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CAPACITY CONSIDERATIONS FOR TWO-LEVEL FRACTIONAL FACTORIAL DESIGNS

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Abstract: This paper tackles the following question: For a two-level experimental design with a given number, $N=2^q$, of runs and a specified resolution R, what is the maximum number, k, of factors that can be accommodated? This problem is intimately connected with other problems that have been extensively studied by previous authors. Prior results are summarized, explained, and extended.

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Key words: Most two-level factors; minimum two-level design; maximum resolution.

1. Introduction

Consider a two-level fractional factorial design of resolution R for investigating k factors in N $(=2^q)$ runs. Given any two of the quantities (R, k, q), it is of interest to ask what value can be achieved for the third quantity. The choices are:

(1) Fix k and q, investigate what maximum value of R is possible. Box and Hunter (1961a, 1961b) solved this for small k and q. For larger designs, upper bounds were sought by Robillard (1968), Fujii (1976), Webb (1968), Margolin (1969), and Fries and Hunter (1980).

(2) Fix R and k, investigate what minimum value of q (and thus $N=2^q$) is possible. This was investigated by Webb (1968) and Margolin (1969).

(3) Fix R and q and seek the maximum possible value of k. Work on this aspect includes Addelman (1965), Draper and Mitchell (1967), Draper and Lin (1988).

Of course, the three problems are essentially equivalent, but sometimes it is easier to tackle a problem in one manner rather than another. Our intention in the present paper is to follow the third approach, and to produce a comprehensive table for the (connected) ranges $\text{III} \le R \le \text{XIII}$, $4 \le k \le 4095$, and $3 \le q \le 12$. This table reproduces the results published to date, and extends them.

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2. Notation and definitions

A 2^k factorial design is one in which k variables or factors, labeled $(\mathbf{1}, \mathbf{2}, \dots, \mathbf{k})$, are each allocated two levels, conventionally ± 1 in coded coordinates, and every possible combination of the $\pm$ signs is run, typically in a randomized, or randomized block, order. A fractional two-level design is one that employs only a fraction of the 2^k runs. Many such designs use a 2^{-p} fraction of the whole 2^k runs and so have been designated 2^{k-p} fractional factorials. However, strictly speaking, any selection of the 2^k runs forms a fractional design, but not necessarily a 2^{k-p} fraction. Two-level factorial and fractional factorial designs have been used for many years, and certainly since Yates (1935). A large compilation of 2^{k-p} designs was made available by the National Bureau of Standards (1957), for example. Alternatives to the classical methods of formation and analysis were given by Box and Hunter (1961a, 1961b); see also Box, Hunter and Hunter (1978) and Box and Draper (1987). We follow the Box and Hunter (1961a) notation and development in this paper. The numbers $\mathbf{1}, \mathbf{2}, \dots, \mathbf{k}$, attached to the factors, are called *letters*. A product of any subset of these variables, or *letters*, is called a *word*. Associated with every 2^{k-p} design is a set of p words, $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_p$, called *generators*. For $p > 1$, a set of generators is not unique, and the same design may be described via different sets of generators. Let $\mathbf{I}$ be the *identity* defined so that, for all words $\mathbf{W}$, $\mathbf{IW} = \mathbf{WI} = \mathbf{W}$ and $\mathbf{W}^2 = \mathbf{I}$. This enables us to write the *product* $\mathbf{UW}$ of two words $\mathbf{U}$ and $\mathbf{W}$ in a minimally reduced form. The set of distinct words formed by all possible products involving the p generators gives the *defining relation*, which contains 2^p terms including the identity term $\mathbf{I}$.

An important characteristic of a 2^{k-p} design is its resolution, defined by Box and Hunter (1961, p. 319) as follows: ''A design of resolution R is one in which no p factor effect is confounded with any other effect containing less than $R-p$ factors. ... In general, the *resolution* of a two-level fractional design is *the length of the shortest word in the defining relation.*''

In fact, one way to characterize a 2_R^{k-p} (''two to the k minus p, resolution R'') fractional factorial design is by its *word length pattern*. Suppose a design D of resolution R has γ_t words of length t in the defining relation of D, where $t = R, R+1, \dots, k$. The vector $\boldsymbol{\gamma} = (\gamma_R, \dots, \gamma_k)$ will be called the *word length pattern* of D. If the word length patterns of two designs are different, the designs are necessarily different. However, different designs can have the same word length pattern, e.g., designs 3.4 and 3.5 of Table 1 in Draper and Mitchell (1968).

A major use of two-level fractional factorials is for screening experiments, in which many factors are examined in relatively few runs, to identify those (few) factors that exert large effects on one or more response variables. If a design's resolution is specified, and if the number of runs is fixed, then the design will accommodate only a certain maximum number of factors. Such a design can be called a *saturated design*.

3. Construction methods

When R and q are given, we seek the maximum possible value of k. The following features were used to obtain values of k.

Computer algorithms. Generators are picked successively in an entry sequence explained below. When a new potential generator is added, the potential defining relation is constructed and checked to see if all its words satisfy the resolution requirements. If not, the potential generator is dropped, and the search continues. As mentioned by Draper and Lin (1988, p. 16), the design produced depends on the chosen generator entry sequence. Two different generator entry sequences were actually used. One was Addelman's (1965, p. 440) approach, in which lower generators were introduced before higher order ones. Generators of the same order were added as long as the desired resolution was maintained. Another generator entry sequence followed the Yates' order. (See Box, Hunter and Hunter, 1978, p. 323; or Box and Draper, 1987, p. 127.)

Deletion. *Deletion* of a variable $\mathbf{d}$ from a design implies the removal of all words in the defining relation that involve the variable $\mathbf{d}$. Such a reduced design is thus a fractional factorial with the same number of runs, but with one less variable, one less generator, and a defining relation of half the previous length. Deletion allows us to obtain a $(k-1)$-factor design of resolution greater than or equal to R from a k-factor design of resolution R in the same number of runs. (The resolution would increase, for example, if there was only one smallest-sized-word and it was removed by a deletion.) Note that if we do not wish to use all of the factors that are possible for specified values of R and $N=2^q$, some factors can simply be deleted. So we can focus on saturated designs without loss of generality.

Erasure. By the *erasure* of a variable is meant (Draper and Mitchell, 1967, p. 1113) the removal of its symbol from the defining relation of a higher resolution design to create the defining relation of a lower resolution design. No words are deleted. Erasure allows us to obtain a k-factor design of resolution $2l-1$ from a $(k+1)$-factor design of resolution $2l$. The reverse of erasure is *attachment* which is essentially *foldover*.

Foldover. *Foldover* was introduced by Box and Hunter (1961a, p. 337). Suppose $\mathbf{A}$ represents the block of signs in the factor columns of a 2_{2l-1}^{k-p} design (ignoring the $\mathbf{I}$ column for the moment). Then if we *foldover* $\mathbf{A}$ to obtain $-\mathbf{A}$, the block of signs

$$\begin{matrix} \mathbf{A} \\ -\mathbf{A} \end{matrix}$$

defines a $2_{2l}^{k-(p-1)}$ design. It is also possible to fold over the $\mathbf{I}$ column as well to introduce a new variable. The folded design can then be written

$$\begin{matrix} \mathbf{A} & \mathbf{I} \\ -\mathbf{A} & -\mathbf{I} \end{matrix}$$

which is a $2_{2l}^{(k+1)-p}$ design. Such a design can also be viewed as obtained by *attaching* an extra variable letter $\mathbf{k+1}$ to each of the 2^{p-1} odd words in the defining relation of the original design. Foldover allows the construction of saturated even resolution ($2l$ say) designs from saturated odd resolution $2l-1$ designs. To obtain the defining relation of a folded design we can either seek the common part of the defining relations of the two pieces (original and folded portion), or we can apply the U–L rule due to Box and Hunter (1961a, p. 328).

U–L Rule: Suppose two generating relations for two designs of the same family have L generators common and of the same (or LIKE) sign and $U=p-L$ generators which are the same apart from a sign change (i.e., are of UNLIKE sign). Then the L generators of like sign together with $U-1$ generators arising from independent even products of the U generators of unlike sign provide $L+U-1=p-1$ generators for the combined design.

Robillard rule. In the case $k=q+1$, the maximum resolution is obviously $R=k=q+1$. In terms of paragraph (3) of Section 1, this means that if $F=q+1$ then k must be $q+1$ and this is the maximum number of factors that can be accommodated. Robillard (1968) showed that, for the case $k=q+2$, $R_{\text{Max}}=[2k/3]$, where $[x]$ is the integer part of x. Thus, according to Robillard's rule, if $q+1 \geq R > [\frac{2}{3}(q+2)]$ then $k=q+1$ is the maximum number of factors that can be accommodated. Otherwise, if $R \leq [\frac{2}{3}(q+2)]$ then at least $q+2$ factors can be accommodated.

4. Designs of various resolutions

We now discuss the existence of saturated designs of resolutions from III to XIII for $q \leq 12$.

Fact 1. *The maximum number of variables, k_{Max}, that can be accommodated in a resolution* III *design of N $(=2^q)$ runs is 2^q-1.*

This can obviously be achieved by assigning new factors to all possible interaction effects generated from the original q basic factors. There are

$$\binom{q}{1}+\binom{q}{2}+\binom{q}{3}+\cdots+\binom{q}{q}=2^q-1$$

of these. For example, if $N=2^3$ and $R=\text{III}$, the basic factors are **1, 2, 3**, and the generators are **4(=12)**, **5(=13)**, **6(=23)** and **7(=123)**. Therefore, $k_{\text{Max}}=2^3-1=7$.

Fact 2. *The maximum number of variables, k_{Max}, that can be accommodated in a resolution* IV *design of N $(=2^q)$ runs is 2^{q-1}.*

This can be achieved by assigning new factors to all possible odd-ordered interaction effects. There are

$$\binom{q}{1}+\binom{q}{3}+\cdots+\binom{q}{q^*}=2^{q-1}$$

of these, where q^* is the largest odd integer not exceeding q. (See Box and Hunter, 1961, p. 341.) For example, if $N=2^4$ and $R=\text{IV}$, the basic factors are **1, 2, 3, 4**, and the generators are **5(=123)**, **6(=124)**, **7(=134)** and **8(=234)**. Therefore, $k_{\text{Max}}=2^{4-1}=8$.

Fact 3. *A saturated design of resolution $R=2l$ can be obtained by folding over a saturated design of resolution $2l-1$ plus an* **I** *column. Also, this procedure can be reversed via erasure of one variable.*

Thus Webb's (1968, p. 297) conjecture for the resolution IV case, proved by Margolin (1969), is re-confirmed and extended.

Fact 2 can be viewed as special case of Fact 3, i.e., folding over a $N=2^{q-1}$ run saturated resolution III design which contains $2^{q-1}-1$ factors results in a $N=2^q$ run saturated resolution IV design containing $(2^{q-1}-1)+1=2^{q-1}$ factors.

The following discussion on designs of resolutions III to XIII will focus on odd resolution cases. All saturated even resolution designs will automatically be constructed by foldover as described in Fact 3.

Resolution III. Fact 1 is applied to this case; $k_{\text{Max}}=N-1=2^q-1$ whenever $N=2^q$. However, finding the minimum aberration design of resolution III for $k<k_{\text{Max}}$ is still an interesting problem, although it will not be considered here.

Resolution V. Box and Hunter (1961) provided k_{Max} for cases $q\leq 7$. Also see Rao (1947). The value $k_{\text{Max}}=17$ for the case $q=8$ was conjectured by Addelman (1965) and later confirmed by Draper and Mitchell (1967). For $q=9$, see Mitchell (1966, p. 104). For cases $10\leq q\leq 12$, see Verhoeff (1987). The k_{Max} values are shown in Table 4.

Resolution VII. The program used by Draper and Lin (1988) is also employed here to produce Table 1, which lists resolution VII designs for $k=7, 8, \ldots, 24$.

Resolution IX. Table 2 shows computer generated saturated resolution IX designs for $k=9, 10, 11, 12, 13, 14$. In all cases $p\leq 2$, which means that Robillard's (1968) rule can be applied to confirm that the k values are maximal.

Table 1
Resolution VII designs

q	N	Type	Generators	k_{Max}
6	64	2_{VII}^{7-1}	$\pm 7 = 123456$	7
7	128	2_{VII}^{8-1}	$\pm 8 = 123456$	8
8	256	2_{VII}^{9-1}	$\pm 9 = 123456$	9
9	512	2_{VII}^{11-2}	$\pm\overline{10} = 123456$ $\pm\overline{11} = 123789$	11
10	1024	2_{VII}^{15-5}	$\pm\overline{11} = 123456$ $\pm\overline{12} = 123789$ $\pm\overline{13} = 14578\overline{10}$ $\pm\overline{14} = 24679\overline{10}$ $\pm\overline{15} = 35689\overline{10}$	15
11	2048	2_{VII}^{23-12}	$\pm\overline{12} = 123456$ $\pm\overline{13} = 123789$ $\pm\overline{14} = 14578\overline{10}$ $\pm\overline{15} = 24679\overline{10}$ $\pm\overline{16} = 35689\overline{10}$ $\pm\overline{17} = 34678\overline{11}$ $\pm\overline{18} = 15679\overline{10}$ $\pm\overline{19} = 24589\overline{11}$ $\pm\overline{20} = 2357\overline{10}\,\overline{11}$ $\pm\overline{21} = 1268\overline{10}\,\overline{11}$ $\pm\overline{22} = 1349\overline{10}\,\overline{11}$ $\pm\overline{23} = 123456789\overline{10}\,\overline{11}$	23
12	4096	2_{VII}^{24-12}	$\pm\overline{13} = 123456$ $\pm\overline{14} = 123789$ $\pm\overline{15} = 14578\overline{10}$ $\pm\overline{16} = 24679\overline{10}$ $\pm\overline{17} = 35689\overline{10}$ $\pm\overline{18} = 34678\overline{11}$ $\pm\overline{19} = 15679\overline{10}$ $\pm\overline{20} = 24589\overline{11}$ $\pm\overline{21} = 2357\overline{10}\,\overline{11}$ $\pm\overline{22} = 1268\overline{10}\,\overline{11}$ $\pm\overline{23} = 1349\overline{10}\,\overline{11}$ $\pm\overline{24} = 25678\overline{12}$	24

Resolution XI. Table 3 shows computer generated saturated resolution XI designs for $k = 11, 12, 13$. Again Robillard's rule confirms that only one generator is possible in all cases.

Resolution XIII. This is the maximum resolution that can be achieved for 2^{12} runs; this implies a 2_{XIII}^{13-1} design with one generator $\overline{13} = 123456789\overline{10}\,\overline{11}\,\overline{12}$.

Table 2
Resolution IX designs

q	$N\ (=2^q)$	Type	Generators	k_{Max}
8	256	2_{IX}^{9-1}	$\pm 9 = \mathbf{12345678}$	9
9	512	2_{IX}^{10-1}	$\pm \overline{10} = \mathbf{12345678}$	10
10	1024	2_{IX}^{11-1}	$\pm \overline{11} = \mathbf{12345678}$	11
11	2048	2_{IX}^{12-1}	$\pm \overline{12} = \mathbf{12345678}$	12
12	4096	2_{IX}^{14-2}	$\pm \overline{13} = \mathbf{12345678}$ $\pm \overline{14} = \mathbf{12349}\overline{10}\,\overline{11}\,\overline{12}$	14

Table 3
Resolution XI designs

q	$N\ (=2^q)$	Type	Generators	k_{Max}
10	1024	2_{XI}^{11-1}	$\pm \overline{11} = \mathbf{123456789}\overline{10}$	11
11	2048	2_{XI}^{12-1}	$\pm \overline{12} = \mathbf{123456789}\overline{10}$	12
12	4096	2_{XI}^{13-1}	$\pm \overline{13} = \mathbf{123456789}\overline{10}$	13

Table 4 summarises the results above. This table can be used in three ways: To solve the first step in *minimum aberration problems* (searching for maximum R, given k and q), to solve *minimum design problems* (searching for minimum N, given k and R) and to solve *most factors problems* (searching for maximum k, given q and R). We next illustrate these three cases more fully with examples.

Table 4
Maximal number of factors in two-level designs of resolution R

q	3	4	5	6	7	8	9	10	11	12
$N\ (=2^q)$	8	16	32	64	128	256	512	1024	2048	4096
Resolution III	7	15	31	63	127	255	511	1023	2047	4095
Resolution IV	4	8	16	32	64	128	256	512	1024	2048
Resolution V	–	5	6	8	11	17	23	32[a]	41[a]	65[a]
Resolution VI	–	–	6	7	9	12	18	24	33[a]	41[a]
Resolution VII	–	–	–	7	8	9	11	15	23	24
Resolution VIII	–	–	–	–	8	9	10	12	16	24
Resolution IX	–	–	–	–	–	9	10	11	12	14
Resolution X	–	–	–	–	–	–	10	11	12	13
Resolution XI	–	–	–	–	–	–	–	11	12	13
Resolution XII	–	–	–	–	–	–	–	–	12	13
Resolution XIII	–	–	–	–	–	–	–	–	–	13

[a] These values are believed to be maximal but, unlike the rest, are not guaranteed.

(1) Suppose $k=25$, $N=512$ $(=2^9)$. Table 4 shows, in the $q=9$ column, that a resolution V plan can accommodate at most 23 factors, whereas a resolution IV plan can accommodate up to 256 factors. Thus $R=\text{IV}$ is the best achievable for $k=25$, $q=9$.

(2) Suppose we choose $k=10$ and $R=\text{V}$; how many runs are needed? In the resolution V row of Table 4, we see that a 64-run design can accommodate eight factors, whereas a 128-run design can accommodate 11 factors. Therefore, $N=128$ $(=2^7)$ runs are needed.

(3) Suppose we choose $N=256$ (i.e., $q=8$) and $R=\text{V}$. In Table 4 the entry in the $q=8$ column and $R=\text{V}$ row gives the maximum number of factors that can be accommodated as 17.

Some of the entries in Table 4 provide corrections to values given by Fries and Hunter (1980, p. 605, Table 2). Appropriate specific changes are given in Table 5. Similar changes are implicit in Franklin (1984, Table 1).

Table 5
Changes to Fries and Hunter (1980, p. 605, Table 2)

k	p	$q=k-p$	R_{Max} in Fries & Hunter	Change R_{Max} to
12	5	7	V	IV
13	5	8	IV	V
13[a]	3	10	VI	VII
14	5	9	V	VI
14	4	10	VI	VII
14[a]	3	11	VII	VIII

[a] In these two cases, Fries and Hunter's (1980, p. 605, Table 2) R_{Max} bounds (1) and (2) must also be increased by one unit.

5. Designs of resolution star

A two-level fractional factorial design will be said to be of resolution R^* ('resolution R star'), if it is of resolution R and there is no word in the defining relation of length $R+1$. Resolution III* designs, namely, resolution III designs with no four-letter word in defining relation, were used by Hartley (1959). For additional discussion, see Westlake (1965) and Draper and Lin (1988). Any design of even resolution, $2l$ say, obtained by foldover is automatically a resolution $(2l)^*$ design because its defining relation will contain no words of odd length.

Foldover also provides information in searches for k_{Max}. For example, suppose a 2_{III}^{k-p} design is folded over. The resulting design has 2^{k-p+1} runs. The folded portion has a word length pattern identical to that of the original design, namely $\gamma=(\gamma_3, 0, \gamma_5, \gamma_6, \ldots, \gamma_k)$. However, the signs of all the odd-length words in its defining relation are reversed from these in the original design. The combined design will thus have a defining relation consisting of all the (common) even-length words and the word length pattern will have $\gamma_i = 0$ for all odd i, while γ_4 remains 0; thus the new $\gamma = (\gamma_6, 0, \gamma_8, 0, \gamma_{10}, \ldots, \gamma_k)$. We thus obtain a resolution VI design. If the resolution III* design that is folded is saturated, so is the resulting resolution VI design. Another way of obtaining a saturated resolution VI design by foldover is to fold a resolution V design together with its **I** column to introduce a new variable, as in Fact 3. These relationships are illustrated in Figure 1. (For the relationship between resolution III* and V designs, see Draper and Lin, 1988.)

The situation described in the foregoing paragraph extends to a general statement as follows.

Fact 4. *If an N $(=2^q)$-run design of resolution $(2l-1)^*$ in k factors is folded over, the combined design is*:

(a) *a $2N$ $(=2^{q+1})$-run design of resolution $2l+2$ if the original design has $\gamma_{2l+2} \neq 0$, or*

(b) *a $2N$ $(=2^{q+1})$-run full factorial design if $k < 2l+2$, or*

(c) *a $2N$ $(=2^{q+1})$-run design of even resolution higher than $2l+2$, otherwise.*

Fact 4 can be established by an argument similar to that above for the III* case which is covered by Fact 4. The general relationship is illustrated in Figure 2. Note that two of the arrows are one-way unlike these in Figure 1. (It is easily confirmed that only one-way arrows are possible when $l > 2$.)

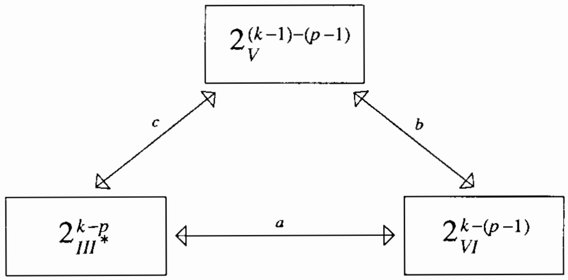

Fig. 1. Relationship among 2_{III}^{k-p}, $2_{\mathrm{V}}^{(k-1)-(p-1)}$ and $2_{\mathrm{VI}}^{k-(p-1)}$ designs. (a) Foldover. (b) Foldover (plus the **I** column) and reverse by erasure. (c) See Draper and Lin (1988).

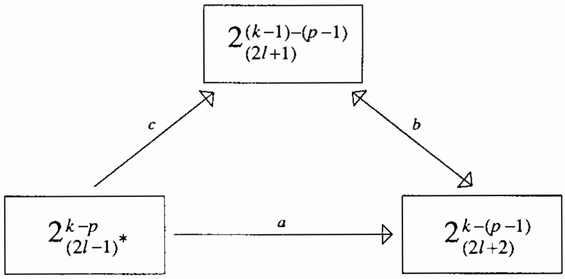

Fig. 2. Relationship among $2_{(2l-1)^*}^{k-p}$, $2_{2l+1}^{(k-1)-(p-1)}$ and $2_{2l+2}^{k-(p-1)}$ designs. (a) Foldover. (b) Foldover (plus the **I** column) and reverse by erasure. (c) See Draper and Lin (1988).

Table 6
Maximal number of factors in two-level designs of resolution R^*

q	3	4	5	6	7	8	9	10	11	12
$N\ (=2^q)$	8	16	32	64	128	256	512	1024	2048	4096
Resolution III*	4	6	7	9	12	18	24	33[a]	42[a]	66[a]
Resolution IV*	4	8	16	32	64	128	256	512	1024	2048
Resolution V*	–	5	6	8	9	10	12	16	23	25[a]
Resolution VI*	–	–	6	7	9	12	18	24	33[a]	42[a]
Resolution VII*	–	–	–	7	8	9	10	12	13	15
Resolution VIII*	–	–	–	–	8	9	10	12	16	24
Resolution IX*	–	–	–	–	–	9	10	11	12	14
Resolution X*	–	–	–	–	–	–	10	11	12	13
Resolution XI*	–	–	–	–	–	–	–	11	12	13
Resolution XII*	–	–	–	–	–	–	–	–	12	13
Resolution XIII*	–	–	–	–	–	–	–	–	–	13

[a] These values are believed to be maximal but, unlike the rest, are not guaranteed.

Table 6 shows the maximum number of factors that can be accommodated in resolution star designs for $q \le 12$. Note that, when R is even, the k values are identical to those in Table 4.

Remarks on Table 6. (1) If a design of resolution star is used (Hartley, 1959; Westlake, 1965), then the relationships among q (or equivalently N), R^*, and k can be found from Table 6. A similar discussion on Table 4 allows us to find R^*_{Max} (when k and q are given), or N_{Min} (when R^* and k are given), or k_{Max} (when q and R^* are specified).

(2) The relationships between designs of resolution $(2l-1)^*$ and resolution $2l+1$ shown in Figure 2, allows a cross check between Table 4 and Table 6. These connecting relationships have been studied by Draper and Lin (1988) for the case $l=2$, and will be further investigated.

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