

# **Fundamentals of Probability: A First Course**

Anirban DasGupta

# Contents

|          |                                                         |           |
|----------|---------------------------------------------------------|-----------|
| <b>1</b> | <b>Introducing Probability</b>                          | <b>5</b>  |
| 1.1      | Experiments and Sample Spaces . . . . .                 | 6         |
| 1.2      | Set Theory Notation and Axioms of Probability . . . . . | 7         |
| 1.3      | How to Interpret a Probability . . . . .                | 10        |
| 1.4      | Calculating Probabilities . . . . .                     | 12        |
| 1.4.1    | Manual Counting . . . . .                               | 13        |
| 1.4.2    | General Counting Methods . . . . .                      | 15        |
| 1.5      | Inclusion Exclusion Formula . . . . .                   | 18        |
| 1.6      | * Bounds on the Probability of a Union . . . . .        | 21        |
| 1.7      | Synopsis . . . . .                                      | 22        |
| 1.8      | Exercises . . . . .                                     | 22        |
| 1.9      | References . . . . .                                    | 28        |
| <b>2</b> | <b>The Birthday and the Matching Problem</b>            | <b>30</b> |
| 2.1      | The Birthday Problem . . . . .                          | 30        |
| 2.1.1    | * Stirling's Approximation . . . . .                    | 31        |
| 2.2      | The Matching Problem . . . . .                          | 32        |
| 2.3      | Synopsis . . . . .                                      | 34        |
| 2.4      | Exercises . . . . .                                     | 34        |
| 2.5      | References . . . . .                                    | 35        |
| <b>3</b> | <b>Conditional Probability and Independence</b>         | <b>36</b> |
| 3.1      | Basic Formulas and First Examples . . . . .             | 36        |
| 3.2      | More Advanced Examples . . . . .                        | 38        |
| 3.3      | Independent Events . . . . .                            | 41        |
| 3.4      | Bayes Theorem . . . . .                                 | 44        |
| 3.5      | Synopsis . . . . .                                      | 47        |
| 3.6      | Exercises . . . . .                                     | 48        |
| <b>4</b> | <b>Integer Valued and Discrete Random Variables</b>     | <b>53</b> |
| 4.1      | Mass Function . . . . .                                 | 53        |
| 4.2      | CDF and Median of a Random Variable . . . . .           | 55        |
| 4.2.1    | Functions of a Random Variable . . . . .                | 62        |
| 4.2.2    | Independence of Random Variables . . . . .              | 64        |
| 4.3      | Expected Value of a Discrete Random Variable . . . . .  | 65        |

|          |                                                               |            |
|----------|---------------------------------------------------------------|------------|
| 4.4      | Basic Properties of Expectations . . . . .                    | 66         |
| 4.5      | Illustrative Examples . . . . .                               | 68         |
| 4.6      | Using Indicator Variables to Calculate Expectations . . . . . | 70         |
| 4.7      | The Tail Sum Method for Calculating Expectations . . . . .    | 71         |
| 4.8      | Variance, Moments, and Basic Inequalities . . . . .           | 73         |
| 4.9      | Illustrative Examples . . . . .                               | 75         |
| 4.9.1    | Variance of a Sum of Independent Random Variables . . .       | 77         |
| 4.10     | Utility of $\mu, \sigma$ as Summaries . . . . .               | 78         |
| 4.10.1   | Chebyshev's Inequality and Weak Law of Large Numbers          | 78         |
| 4.10.2   | * Better Inequalities . . . . .                               | 80         |
| 4.11     | * Other Fundamental Moment Inequalities . . . . .             | 81         |
| 4.11.1   | * Applying Moment Inequalities . . . . .                      | 83         |
| 4.12     | Truncated Distributions . . . . .                             | 84         |
| 4.13     | Synopsis . . . . .                                            | 85         |
| 4.14     | Exercises . . . . .                                           | 87         |
| 4.15     | References . . . . .                                          | 92         |
| <b>5</b> | <b>Generating Functions</b>                                   | <b>93</b>  |
| 5.1      | Generating Functions . . . . .                                | 93         |
| 5.2      | Moment Generating Functions and Cumulants . . . . .           | 97         |
| 5.2.1    | * Cumulants . . . . .                                         | 99         |
| 5.3      | Synopsis . . . . .                                            | 101        |
| 5.4      | Exercises . . . . .                                           | 101        |
| 5.5      | References . . . . .                                          | 103        |
| <b>6</b> | <b>Standard Discrete Distributions</b>                        | <b>104</b> |
| 6.1      | Introduction to Special Distributions . . . . .               | 104        |
| 6.2      | Discrete Uniform Distribution . . . . .                       | 107        |
| 6.3      | Binomial Distribution . . . . .                               | 108        |
| 6.4      | Geometric and Negative Binomial Distribution . . . . .        | 113        |
| 6.5      | Hypergeometric Distribution . . . . .                         | 116        |
| 6.6      | Poisson Distribution . . . . .                                | 118        |
| 6.6.1    | Mean Absolute Deviation and the Mode . . . . .                | 123        |
| 6.7      | Poisson Approximation to Binomial . . . . .                   | 124        |
| 6.8      | * Miscellaneous Poisson Approximations . . . . .              | 127        |
| 6.9      | Benford's Law . . . . .                                       | 130        |

|          |                                                                    |            |
|----------|--------------------------------------------------------------------|------------|
| 6.10     | Distribution of Sums and Differences . . . . .                     | 131        |
| 6.10.1   | * Distribution of Differences . . . . .                            | 133        |
| 6.11     | * Discrete Does Not Mean Integer Valued . . . . .                  | 134        |
| 6.12     | Synopsis . . . . .                                                 | 135        |
| 6.13     | Exercises . . . . .                                                | 136        |
| 6.14     | References . . . . .                                               | 141        |
| <b>7</b> | <b>Continuous Random Variables</b>                                 | <b>143</b> |
| 7.1      | The Density Function and the CDF . . . . .                         | 143        |
| 7.1.1    | Quantiles . . . . .                                                | 149        |
| 7.2      | Generating New Distributions from Old. . . . .                     | 151        |
| 7.3      | Normal and Other Symmetric Unimodal Densities . . . . .            | 154        |
| 7.4      | Functions of a Continuous Random Variable . . . . .                | 157        |
| 7.4.1    | Quantile Transformation . . . . .                                  | 162        |
| 7.4.2    | Cauchy density . . . . .                                           | 163        |
| 7.5      | Expectation of Functions and Moments . . . . .                     | 164        |
| 7.6      | The Tail Probability Method for Calculating Expectations . . . . . | 172        |
| 7.6.1    | * Survival and Hazard Rate . . . . .                               | 173        |
| 7.6.2    | * Moments and the Tail . . . . .                                   | 173        |
| 7.7      | * Moment Generating Function and Fundamental Tail Inequalities .   | 175        |
| 7.7.1    | * Chernoff-Bernstein Inequality . . . . .                          | 176        |
| 7.7.2    | * Lugosi's Improved Inequality . . . . .                           | 178        |
| 7.8      | * Jensen and Other Moment Inequalities and a Paradox . . . . .     | 179        |
| 7.9      | Synopsis . . . . .                                                 | 181        |
| 7.10     | Exercises . . . . .                                                | 184        |
| 7.11     | References . . . . .                                               | 188        |
| <b>8</b> | <b>Some Special Continuous Distributions</b>                       | <b>189</b> |
| 8.1      | Uniform Distribution . . . . .                                     | 189        |
| 8.2      | Exponential and Weibull Distributions . . . . .                    | 191        |
| 8.3      | Gamma and Inverse Gamma Distributions . . . . .                    | 195        |
| 8.4      | Beta Distribution . . . . .                                        | 200        |
| 8.5      | Extreme Value Distributions . . . . .                              | 203        |
| 8.6      | * Exponential Density and the Poisson Process . . . . .            | 206        |
| 8.7      | Synopsis . . . . .                                                 | 209        |
| 8.8      | Exercises . . . . .                                                | 210        |

|           |                                                                     |            |
|-----------|---------------------------------------------------------------------|------------|
| 8.9       | References . . . . .                                                | 214        |
| <b>9</b>  | <b>Normal Distribution</b>                                          | <b>216</b> |
| 9.1       | Definition and Basic Properties . . . . .                           | 216        |
| 9.2       | Working with a Normal Table . . . . .                               | 220        |
| 9.3       | Additional Examples and the Lognormal Density . . . . .             | 221        |
| 9.4       | Sums of Independent Normal Variables . . . . .                      | 225        |
| 9.5       | Mills Ratio and Approximations for the Standard Normal CDF . . .    | 227        |
| 9.6       | Synopsis . . . . .                                                  | 229        |
| 9.7       | Exercises . . . . .                                                 | 231        |
| 9.8       | References . . . . .                                                | 234        |
| <b>10</b> | <b>Normal Approximations and Central Limit Theorem</b>              | <b>236</b> |
| 10.1      | Some Motivating Examples . . . . .                                  | 236        |
| 10.2      | Central Limit Theorem . . . . .                                     | 238        |
| 10.3      | Normal Approximation to Binomial . . . . .                          | 240        |
| 10.3.1    | Continuity Correction . . . . .                                     | 241        |
| 10.3.2    | A New Rule of Thumb . . . . .                                       | 245        |
| 10.4      | Examples of the General CLT . . . . .                               | 248        |
| 10.5      | Normal Approximation to Poisson and Gamma . . . . .                 | 253        |
| 10.6      | * Convergence of Densities and Higher Order Approximations . . .    | 256        |
| 10.6.1    | * Refined Approximations . . . . .                                  | 257        |
| 10.7      | Practical Recommendations for Normal Approximations . . . . .       | 260        |
| 10.8      | Synopsis . . . . .                                                  | 261        |
| 10.9      | Exercises . . . . .                                                 | 262        |
| 10.10     | References . . . . .                                                | 267        |
| <b>11</b> | <b>Multivariate Discrete Distributions</b>                          | <b>269</b> |
| 11.1      | Bivariate Joint Distributions and Expectations of Functions . . . . | 269        |
| 11.2      | Conditional Distributions and Conditional Expectations . . . . .    | 276        |
| 11.2.1    | Examples on Conditional Distributions and Expectations              | 278        |
| 11.3      | Using Conditioning to Evaluate Mean and Variance . . . . .          | 282        |
| 11.4      | Covariance and Correlation . . . . .                                | 285        |
| 11.5      | Multivariate Case . . . . .                                         | 290        |
| 11.5.1    | * Joint MGF . . . . .                                               | 291        |
| 11.5.2    | Multinomial Distribution . . . . .                                  | 293        |

|           |                                                                   |            |
|-----------|-------------------------------------------------------------------|------------|
| 11.6      | Synopsis . . . . .                                                | 296        |
| 11.7      | Exercises . . . . .                                               | 298        |
| <b>12</b> | <b>Multidimensional Densities</b>                                 | <b>303</b> |
| 12.1      | Joint Density Function and Its Role . . . . .                     | 303        |
| 12.2      | Expectation of Functions . . . . .                                | 313        |
| 12.3      | Bivariate Normal . . . . .                                        | 318        |
| 12.4      | Conditional Densities and Expectations . . . . .                  | 323        |
| 12.4.1    | Examples on Conditional Densities and Expectations . . .          | 325        |
| 12.5      | Bivariate Normal Conditional Distributions . . . . .              | 331        |
| 12.6      | Order Statistics . . . . .                                        | 333        |
| 12.6.1    | Basic Distribution Theory . . . . .                               | 334        |
| 12.6.2    | * More Advanced Distribution Theory . . . . .                     | 337        |
| 12.7      | Synopsis . . . . .                                                | 341        |
| 12.8      | Exercises . . . . .                                               | 345        |
| 12.9      | References . . . . .                                              | 350        |
| <b>13</b> | <b>Convolutions and Transformations</b>                           | <b>351</b> |
| 13.1      | Convolutions and Examples . . . . .                               | 351        |
| 13.2      | Products and Quotients and the $t$ and $F$ Distribution . . . . . | 356        |
| 13.3      | Transformations . . . . .                                         | 361        |
| 13.4      | Applications of Jacobian Formula . . . . .                        | 362        |
| 13.5      | Polar Coordinates in Two Dimensions . . . . .                     | 364        |
| 13.6      | Synopsis . . . . .                                                | 367        |
| 13.7      | Exercises . . . . .                                               | 369        |
| 13.8      | References . . . . .                                              | 373        |
| <b>14</b> | <b>Markov Chains and Applications</b>                             | <b>374</b> |
| 14.1      | Notation and Basic Definitions . . . . .                          | 375        |
| 14.2      | Chapman-Kolmogorov Equation . . . . .                             | 381        |
| 14.3      | Communicating Classes . . . . .                                   | 386        |
| 14.4      | * Gambler's Ruin . . . . .                                        | 388        |
| 14.5      | * First Passage, Recurrence and Transience . . . . .              | 390        |
| 14.6      | Long Run Evolution and Stationary Distributions . . . . .         | 397        |
| 14.7      | Synopsis . . . . .                                                | 405        |
| 14.8      | Exercises . . . . .                                               | 406        |

|           |                                                                 |            |
|-----------|-----------------------------------------------------------------|------------|
| 14.9      | References . . . . .                                            | 416        |
| <b>15</b> | <b>Urn Models in Physics and Genetics</b>                       | <b>417</b> |
| 15.1      | Stirling Numbers and Their Basic Properties . . . . .           | 417        |
| 15.2      | Urn Models in Quantum Mechanics . . . . .                       | 420        |
| 15.3      | * Poisson Approximations . . . . .                              | 424        |
| 15.4      | Pólya's Urn . . . . .                                           | 426        |
| 15.5      | Pólya-Eggenberger Distribution . . . . .                        | 429        |
| 15.6      | * de Finetti's Theorem and Pólya Urns . . . . .                 | 430        |
| 15.7      | Urn Models in Genetics . . . . .                                | 432        |
| 15.7.1    | Wright-Fisher Model . . . . .                                   | 433        |
| 15.7.2    | Time until Allele Uniformity . . . . .                          | 435        |
| 15.8      | Mutation and Hoppe's Urn . . . . .                              | 436        |
| 15.9      | * The Ewens Sampling Formula . . . . .                          | 440        |
| 15.10     | Synopsis . . . . .                                              | 443        |
| 15.11     | Exercises . . . . .                                             | 445        |
| 15.12     | References . . . . .                                            | 449        |
| <b>16</b> | <b>Appendix I: Supplementary Homework and Practice Problems</b> | <b>451</b> |
| 16.1      | Word Problems . . . . .                                         | 451        |
| 16.2      | True-False Problems . . . . .                                   | 471        |
| <b>17</b> | <b>Appendix II</b>                                              | <b>478</b> |
| 17.1      | Glossary of Symbols . . . . .                                   | 478        |
| 17.2      | Formula Summaries . . . . .                                     | 481        |
| 17.2.1    | Moments and MGFs of Common Distributions . . . . .              | 481        |
| 17.2.2    | Useful Mathematical Formulas . . . . .                          | 487        |
| 17.2.3    | Useful Calculus Facts . . . . .                                 | 488        |
| 17.3      | Tables . . . . .                                                | 489        |
| 17.3.1    | Normal Table . . . . .                                          | 489        |
| 17.3.2    | Poisson Table . . . . .                                         | 491        |