designs for $k = 7, 8$, and 9 with $2^q = 32, 32$, and 64 runs, respectively. In fact, for 32 runs ($q = 5$), seven is the maximum number of factors possible for a resolution III* design, so Westlake's entry for eight factors and 32 runs is an error. This error was also made by Draper (1985, p. 174) and Box and Draper (1987, p. 521). Both eight factors and nine factors require 64 runs ($q = 6$) for resolution III*, and nine is the maximum possible for 64 runs ($q = 6$). For 10 to 12 factors, 128 runs ($q = 7$) are needed to get a resolution III* design, 12 being the maximum possible. Examples of such resolution III* designs are the following:

1. An $N = 2^5$, $k = 7$, $2^{7-2}_{\mathrm{III}^*}$ design; $\mathbf{I} = \mathbf{126} = \mathbf{347}$ $(= \mathbf{123467})$ with $\boldsymbol{\gamma} = (2, 0, 0, 1)$.
2. An $N = 2^6$, $k = 9$, $2^{9-3}_{\mathrm{III}^*}$ design; $\mathbf{I} = \mathbf{127} = \mathbf{348} = \mathbf{569}$ $(= \mathbf{123478} = \mathbf{125679} = \mathbf{345678} = \mathbf{123456789})$, with $\boldsymbol{\gamma} = (3, 0, 0, 3, 0, 0, 1)$. For $k = 8$, delete any variable, for example, **9**.
3. An $N = 2^7$, $k = 12$, $2^{12-5}_{\mathrm{III}^*}$ design; $\mathbf{I} = \mathbf{128} = \mathbf{13579} = \mathbf{2345\overline{10}} = \mathbf{1346\overline{11}} = \mathbf{1234567\overline{12}}$ (= their products) with $\boldsymbol{\gamma} = (1, 0, 9, 12, 3, 3, 3, 0, 0, 0)$. (Here we use an overbar to distinguish between, e.g., "twelve" and "one-two.")
4. An $N = 2^7$, $k = 12$, $2^{12-5}_{\mathrm{III}^*}$ design; $\mathbf{I} = \mathbf{128} = \mathbf{349} = \mathbf{56\overline{10}} = \mathbf{1357\overline{11}} = \mathbf{2467\overline{12}}$ (= their products) with $\boldsymbol{\gamma} = (3, 0, 3, 12, 9, 3, 1, 0, 0, 0)$. For $k = 11$, delete any variable, for example, **8**. For $k = 10$, delete any two variables, for example, **8** and **9**.

Note that the word-length patterns of the two given $2^{12-5}_{\mathrm{III}^*}$ designs are different. It is possible to show [see Corollary 3 and, for example, Box and Hunter (1961, p. 449)] that no more than 12 factors can be accommodated in a 2^7-run design of resolution III*. Thus a unique design does not exist, even for the case involving the maximum number of factors. The first of these designs has less *aberration* than the second, however. Fries and Hunter (1980) described the idea of *aberration*: "When comparing two designs using resolution as the criterion, one considers the lengths of the shortest word in each defining relation. If these lengths are equal, the two designs are regarded as being equivalent. With aberration as the criterion, however, one continues to examine the length of the next shortest word in each defining relation until one design is ranked superior to the other" (p. 602). If we apply this rule, clearly design 3 is preferred over design 4.

3. RESOLUTION V DESIGNS FROM RESOLUTION III* DESIGNS

We now discuss how to derive resolution V designs from resolution III* designs via a general theorem.

Theorem 1. Any k-factor two-level fractional factorial design of resolution III* forms a base that can be converted into a $(k - 1)$-factor design of resolution V in the same number of runs.

Proof. Suppose that $x_1, x_2, \ldots, x_k$ are the k factors that form the given resolution III* design. Then the matrix $\mathbf{X} = [1, x_1x_2, x_1x_3, \ldots, x_{(k-1)}x_k]$, where the symbols within the brackets are column designations, must be of full rank, since two-factor interactions are not confounded with one another. Select any of the factors—for example, x_l, and define $y_i = x_l x_i$ when $i = 1 \ldots k (i \neq l)$. Then $y_i y_j = x_i x_j$ for all $1 \leq i < j \leq k (i, j \neq l)$. If we select $x_l = x_k$, the matrix

$$\mathbf{Y} = [1, y_1, y_2, \ldots, y_{(k-1)}, y_1 y_2, y_1 y_3, \ldots, y_{(k-2)} y_{(k-1)}]$$
$$= [1, x_1 x_k, x_2 x_k, \ldots, x_{(k-1)} x_k, x_1 x_2, x_1 x_3, \ldots, x_{(k-2)} x_{(k-1)}],$$

which is a reordering of the columns of $\mathbf{X}$ and is therefore of full rank; that is, main effects and two-factor interactions in the y's are not confounded with one another. Thus $y_1, y_2, \ldots, y_{(k-1)}$ form a $(k - 1)$-factor design of at least resolution V.

Corollary 1. If k is the maximum number of factors that can be accommodated in a resolution III* design, then the maximum number of factors that can be accommodated in a resolution V design with the same number of runs is at least $(k - 1)$.

In the theorem, we have used variable k as the conversion multiplier. In fact, any letter could be used. For convenience in presentation, we use the letter **1** in the following examples to illustrate the application of Theorem 1. These employ designs 1, 2, 3, and 4 given previously. In all cases, the resolution V design has one fewer factor than the resolution III* design, but the same number of runs. The numbers 1–4 correspond with the design numbers in Section 2; Example 5 is new. The examples are as follows:

1. An $N = 2^5$, $k = 7$, $2^{7-2}_{\mathrm{III}^*}$ design; the original seven factors are **1, 2, 3, 4, 5, 6**(= **12**), **7**(= **34**). We now set **a** = **12**, **b** = **13**, **c** = **14**, **d** = **15**, **e** = **16**, and **f** = **17** and consider what design is formed for these six factors (**a**, **b**, **c**, **d**, **e**, **f**). Theorem 1 says that this derived design must be of resolution V. One (somewhat tedious) way to actually confirm this, would be to take all possible products of **a**, **b**, . . . , **f**: only one, namely **abcef**, reduces to I. This can also be seen more quickly by taking **a** = **12**, **b** = **13**, **c** = **14**, **d** = **15**, and **e** = **16** = **2** as *basic columns*. (They are obviously orthogonal to one another because **1, 2, 3, 4, 5**, and **6** are.) Then **f** = **17** = **134** = **abce** is a single generator for the new design. We have thus defined a 32-run 2^{6-1}_{V} design generated by **I** = **abcef** with $\boldsymbol{\gamma} = (1, 0)$.

2. A 2^{8-2}_{V} design with $\boldsymbol{\gamma} = (2, 1, 0, 0)$ may be obtained similarly via design 2.

3. An $N = 2^7$, $k = 12$, $2^{12-5}_{\mathrm{III}^*}$ design. Examining the defining relation, we see that, by deleting any one variable from the only three-letter word—for example, variable **8**—we obtain a 2^{11-4}_{V} design right away. This is always true for any resolution III* design with only one three-letter word. The general method of Theorem 1 could also be applied here, however,—for example, by setting **a** = **12**, **b** = **13**, **c** = **14**, **d** = **15**, **e** = **16**, **f** = **17**, and **g** = **18** = **2** as basic columns and then **h** = **19** = **357** = **abdfg**, **i** = $\mathbf{1\overline{10}}$ = **12345** = **bcdg**, **j** = $\mathbf{1\overline{11}}$ = **346** = **abceg**, and **k** = $\mathbf{1\overline{12}}$ = **234567** = **abcdef**. The resulting defining relation is I = **abdfgh** = **bcdgi** = **abcegj** = **abcdefk** = (their products) with $\boldsymbol{\gamma} = (6, 6, 2, 1, 0, 0, 0)$, so the 11 factors $(a, b, \ldots, k)$ form a 2^{11-4}_{V} design.

4. A 2^{11-4}_{V} design with $\boldsymbol{\gamma} = (6, 6, 2, 1, 0, 0, 0)$ can be obtained similarly via design 4.

5. A $2^{18-10}_{\mathrm{III}^*} \rightarrow 2^{17-9}_{\mathrm{V}}$ design. Addelman (1965) obtained a specific 2^{17-9}_{V} design, which was shown to be unique by Draper and Mitchell (1967). We now show how this design can be obtained from a specific $2^{18-10}_{\mathrm{III}^*}$ design, using Theorem 1. The left portion of Table 1 shows a $2^{18-10}_{\mathrm{III}^*}$ design generated via a computer program written specifically to seek such designs. The right portion of Table 1 is obtained by applying Theorem 1 using **1** as the conversion multiplier, thus providing a 2^{17-9}_{V} design.

4. RESOLUTION III* DESIGNS FROM RESOLUTION V DESIGNS

An obvious question is whether or not we can simply add one new generator of *word-length three*

Table 1. Obtaining a 2^{17-9}_{V} Design From a $2^{18-10}_{III^*}$ Design

III*	V
1	
2	**a = 12**
3	**b = 13**
4	**c = 14**
5	**d = 15**
6	**e = 16**
7	**f = 17**
8	**g = 18**
9 = 12	**h = 19 = 2**
$\mathbf{\overline{10}}$ **= 1345**	**i =** $\mathbf{1\overline{10}}$ **= 345 = abcdh**
$\mathbf{\overline{11}}$ **= 2346**	**j =** $\mathbf{1\overline{11}}$ **= 12346 = bceh**
$\mathbf{\overline{12}}$ **= 2357**	**k =** $\mathbf{1\overline{12}}$ **= 12357 = bdfh**
$\mathbf{\overline{13}}$ **= 1467**	**l =** $\mathbf{1\overline{13}}$ **= 467 = acefh**
$\mathbf{\overline{14}}$ **= 2458**	**m =** $\mathbf{1\overline{14}}$ **= 12458 = cdgh**
$\mathbf{\overline{15}}$ **= 1568**	**n =** $\mathbf{1\overline{15}}$ **= 568 = adegh**
$\mathbf{\overline{16}}$ **= 3478**	**o =** $\mathbf{1\overline{16}}$ **= 13478 = abcfgh**
$\mathbf{\overline{17}}$ **= 14678**	**p =** $\mathbf{1\overline{17}}$ **= 4678 = cefg**
$\mathbf{\overline{18}}$ **= 2345678**	**q =** $\mathbf{1\overline{18}}$ **= 12345678 = abcdefg**

NOTE: The word-length pattern for the $2^{18-10}_{\mathrm{III}^*}$ design in variables (1, 2, . . . , $\overline{18}$) is γ_{III^*} = (3, 0, 36, 102, 117, 153, 200, 153, 117, 102, 36, 0, 3, 0, 0, 1). The word-length pattern for the derived 2^{17-9}_{V} design in variables $(a, b, \ldots, q)$ is γ_{V} = (34, 68, 68, 85, 85, 68, 68, 34, 0, 0, 0, 0, 1). This is the same as Addelman's (1965) design.

to a $(k - 1)$-factor resolution V design to obtain a k-factor resolution III* design for the converse of Theorem 1. In general, the answer is no, but a less straightforward converse is always possible.

Converse of Theorem 1. Any $(k - 1)$-factor two-level fractional factorial design of resolution V can be converted into a k-factor design of resolution III* in the same number of runs.

Proof. Our method involves the addition of one factor and a redefinition of all the factors. Suppose that $x_1, x_2, \ldots, x_{(k-1)}$ form a resolution V design. Then the matrix $\mathbf{X} = [1, x_1, \ldots, x_{(k-1)}, x_1x_2, \ldots, x_{(k-2)}x_{(k-1)}]$ is of full rank, since all main effects and two-factor interactions are not confounded with one another. Define $y_k = x_1x_2$ (for example; actually, any two-factor interaction could be selected here), and set $y_i = x_1x_2x_i$ when $i = 1 \ldots (k - 1)$. Then $y_iy_j = x_ix_j$ for $1 \le i < j \le (k - 1)$ and $y_iy_k = x_i$ for $1 \le i \le (k - 1)$. Therefore,

$$\begin{aligned}\mathbf{Y} &= [1, y_1y_2, \ldots, y_{(k-1)}y_k] \\ &= [1, y_1y_2, \ldots, y_{(k-2)}y_{(k-1)}, \\ &\quad y_ky_1, \ldots, y_ky_{(k-1)}] \\ &= [1, x_1x_2, \ldots, x_{(k-2)}x_{(k-1)}, \\ &\quad x_1, \ldots, x_{(k-1)}]\end{aligned}$$

which is a reordering of the columns of $\mathbf{X}$ and so is of full rank. Thus no two-factor interaction in the y's is confounded with any other two-factor interaction in the y's. Since $y_1y_2 = x_1x_2 = y_k$, however, the new design must be of resolution III. Thus $y_1, y_2, \ldots, y_k$ form a k-factor design of resolution III*.

Corollary 2. If $(k - 1)$ is the maximum number of factors that can be accommodated in a resolution V design, then the maximum number of factors that can be accommodated in a resolution III* design with the same number of runs is at least k.

Corollaries 1 and 2 together imply Corollary 3.

Corollary 3. The maximum number of factors that can be accommodated in a resolution III* design with N runs exceeds the maximum number of factors that can be accommodated in a resolution V design with N runs by exactly 1.

We now obtain the following resolution III* designs by converting the saturated resolution V designs in Box and Hunter (1961). The conversion

Table 2. Obtaining a $2_{III^*}^{18-10}$ Design From a 2_V^{17-9} Design

V	III*
1	**a = 121 = 2**
2	**b = 122 = 1**
3	**c = 123**
4	**d = 124**
5	**e = 125**
6	**f = 126**
7	**g = 127**
8	**h = 128**
9 = 1234	**i = 129 = 34 = cd**
$\overline{10}$ = 1256	**j = 12$\overline{10}$ = 56 = ef**
$\overline{11}$ = 1278	**k = 12$\overline{11}$ = 78 = gh**
$\overline{12}$ = 1357	**l = 12$\overline{12}$ = 2357 = bceg**
$\overline{13}$ = 12368	**m = 12$\overline{13}$ = 368 = abcfh**
$\overline{14}$ = 13458	**n = 12$\overline{14}$ = 23458 = acdeh**
$\overline{15}$ = 14567	**o = 12$\overline{15}$ = 24567 = adefg**
$\overline{16}$ = 24678	**p = 12$\overline{16}$ = 14678 = bdfgh**
$\overline{17}$ = 345678	**q = 12$\overline{17}$ = 12345678 = abcdefgh**
	r = 12 = ab (new factor)

NOTE: The word-length pattern for the $2_{III^*}^{18-10}$ design in variables (**a**, **b**, . . . , **r**) is γ_{III^*} = (4, 0, 30, 102, 132, 153, 180, 153, 132, 102, 30, 0, 4, 0, 0, 1). This is different from the $2_{III^*}^{18-10}$ design in Table 1.

multiplier used is **12**, although any two-letter word can be used.

6. An $N = 2^4$, $k = 5$, 2_V^{5-1} design; the original five factors are **1**, **2**, **3**, **4**, **5**(= **1234**). Set **a** = (**12**)**1** = **2**, **b** = (**12**)**2** = **1**, **c** = (**12**)**3**, **d** = (**12**)**4**, **e** = (**12**)**5** = **34** = **cd**, and **f** = **12** = **ab** (a new factor) and consider what design is formed for these six factors (**a**, **b**, **c**, **d**, **e**, **f**). If we take **a**, **b**, **c**, **d** as basic columns, then **e** = **cd** and **f** = **ab** will be two generators. This establishes the defining relation **I** = **cde** = **abf**(= **abcdef**), so these six factors (**a**, **b**, **c**, **d**, **e**, **f**) form a $2_{III^*}^{6-2}$ design with γ = (2, 0, 0, 1). This is essentially Hartley's (1959, p. 614) design.

Similarly, a $2_{III^*}^{7-2}$ design in which γ = (1, 0, 1, 1, 0), a $2_{III^*}^{9-3}$ design in which γ = (3, 0, 0, 3, 0, 0, 1), and a $2_{III^*}^{12-5}$ design in which γ = (2, 0, 6, 12, 6, 3, 2, 0, 0, 0) can be obtained [corresponding to the designs listed in Box and Hunter (1961, p. 450)].

7. A ($2_V^{17-9} \rightarrow 2_{III^*}^{18-10}$) design. The left portion of Table 2 shows Addelman's (1965, p. 441) 2_V^{17-9} design. Note that his generators are different from those shown in Table 1, although the design is, of course, identical (Draper and Mitchell 1967). The right portion of Table 2 shows the resolution III*

Table 3. Run Size of Two-Level Fractional Factorial Portion of Smallest Possible Composite Designs of Resolutions III and V*

k	6	7	8	9	10	11	12
Resolution III* design	2^{6-2}	2^{7-2}	2^{8-2}	2^{9-3}	2^{10-3}	2^{11-4}	2^{12-5}
Resolution V design	2^{6-1}	2^{7-1}	2^{8-2}	2^{9-2}	2^{10-3}	2^{11-4}	2^{12-4}

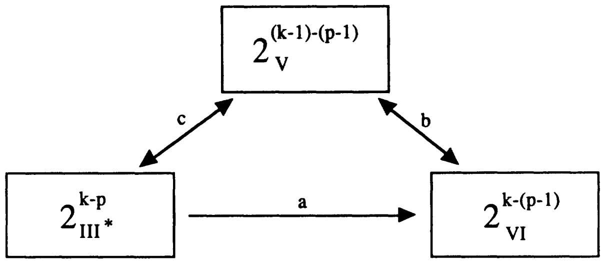

Figure 1. Relationship Among $2^{k-p}_{III^*}$, $2^{(k-1)-(p-1)}_{V}$, and $2^{k-(p-1)}_{VI}$ Designs: a, Foldover; b, Foldover (plus the I column) and Reverse by Erasure; c, Theorem 1 and Its Converse.

design obtained by applying the converse of Theorem 1.

Note 1. For $q = 7$, the maximum number of factors that can be accommodated in a resolution V design is 11; thus there is no resolution V design with 12 factors in 2^7 runs. An entry for the case in which $k = 12$ and $p = 5$ (i.e., $q = k - p = 7$) in table 2 of Fries and Hunter (1980, p. 605) showing $R_{\max} = 5$ is thus erroneous and should be corrected to indicate resolution IV. In their same table 2, Fries and Hunter (1980, p. 605) showed, for the case in which $k = 13$ and $p = 5$ (i.e., $q = 8$), $R_{\max} = 4$. This should be corrected to indicate resolution V. Similar changes are implicit in Franklin (1984, table 1).

Note 2. Any resolution III* design for k factors can be combined with a set of $2k$ axial points to form a second-order design of composite type, as shown by Hartley (1959). Some possibilities are shown in Table 3 and compared with the "usual" resolution V fractional factorial design (Box and Hunter 1957). Note that no savings in the number of factorial runs is achieved when $k = 8$, 10, and 11, but the number of factorial runs is halved when $k = 6$, 7, 9, and 12.

5. EXTENSIONS AND APPLICATIONS

In general, we define a resolution "odd-star" design as a design of odd resolution—$R = (2l - 1)$, say—for which no $(R + 1)$-length word exists in its defining relation. Thus the word-length pattern is $(\gamma_R, 0, \gamma_{(R+2)}, \ldots, \gamma_k)$. The following result can be

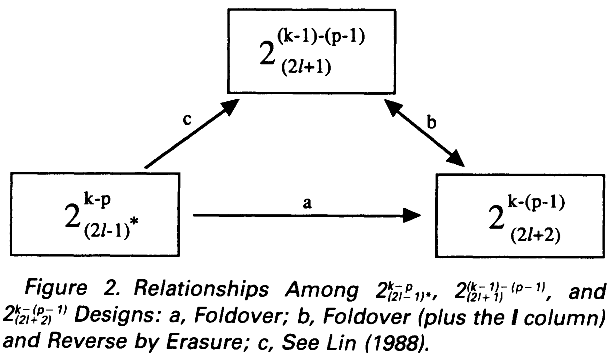

Figure 2. Relationships Among $2^{k-p}_{(2l-1)^*}$, $2^{(k-1)-(p-1)}_{(2l+1)}$, and $2^{k-(p-1)}_{(2l+2)}$ Designs: a, Foldover; b, Foldover (plus the I column) and Reverse by Erasure; c, See Lin (1988).

proved in a manner similar to that used in the proof of Theorem 1.

Theorem 2. Any k-factor two-level fractional factorial design of resolution $(2l - 1)^*$ forms a base that can be converted into a $(k - 1)$-factor design of resolution $(2l + 1)$ in the same number of runs.

The converse of Theorem 2 is not always true, however, (see Lin 1988). The relationships are illustrated in Figure 1 (for $l = 2$) and in Figure 2 (for $l \geq 3$). The arrows indicate where passage is possible. In the figures, *foldover* of a design means adding, to the original design, the design that is obtained by switching the signs of all the variables (see Box and Hunter 1961, p. 337), and *erasure* of a variable means the removal of its symbol from the defining relation of a higher-resolution design to create the defining relation of a lower-resolution design. The ideas described previously can also be extended to general p-level designs ($p \geq 3$) in a straightforward manner, using modified definitions and notation.

Applications of resolution III* designs were suggested by Box and Jones (1989) in the following context. They wished to examine experiments conducted with both *design factors* (also called control or inner-array factors) and *environmental factors* (also called noise or outer-array factors). Large numbers of runs are typically needed in such designs, because each inner-array design point has an outer-array design at its location. Box and Jones (1989) assumed a second-order model and showed that to minimize a certain criterion of interest it is not actually necessary to estimate all of the second-order parameters individually. By using a central composite design, based on a resolution III* design but modified according to one's estimation needs, a considerable economy of experimentation can be achieved. The results of our article facilitate such an application by making it easy to obtain resolution III* designs from tables of resolution V designs.

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REFERENCES

Addelman, S. (1965), "The Construction of a 2^{17-9} Resolution V Plan in Eight Blocks of 32," *Technometrics*, 7, 439–443.

Bose, R. C. (1947), "Mathematical Theory of the Symmetrical Factorial Designs," *Sankhyā*, 8, 107–166.

Box, G. E. P., and Draper, N.R. (1987), *Empirical Model-Building and Response Surfaces*, New York: John Wiley.

Box, G. E. P., and Hunter, J. S. (1957), "Multi-Factor Experimental Designs for Exploring Response Surfaces," *The Annals of Mathematical Statistics*, 28, 195–241.

——— (1961), "The 2^{k-p} Fractional Factorial Designs" (Parts 1 and 2), *Technometrics*, 3, 311–351, 449–458.

Box, G. E. P., Hunter, W. G., and Hunter, J. S. (1978), *Statistics for Experimenters*, New York: John Wiley.

Box, G. E. P., and Jones, S. P. (1989), "Designs for Minimizing the Effect of Environmental Variables," unpublished paper presented at the 33rd Annual Fall Technical Conference of the American Society for Quality Control, Houston, Texas, October 27.

Draper, N. R. (1985), "Small Composite Designs," *Technometrics*, 27, 173–180.

Draper, N. R., and Mitchell, T. J. (1967), "The Construction of Saturated 2_R^{k-p} Designs," *The Annals of Mathematical Statistics*, 38, 1110–1126.

Franklin, M. F. (1984), "Construction Tables of Minimum Aberration p^{n-m} Designs," *Technometrics*, 26, 225–232.

Fries, A., and Hunter W. G. (1980), "Minimum Aberration 2^{k-p} Designs," *Technometrics*, 22, 601–608.

Hartley, H. O. (1959), "Smallest Composite Designs for Quadratic Response Surfaces," *Biometrics*, 15, 611–624.

Lin, D. K. J. (1988), "Small Response Surface Designs," unpublished Ph.D. thesis, University of Wisconsin-Madison, Dept. of Statistics.

National Bureau of Standards (1957), *Fractional Factorial Experimental Designs for Factors at Two Levels* (National Bureau of Standards Applied Mathematics Series, Vol. 48), Washington, DC: U.S. Government Printing Office.

Rao, C. R. (1947), "Factorial Arrangements Derivable From Combinatorial Arrangements of Arrays," *Journal of the Royal Statistical Society*, Ser. B, 9, 128–139.

Westlake, W. J. (1965), "Composite Designs Based on Irregular Fractions of Factorials," *Biometrics*, 21, 324–336.

Yates, F. (1935), "Complex Experiments," *Journal of the Royal Statistical Society*, Ser. B, 2, 181–223.