Journal of Statistical Planning and Inference ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Hamburger moment problem for powers and products of random variables

Jordan Stoyanov [a,*], Gwo Dong Lin [b], Anirban DasGupta [c]

[a] *School of Mathematics & Statistics, Newcastle University, Newcastle upon Tyne, NE1 7RU, UK*
[b] *Institute of Statistical Science, Academia Sinica, Taipei 11529, Taiwan, ROC*
[c] *Department of Statistics, Purdue University, West Lafayette, IN 47907-2068, USA*

A R T I C L E I N F O

MSC (2010):
60E05
44A60

Keywords:
Hamburger moment problem
Powers and products of random variables
Carleman's condition
Cramér's condition
Hardy's condition
Krein's condition
Normal distribution
Laplace distribution
Double generalized gamma distribution
Logistic distribution

A B S T R A C T

We present new results on the Hamburger moment problem for probability distributions and apply them to characterize the moment determinacy of powers and products of i.i.d. random variables with values in the whole real line. Detailed proofs of all results are given followed by comments and examples. We also provide new and more transparent proofs of a few known results. E.g., we give a new and short proof that the product of three or more i.i.d. normal random variables is moment-indeterminate. The illustrations involve specific distributions such as the double generalized gamma (DGG), normal, Laplace and logistic. We show that sometimes, but not always, the power and the product of i.i.d. random variables (of the same odd 'order') share the same moment determinacy property. This is true for the DGG and the logistic distributions.

The paper also treats two unconventional types of problems: products of independent random variables of different types and a random power of a given random variable. In particular, we show that the product of Laplace and logistic random variables, the product of logistic and exponential random variables, the product of normal and χ^2 random variables, and the random power Z^N, where $Z \sim \mathcal{N}$ and N is a Poisson random variable, are all moment-indeterminate.

1. Introduction

Our goal in this paper is to first obtain some general results on moment determinacy of probability distributions. Then we apply these results to study the moment determinacy of powers and products of i.i.d. random variables taking values in the whole real line $\mathbb{R}$. Hence, we deal with the Hamburger moment problem. This problem has a rich history and is considered to be a classical problem in mathematics and probability, see Akhiezer (1965), Shohat and Tamarkin (1943) or Simon (1998). The review paper by Diaconis (1987) is a very readable and useful account on the topic showing the role of moments in probability and statistics.

The powers and products are relatively easy nonlinear transformations of random data, however their study leads to challenging problems, see, e.g., Slud (1993), DasGupta (1997), Lin and Huang (1997), Galambos and Simonelli (2004), Berg (1988, 2005), Pakes (2001), and Lin and Stoyanov (2013). Interestingly, these transformations are the basis of stochastic

* Corresponding author. Tel.: +44 191 2227239; fax: +44 191 2228020.
E-mail addresses: stoyanovj@gmail.com (J. Stoyanov), gdlin@stat.sinica.edu.tw (G.D. Lin), dasgupta@stat.purdue.edu (A. DasGupta).

0378-3758/$ - see front matter
http://dx.doi.org/10.1016/j.jspi.2013.11.002

models of complex practical phenomena. Among the existing works, we mention here just a few sources: Carmona and Molchanov (1993), Frisch and Sornette (1997), Galambos and Simonelli (2004), and De Abreu (2010). Hence, our results and the specific distributions considered as examples are not of academic interest only.

The Stieltjes moment problem for powers and products of positive random variables was studied recently by Lin and Stoyanov (2013). It is more delicate to deal with variables taking values in $\mathbb{R}$. Some ideas and techniques used in the Stieltjes case, after being appropriately extended and considered together with additional arguments, allow us to derive results in the Hamburger case.

We use traditional notions, notations and terms such as Cramér's condition, Carleman's condition, Krein's condition, and a few more, without giving their definitions. The reader can consult several available sources; among them is Stoyanov (2013). In particular, we use the abbreviations 'M-det' and 'M-indet' for a random variable and also for its distribution which is moment-determinate or moment-indeterminate, respectively.

In the sequel we use the symbol $\simeq$ for asymptotic equivalence of real-valued functions or sequences with the usual meaning of the symbols $\mathcal{O}$ and o. We write $X \sim F$ to say that X is a random variable with distribution function F.

The principal contributions of this paper are the following:

(a) We formalize the notion of 'growth rate' of the even order moments and prove general results that establish that slow growth leads to moment determinacy (Theorems 1, 2, and 2′), and fast growth to moment indeterminacy (Theorem 3).
(b) We apply these general results to obtain new determinacy and indeterminacy results for powers and products, and also to provide shorter and transparent proofs of some known and by now classic results (Theorem 4, Propositions 1 and 1′, and Corollaries 2 and 3).
(c) We give several diverse illustrative examples of application of our general results in special parametric families of distributions, such as Laplace, double generalized gamma, logistic (Theorems 5 and 6, and Lemma 4).
(d) We provide a set of initial results (Theorems 7–10 and Corollary 5) for the new problems asking how to characterize the moment (in)determinacy of random powers, products of a random number of i.i.d. random variables, and products of independent random variables of different types, e.g., Laplace, logistic, normal, half-normal, χ^2 and exponential.

2. General theorems: proofs, comments and corollaries

Suppose that X is a random variable taking values in the whole real line $\mathbb{R}=(-\infty,\infty)$ and that all moments of X are finite. This means that $\mathbf{E}[|X|^k]<\infty$ for any $k=1,2,\ldots$ and we denote by $m_k=\mathbf{E}[X^k]$ the kth order moment of X. To avoid trivial cases, we assume that X is not degenerate at any point, so all even order moments are positive: $m_{2k}>0$. A classical question to ask is about the uniqueness of X, or, equivalently, of its distribution, in terms of the moment sequence $\{m_k, k=1,2,\ldots\}$. This is the determinacy question in the Hamburger moment problem, see Akhiezer (1965), Shohat and Tamarkin (1943) or Simon (1998).

It turns out some conclusions about a distribution on $\mathbb{R}$ can be drawn from the subsequence $\{m_{2k}, k=1,2,\ldots\}$ of the even order moments instead of involving the entire moment sequence $\{m_k, k=1,2,\ldots\}$.

Let us start with a simple preliminary statement.

Lemma 1. *For each* $k\geq 1$, *we have*

(i) $m_{2k}\leq m_{2(k+1)}$ *assuming that* $m_2\geq 1$;
(ii) $m_2 m_{2k}\leq m_{2(k+1)}$;
(iii) $m_{2(k+1)}/m_{2k}\leq m_{2(k+2)}/m_{2(k+1)}$.

Proof. Notice that claim (i) means that the subsequence $\{m_{2k},\ k=1,2,\ldots\}$ is increasing in k if $m_2\geq 1$. Both claims (i) and (ii) easily follow by applying Lyapunov's inequality, see, e.g., Shiryaev (1996) or DasGupta (2008). Claim (iii) is a consequence of Hölder's inequality. □

2.1. Slow growth rate of the moments implies moment determinacy

We start with a random variable $X\sim F$ with values in $\mathbb{R}$ and all moments finite and let $\{m_k,\ k=1,2,\ldots\}$ be its moment sequence. Since the odd order moments can be negative or equal to zero, it is reasonable to use the even order moments in deriving properties of F. We want to introduce and exploit a number, say ρ, which characterizes the 'growth rate' of the even order moments and is related to the moment determinacy of F. For this purpose, consider the ratio

$$\Delta_{k+1}:=m_{2(k+1)}/m_{2k}$$

From Lemma 1 (iii), Δ_{k+1} is increasing in k. Suppose that there exist two numbers, $\rho\geq 0$ and $C_\rho\in(0,\infty)$, such that

$$\Delta_{k+1}\simeq C_\rho (k+1)^\rho \quad \text{as } k\to\infty. \tag{1}$$

In such a case we refer to ρ as the ‘growth rate’ of the even order moments of X and F. However, instead of (1), the following weaker condition

$$m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^{\rho}) \quad \text{for large } k \tag{2}$$

will be good enough for us to characterize the moment determinacy of F (see Theorem 1 below).

For X with values in $\mathbb{R}$ or its subset, ρ can be zero or positive, or maybe such a ρ does not exist. The latter is true for the lognormal distribution. We will see below that the value of ρ is essential for the moment (in)determinacy of F.

Theorem 1. *Suppose the ratio of the even order moments of X is $m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^2)$ as $k\to\infty$, i.e., (2) holds with $\rho=2$. Then X satisfies Carleman's condition and hence is M-det.*

Proof. It easily follows from the assumption and Lyapunov's inequality that there exists a constant $c>0$ such that

$$m_{2k}^{2(k+1)/(2k)} \le m_{2(k+1)} \le c(k+1)^2 m_{2k} \quad \text{for all large } k.$$

This implies

$$m_{2k}^{1/k} \le c(k+1)^2 \quad \text{for all large } k,$$

and hence

$$m_{2k}^{-1/(2k)} \ge c^{-1/2}(k+1)^{-1} \quad \text{for large } k.$$

Therefore, X satisfies Carleman's condition $\sum_{k=1}^{\infty} m_{2k}^{-1/(2k)} = \infty$, which is sufficient for X to be M-det. □

Remark 1. The power $\rho=2$ in the condition of Theorem 1 is the best possible in the following sense. For each $\varepsilon>0$, there exists a random variable ξ such that its even order moments satisfy the condition $m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^{2+\varepsilon})$ as $k\to\infty$, and ξ is M-indet. To see this, consider a random variable ξ with the density $f(x)=c\exp(-|x|^{\beta})$, $x\in\mathbb{R}$. In fact, $\xi\sim DGG(1,\beta,1)$, where DGG is the class of the double generalized gamma distributions; for more details see Section 3 below. By using the well-known Stirling's approximation for the gamma-function: $\Gamma(x+1)=x\Gamma(x)\simeq\sqrt{2\pi}x^{x+1/2}e^{-x}$ as $x\to\infty$ (see, e.g., Whittaker and Watson, 1927, p. 253), we obtain

$$\frac{\mathbf{E}[\xi^{2(k+1)}]}{\mathbf{E}[\xi^{2k}]} = \frac{\Gamma((1+2(k+1))/\beta)}{\Gamma((1+2k)/\beta)} = \mathcal{O}\left((k+1)^{2/\beta}\right) \quad \text{as } k\to\infty.$$

For each $\varepsilon>0$ and $\beta=1/(1+\varepsilon/2)$ we find $\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}] = \mathcal{O}((k+1)^{2+\varepsilon})$ as $k\to\infty$. However, for $\beta<1$ the Krein quantity of the density f is finite, i.e., $\mathrm{K}[f] := \int_{-\infty}^{\infty}(-\log f(x)/(1+x^2))\,dx<\infty$. This is a sufficient condition for ξ to be M-indet. The M-indet property of ξ also follows from Lemma 4 below.

Under the same assumption as that in Theorem 1, we have an even stronger statement, see Theorem 2 below. Note that its proof does not use Lyapunov's inequality, and that Cramér's condition implies Carleman's condition, because its equivalent condition $\limsup_{k\to\infty}(1/2k)m_{2k}^{1/(2k)}<\infty$ implies $\sum_{k=1}^{\infty} m_{2k}^{-1/(2k)}=\infty$, see Shiryaev (1996). Recall that the converse is not in general true.

Theorem 2. *Suppose X is a random variable with all moments finite, such that the condition in Theorem 1 holds: $m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^2)$ as $k\to\infty$. Then X satisfies Cramér's condition, and hence is M-det.*

To prove Theorem 2, we apply a characterization of Hardy's condition, which was established in Stoyanov and Lin (2012, Theorem 3). In Lemma 2 below, c and c_0 are constants, and for reader's convenience we state Hardy's condition explicitly.

Lemma 2. *Let U be a nonnegative random variable. Then for a real number $a\in(0,1]$, $\mathbf{E}[\exp(cU^a)]<\infty$ for some $c>0$ iff $\mathbf{E}[U^k]\le c_0^k\Gamma(k/a+1)$, $k=1,2,\ldots$, for some $c_0>0$ (independent of k). In particular, U satisfies Hardy's condition, i.e., $\mathbf{E}[\exp(c\sqrt{U})]<\infty$ for some $c>0$, iff $\mathbf{E}[U^k]\le c_0^k(2k)!$, $k=1,2,\ldots$, for some $c_0>0$ (independent of k).*

Proof of Theorem 2. By the assumption, there exists a constant $c_*\ge m_2>0$ such that

$$m_{2(k+1)} \le c_*(k+1)^2 m_{2k} \quad \text{for } k=0,1,2,\ldots,$$

where $m_0\equiv 1$. This implies that

$$m_{2(k+1)} \le (c_*/2)(2k+2)(2k+1)m_{2k} \quad \text{for } k=0,1,2,\ldots,$$

and hence $m_{2(k+1)} \le (c_*/2)^{k+1}\Gamma(2k+3)m_0$ for $k=0,1,2,\ldots$.

Taking $c_0=c_*/2$, we obtain

$$m_{2(k+1)} \le c_0^{k+1}\Gamma(2k+3) \quad \text{for } k=0,1,2,\ldots,$$

or, equivalently,

$$m_{2k} \le c_0^k\Gamma(2k+1) \quad \text{for } k=1,2,\ldots.$$

Hence $U:=|X|^2$ satisfies Hardy's condition by Lemma 2: $\mathbf{E}[e^{c|X|}] < \infty$ for some constant $c > 0$. This means that X itself satisfies Cramér's condition. (Another approach is to prove directly that $\limsup_{k\to\infty}(1/2k)m_{2k}^{1/(2k)} < \infty$.) This completes the proof. □

Denote by $\lfloor a \rfloor$ the largest integer less than or equal to a real number a. We now slightly extend Theorem 2 as follows.

Theorem $2'$**.** *Suppose X is a random variable such that for some real $a \ge 1$ its even order moments satisfy the relation $m_{2(k+1)}/m_{2k} = \mathcal{O}((k+1)^{2/a})$ as $k \to \infty$. Then the power of X of order $\lfloor a \rfloor$, i.e., $X^{\lfloor a \rfloor}$, satisfies Cramér's condition and hence is M-det.*

Proof. Note that

$$\frac{\mathbf{E}[(X^{\lfloor a \rfloor})^{2(k+1)}]}{\mathbf{E}[(X^{\lfloor a \rfloor})^{2k}]} = \frac{\mathbf{E}[X^{2\lfloor a \rfloor k+2\lfloor a \rfloor}]}{\mathbf{E}[X^{2\lfloor a \rfloor k+2\lfloor a \rfloor-2}]}\frac{\mathbf{E}[X^{2\lfloor a \rfloor k+2\lfloor a \rfloor-2}]}{\mathbf{E}[X^{2\lfloor a \rfloor k+2\lfloor a \rfloor-4}]}\cdots\frac{\mathbf{E}[X^{2\lfloor a \rfloor k+2}]}{\mathbf{E}[X^{2\lfloor a \rfloor k}]}$$
$$= \mathcal{O}((k+1)^{(2/a)\lfloor a \rfloor}) = \mathcal{O}((k+1)^2) \quad \text{as } k \to \infty.$$

Hence, by Theorem 2, $X^{\lfloor a \rfloor}$ satisfies Cramér's condition and is M-det. □

2.2. *Fast growth rate of the moments implies moment indeterminacy*

We present below a result showing that if the moments grow ‘fast’, i.e., the growth rate is $\rho > 2$ (and this happens if the tails are heavy), then under one additional condition, the distribution is M-indet. The heaviness of the tail as a cause of the moment indeterminacy is well-known and recognized in the moments theory literature. Thus our Theorem 3 below serves as a mathematical formalism of this; we may call this an ‘intuitive principle’.

Theorem 3. *Suppose that the ratio of the even order moments of the random variable X is $m_{2(k+1)}/m_{2k} \ge c(k+1)^{2+\varepsilon}$ for large k, where c and ε are positive constants. Assume further that X has a symmetric density f (about zero) satisfying the condition: for some $x_0 > 0$, f is positive and differentiable on $[x_0, \infty)$ and*

$$L_f(x) := -\frac{xf'(x)}{f(x)} \nearrow \infty \quad \text{as } x_0 < x \to \infty. \tag{3}$$

Then X is M-indet.

Proof. Without loss of generality we can assume that $m_{2(k+1)}/m_{2k} \ge c(k+1)^{2+\varepsilon}$ for each $k \ge 1$. We write a chain of inequalities and multiply them to obtain

$$m_{2(k+1)} \ge c^k((k+1)!)^{2+\varepsilon}m_2 \quad \text{for } k = 1, 2, \ldots.$$

Taking $c_0 = \min\{c, m_2\}$, we have

$$m_{2(k+1)} \ge c_0^{k+1}((k+1)!)^{2+\varepsilon} \quad \text{for } k = 1, 2, \ldots,$$

or, equivalently,

$$m_{2k} \ge c_0^k(k!)^{2+\varepsilon} = c_0^k(\Gamma(k+1))^{2+\varepsilon} \quad \text{for } k = 2, 3, \ldots.$$

As in Remark 1, we use Stirling's approximation $\Gamma(x+1) = x\Gamma(x) \simeq \sqrt{2\pi}x^{x+1/2}e^{-x}$ as $x \to \infty$ to conclude that for some constant $c_* > 0$

$$m_{2k}^{-1/(2k)} \le c_0^{-1/2}(\Gamma(k+1))^{-(2+\varepsilon)/(2k)} \simeq c_*\, k^{-1-\varepsilon/2} \quad \text{for all large } k.$$

This implies that the Carleman quantity for the moments of f is finite:

$$\mathrm{C}[f] := \sum_{k=1}^{\infty} m_{2k}^{-1/(2k)} < \infty.$$

Following the proof of Theorem 2 in Lin (1997) we have, by (3), that for some $x_0^* > x_0$, the logarithmic integral (Krein quantity of f) over the domain $\{x : |x| \ge x_0^*\}$ is finite:

$$\mathrm{K}[f] := \int_{|x| \ge x_0^*} \frac{-\log f(x)}{1+x^2}dx < \infty.$$

According to a result of Pedersen (1998, p. 92), this is a sufficient condition for X to be M-indet on $\mathbb{R}$ (see also Pakes, 2001). The proof is complete. □

Example 1. Let us illustrate the conditions and the claim of Theorem 3. We use, as in Remark 1, a particular case of a double generalized gamma distribution. Consider a random variable $\xi \sim DGG(1, \beta, 1)$, where $0 < \beta < 1$. The density of ξ is $f(x) = c\exp(-|x|^\beta)$, $x \in \mathbb{R}$. First, the distribution of ξ is heavy tailed, however all moments of ξ are finite. Neither Carmér's condition nor Carleman's condition is valid for ξ. Let us find the growth rate of the moments $m_{2k} = \mathbf{E}[\xi^{2k}]$. We have

$$\frac{m_{2(k+1)}}{m_{2k}} = \frac{\Gamma((1+2(k+1))/\beta)}{\Gamma((1+2k)/\beta)} \simeq \left(\frac{2}{\beta}\right)^{2/\beta} (k+1)^{2/\beta} \quad \text{as } k \to \infty.$$

Choosing $\beta = 1/(1+\varepsilon/2)$ we obtain that the growth rate of the even order moments of ξ is $\rho = 2+\varepsilon$. We next use the explicit form of the density f of ξ and see that indeed the ratio $-xf'(x)/f(x) = \beta x^\beta$ converges monotonically to infinity as $x \to \infty$. Hence all conditions in Theorem 3 are satisfied and ξ is M-indet.

Another way to show the M-indet of ξ is to use directly the Krein criterion (see Remark 1 above); it is enough to check that the Krein quantity K[f] is finite for each $0 < \beta < 1$.

To mention finally that these two kinds of arguments, to use Theorem 3, and to apply the Krein criterion, are different from the method used in Shohat and Tamarkin (1943, p. 22) to establish the same property of ξ.

2.3. *Determinacy of powers and products of random variables*

The previous general results will be used now to characterize the moment determinacy of powers and products of random variables. For $\xi_1, \xi_2, \ldots, \xi_n$ being independent copies of a random variable ξ with values in $\mathbb{R}$, we introduce the power X_n and the product Y_n as usual:

$$X_n = \xi^n \quad \text{and} \quad Y_n = \xi_1 \cdots \xi_n, \quad n = 1, 2, \ldots.$$

Notice first that by Lyapunov's inequality, we have the following relation for the moments of the power X_n and the product Y_n: for positive integers k and n,

$$\mathbf{E}[X_n^{2k}] = \mathbf{E}[\xi^{2nk}] \ge (\mathbf{E}[\xi^{2k}])^n = \mathbf{E}[Y_n^{2k}].$$

This relation suggests that some determinacy type properties will eventually be transferable in the 'direction' from X_n to Y_n.

Corollary 1. (i) *If the random variable ξ and the integer n are such that $X_n = \xi^n$ satisfies Carleman's condition (and hence is M-det), i.e., $\sum_{k=1}^{\infty} (\mathbf{E}[X_n^{2k}])^{-1/(2k)} = \infty$, then so does $Y_n = \xi_1 \cdots \xi_n$.*

(ii) *Suppose that X_n satisfies Cramér's condition (and hence is M-det), i.e., X_n has a moment generating function, or, equivalently, the moments of X_n satisfy:*

$$\limsup_{k\to\infty} \frac{1}{2k} (\mathbf{E}[X_n^{2k}])^{1/(2k)} < \infty.$$

The same holds for Y_n. Namely, Y_n satisfies Cramér condition, and hence is M-det.

We now concentrate on the M-det property of the product Y_n.

Proposition 1. *If the random variable ξ and the integer n, the number of independent factors $\xi_1, \ldots, \xi_n$, are such that*

$$\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}] = \mathcal{O}((k+1)^{2/n}) \quad \text{as } k \to \infty,$$

then the product $Y_n = \xi_1 \cdots \xi_n$ satisfies Cramér's condition and hence is M-det.

Proof. From the assumption on ξ and the independence of ξ_j we derive that

$$\mathbf{E}[Y_n^{2(k+1)}]/\mathbf{E}[Y_n^{2k}] = (\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}])^n = \mathcal{O}((k+1)^2) \quad \text{as } k \to \infty.$$

Therefore, by Theorem 2, Y_n satisfies Cramér's condition and is M-det. □

As in Theorem 2$'$ above, we can slightly extend Proposition 1, too.

Proposition 1$'$. *Let $a \ge 1$. If the random variable ξ is such that*

$$\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}] = \mathcal{O}((k+1)^{2/a}) \quad \text{as } k \to \infty,$$

then the product $Y_{\lfloor a \rfloor} = \xi_1 \cdots \xi_{\lfloor a \rfloor}$ satisfies Cramér's condition and is M-det.

Proof. Take $n = \lfloor a \rfloor \le a$ in Proposition 1. □

Remark 2. In Proposition 1, the power $\rho = 2/n$ is the best possible. Indeed, we can show that for each $\varepsilon > 0$, there is a random variable ξ such that $\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}] = \mathcal{O}((k+1)^{2/n+\varepsilon})$ as $k \to \infty$, however Y_n is M-indet. To show this, consider a random variable $\xi \sim DGG(1, \beta, 1)$. For each $\varepsilon > 0$, take $\beta = 2/(2/n+\varepsilon)$, then

$$\frac{\mathbf{E}[\xi^{2(k+1)}]}{\mathbf{E}[\xi^{2k}]} = \mathcal{O}\left((k+1)^{2/n+\varepsilon}\right) \quad \text{as } k \to \infty.$$

However, since $n > \beta$, the product $Y_n = \xi_1 \cdots \xi_n$ is M-indet (see Corollary 4 below).

Let us present a theorem, which in a sense is converse to Proposition 1. It is concerned with the M-indet property of the product Y_n. It will be clear that important are properties of both the density f and the tail $\overline{F}(x)=1-F(x),\ x>0$.

Theorem 4. *Let $\xi\sim F$, where F is absolutely continuous with density f which is symmetric (about 0) and strictly positive on $\mathbb{R}$, and let ξ have finite moments of all positive integer orders. Assume further that*

(i) $f(x)$ *is decreasing in* $x\geq 0$;

(ii) *there exist constants* $x_0\geq 1$ *and* $A>0$ *such that*

$$f(x)/\overline{F}(x)\geq A/x \quad \text{for } x\geq x_0; \tag{4}$$

(iii) *for some constants* $B>0,\ \alpha>0,\ \beta>0$ *and real* γ *we have*

$$\overline{F}(x)\geq Bx^{\gamma}e^{-\alpha x^{\beta}} \quad \text{for } x\geq x_0. \tag{5}$$

Then for any positive integer number $n>\beta$, the product $Y_n=\xi_1\cdots\xi_n$ has a finite Krein quantity and hence Y_n is M-indet.

Remark 3. The condition (ii) above is equivalent to $\liminf_{x\to\infty} L_{\overline{F}}(x)>0$, where

$$L_{\overline{F}}(x):=-x\frac{d}{dx}\log\overline{F}(x)=\frac{xf(x)}{\overline{F}(x)}, \quad x\geq x_0.$$

To prove Theorem 4, we need the following Lemma 3 from Lin and Stoyanov (2013).

Lemma 3. *Under condition (4), the following relation holds:*

$$\int_x^{\infty}\frac{f(u)}{u}du\geq\frac{A}{1+A}\frac{\overline{F}(x)}{x} \quad \text{for } x>x_0.$$

We also need the following two elementary facts:

Fact 1. Let X and Y be two independent and nondegenerate random variables such that X is symmetric about a and Y is symmetric about b. Then XY is symmetric about ab iff $ab=0$. Details can be seen in Hamedani and Walter (1985).

Fact 2. Let X and Y be two independent random variables symmetric about 0. Further, assume that X and Y have the density functions f and g, respectively. Then the product XY is symmetric about 0 and has a density function h given by

$$h(x)=2\int_0^{\infty}\frac{f(t)}{t}g\left(\frac{x}{t}\right)dt \quad \text{for } x>0.$$

Proof of Theorem 4. The density g_n of Y_n is symmetric about 0 (from Fact 1) and, in view of Fact 2 and by induction, g_n can be written as follows: for $x>0$,

$$g_n(x)=2^{n-1}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\frac{f(u_1)}{u_1}\frac{f(u_2)}{u_2}\cdots\frac{f(u_{n-1})}{u_{n-1}}f\left(\frac{x}{u_1u_2\cdots u_{n-1}}\right)du_1\,du_2\cdots du_{n-1}.$$

Hence $g_n(x)>0$ and decreases on $(0,\infty)$. For any $a>0$, we have

$$\begin{aligned} g_n(x)&\geq 2^{n-1}\int_a^{\infty}\int_a^{\infty}\cdots\int_a^{\infty}\frac{f(u_1)}{u_1}\frac{f(u_2)}{u_2}\cdots\frac{f(u_{n-1})}{u_{n-1}}f\left(\frac{x}{u_1u_2\cdots u_{n-1}}\right)du_1\,du_2\cdots du_{n-1}\\ &\geq 2^{n-1}\int_a^{\infty}\int_a^{\infty}\cdots\int_a^{\infty}\frac{f(u_1)}{u_1}\frac{f(u_2)}{u_2}\cdots\frac{f(u_{n-1})}{u_{n-1}}f\left(\frac{x}{a^{n-1}}\right)du_1\,du_2\cdots du_{n-1}\\ &=2^{n-1}f\left(\frac{x}{a^{n-1}}\right)\left(\int_a^{\infty}\frac{f(u)}{u}du\right)^{n-1}, \quad x>0. \end{aligned}$$

The above second inequality follows from the monotone property of f. Taking $a=x^{1/n}>x_0$, we have, by (4) and (5) and Lemma 3, that

$$\begin{aligned} g_n(x)&\geq 2^{n-1}f\left(x^{1/n}\right)\left(\int_{x^{1/n}}^{\infty}\frac{f(u)}{u}du\right)^{n-1}\geq 2^{n-1}f\left(x^{1/n}\right)\left(\frac{A}{1+A}\frac{\overline{F}(x^{1/n})}{x^{1/n}}\right)^{n-1}\\ &\geq 2^{n-1}\left(\frac{A}{1+A}\right)^{n-1}x^{-(1-1/n)}\frac{f(x^{1/n})}{\overline{F}(x^{1/n})}\left(\overline{F}(x^{1/n})\right)^{n}\\ &\geq C_nx^{\gamma-1}e^{-n\alpha x^{\beta/n}}, \end{aligned}$$

where $C_n = 2^{n-1}(A/(1+A))^{n-1}AB^n$. Thus we can evaluate the Krein quantity for g_n :

$$\mathrm{K}[g_n] = 2\int_0^\infty \frac{-\log g_n(x)}{1+x^2}dx = 2\int_0^{x_0^n} \frac{-\log g_n(x)}{1+x^2}dx + 2\int_{x_0^n}^\infty \frac{-\log g_n(x)}{1+x^2}dx$$

$$\le 2(-\log g_n(x_0^n))\int_0^{x_0^n} \frac{1}{1+x^2}dx + 2\int_{x_0^n}^\infty \frac{-\log g_n(x)}{1+x^2}dx < \infty \quad \text{if } n > \beta.$$

This implies that Y_n is M-indet for $n > \beta$ (see, e.g., Lin, 1997, Theorem 1), which is the claim in Theorem 4. □

We present below two corollaries. Corollary 2, involving normally distributed random variables, was proved by Berg (2005) by using a rather different approach. A special case of Corollary 3, namely, the moment indeterminacy of the product Y_2 of Laplace random variables was shown by DasGupta (1997).

Corollary 2. *Let $\xi \sim \mathcal{N}(0,1)$ with Φ and φ used below for the standard normal distribution function and its density. Then the product $Y_n = \xi_1 \cdots \xi_n$ is M-det iff $n=1$ or $n=2$. Consequently, the product of three or more independent normal random variables is M-indet.*

Proof. Since $\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}] = (2k+1)!!/(2k-1)!! = 2k+1$, we conclude, by Proposition 1, that Y_n is M-det if $n=1,2$. On the other hand, we have $\varphi(x)/(1-\Phi(x)) \simeq x$ as $x \to \infty$, and hence for $n \ge 3$, the product $Y_n = \xi_1 \cdots \xi_n$ is M-indet by taking $\beta = 2$ in Theorem 4. □

Corollary 3. *Let ξ have the standard Laplace distribution. Then the product $Y_n = \xi_1 \cdots \xi_n$ is M-det iff $n=1$. Consequently, the product of two or more independent Laplace random variables is M-indet.*

Proof. In this case $f(x) = \frac{1}{2}e^{-x}, x > 0$, and $\overline{F}(x) = \frac{1}{2}e^{-x} = f(x),\ x > 0$. The conditions in Theorem 4 are satisfied by taking $\beta = 1$, and hence Y_n is M-indet for $n = 2, 3, \ldots$. □

Remark 4. Let ξ have the standard Laplace distribution. Then $X_2 = \xi^2$ is M-det, while by Corollary 3, $Y_2 = \xi_1\xi_2$ is M-indet. That is, the power of a Laplace random variable and the product of independent Laplace random variables (both of 'order' two) do not share the same moment determinacy property.

It is useful to mention that in fact the density g_2 of Y_2 is symmetric and has the following explicit form: for $x > 0$,

$$g_2(x) = \frac{1}{2}\int_0^\infty t^{-1}e^{-t-x/t}\,dt = K_0(2\sqrt{x}) \simeq \frac{\sqrt{\pi}}{2}x^{-1/4}e^{-2\sqrt{x}} \quad \text{as } x \to \infty,$$

here K_0 is the so-called modified Bessel function of the second kind. For completeness, we recall the definition of $K_0(x)$ and its approximation:

$$K_0(x) = \frac{1}{2}\int_0^\infty t^{-1}e^{-t-x^2/(4t)}\,dt, \quad x > 0,$$

$$= \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 - \frac{1}{8x}\left(1 - \frac{9}{16x}\left(1 - \frac{25}{24x}\right)\right) + o(x^{-3})\right] \quad \text{as } x \to \infty$$

(see, e.g., Glasser et al., 2012; Malham, 2005, pp. 37–38).

Remark 5. Let ξ have the standard normal distribution $\mathcal{N}(0,1)$. Then the density g_2 of the product $Y_2 = \xi_1\xi_2$ is symmetric and has the explicit form: for $x > 0$,

$$g_2(x) = \frac{1}{\pi}K_0(x) \simeq \frac{1}{\sqrt{2\pi}}x^{-1/2}e^{-x} \quad \text{as } x \to \infty$$

(see also DasGupta, 1997). In this case, we say that Y_2 obeys the standard Bessel distribution.

3. Various illustrations involving specific distributions on $\mathbb{R}$

3.1. Double generalized gamma distributions

On two occasions, Remark 1 and Example 1 above, we used particular cases of the double generalized gamma distribution. We treat here the general case.

We use the notation $\xi \sim DGG(\alpha, \beta, \gamma)$ to tell that the random variable ξ has density f of the form:

$$f(x) = c|x|^{\gamma-1}\,e^{-\alpha|x|^\beta}, \quad x \in \mathbb{R}, \tag{6}$$

here $\alpha,\ \beta,\ \gamma > 0$, $f(0) = 0$ if $\gamma \ne 1$, and $c = \beta\alpha^{\gamma/\beta}/(2\Gamma(\gamma/\beta))$ is the norming constant.

We start with the following known result (see, e.g., Lin and Huang, 1997):

(i) for real $s>0$, $|\xi|^s$ is M-det iff $s \le 2\beta$;
(ii) for odd integer n, $X_n=\xi^n$ is M-det iff $n \le \beta$.

Example 2 (*M-det case*). Let $\xi_1,\dots,\xi_n$ be n independent copies of a random variable $\xi \sim DGG(\alpha,\beta,\gamma)$. Then for any positive integer number $n \le \beta$, both the power $X_n=\xi^n$ and the product $Y_n=\xi_1\cdots\xi_n$ are M-det. To see this, we first calculate that

$$\frac{\mathbf{E}[X_n^{2(k+1)}]}{\mathbf{E}[X_n^{2k}]}=\frac{\mathbf{E}[\xi^{2n(k+1)}]}{\mathbf{E}[\xi^{2nk}]}=\frac{\Gamma((\gamma+2n(k+1))/\beta)}{\alpha^{2n/\beta}\Gamma((\gamma+2nk)/\beta)}$$
$$\simeq [2n/(\alpha\beta)]^{2n/\beta}(k+1)^{2n/\beta} \quad \text{as } k\to\infty.$$

We have used Stirling's approximation: $\Gamma(x)\simeq\sqrt{2\pi}x^{x-1/2}e^{-x}$ as $x\to\infty$.

Then by Theorem 1, the power $X_n=\xi^n$ is M-det if $n\le\beta$, and by Proposition 1, the product $Y_n=\xi_1\cdots\xi_n$ is M-det if $2/\beta\le 2/n$, or if $n\le\beta$, because $\mathbf{E}[\xi^{2(k+1)}]/\mathbf{E}[\xi^{2k}]=\mathcal{O}((k+1)^{2/\beta})$ as $k\to\infty$.

Example 3 (*M-indet case*). Let $\xi\sim DGG(\alpha,\beta,1)$, so its density, see (6) with $\gamma=1$, is $f(x)=ce^{-\alpha|x|^\beta}$, $x\in\mathbb{R}$, where $\alpha,\beta>0$ and c is a norming constant. Then for any positive integer $n>\beta$, the product $Y_n=\xi_1\cdots\xi_n$ is M-indet. To see this, note that $f(x)/\overline{F}(x)\simeq\alpha\beta x^{\beta-1}$ and that $\overline{F}(x)\simeq[c/(\alpha\beta)]x^{1-\beta}e^{-\alpha x^\beta}$ as $x\to\infty$. The density f satisfies the conditions (i) and (ii) in Theorem 4 and hence Y_n is M-indet if $n>\beta$.

Corollary 4. *Let $\xi\sim DGG(\alpha,\beta,1)$; its density is explicitly written in Example 3. Then the product $Y_n=\xi_1\cdots\xi_n$ is M-det iff $n\le\beta$.*

In summary, we have the following neat result about the class of distributions $DGG(\alpha,\beta,\gamma)$ with $\gamma=1$.

Lemma 4. *Let n be a positive odd integer and $\xi_1,\dots,\xi_n$ independent copies of the random variable $\xi\sim DGG(\alpha,\beta,1)$. Then the power $X_n=\xi^n$ is M-det iff the product $Y_n=\xi_1\cdots\xi_n$ is M-det and this is true iff $n\le\beta$.*

We now consider a more general case: γ is any positive odd integer.

Theorem 5. *Let n be a positive odd integer and $\xi\sim DGG(\alpha,\beta,\gamma)$ with $\alpha,\beta>0$ and γ a positive odd integer. Then X_n is M-det iff Y_n is M-det and this is true iff $n\le\beta$. In other words, for odd n and odd γ, the power X_n and the product Y_n have the same moment determinacy property.*

Proof. Define $\eta=\xi^\gamma$, $\eta_i=\xi_i^\gamma$, $i=1,\dots,n$, $X_n^*=\eta^n=(\xi^n)^\gamma=X_n^\gamma$ and $Y_n^*=\eta_1\cdots\eta_n=(\xi_1\cdots\xi_n)^\gamma=Y_n^\gamma$. Since $\eta\sim DGG(\alpha,\beta/\gamma,1)$, Lemma 4 implies that X_n^* is M-det iff Y_n^* is M-det and this is true iff $n\le\beta/\gamma$. Next, note that for each $x>0$, we have $\mathbf{P}[X_n^*>x]=\mathbf{P}[X_n>x^{1/\gamma}]$ and $\mathbf{P}[Y_n^*>x]=\mathbf{P}[Y_n>x^{1/\gamma}]$. This implies that any distributional property shared by X_n^* and Y_n^* can be transferred to a similar property shared by X_n and Y_n, and vice versa. Therefore, X_n is M-det iff Y_n is M-det which is true iff $n\le\beta$, because for odd n, X_n is M-det iff $n\le\beta$ (see Lin and Huang, 1997). This proves Theorem 5. □

Remark 6. It seems hard to prove directly the statement in Theorem 5 for the product $Y_n=\xi_1\cdots\xi_n$. However, with the help of the result for the power $X_n=\xi^n$, we are able to characterize the moment determinacy of Y_n.

3.2. Logistic distributions

Recall that the random variable ξ has the standard logistic distribution if its density, say f, is given by

$$f(x)=\frac{e^{-x}}{(1+e^{-x})^2}=\frac{1}{2+e^x+e^{-x}}, \quad x\in\mathbb{R}.$$

This case is interesting for several reasons including the fact that the logistic distribution does not belong to the family of DGG distributions.

Theorem 6. *Let ξ be a standard logistic random variable. Then*

(a) *The power $X_n=\xi^n$ is M-det for $n=1, 2$ and is M-indet for $n=3,5,7,\dots$.*
(b) *The product $Y_n=\xi_1\cdots\xi_n$ is M-indet for any integer $n\ge 2$.*

Consequently, for odd n, the power X_n and the product Y_n have the same moment determinacy property.

Proof. The statement in (a) for X_n is known and is due to Lin and Huang (1997). Below, see Theorem LH, we provide a new proof. Thus it remains to prove the statement for Y_n. Note first that $\overline{F}(x)=e^{-x}/(1+e^{-x})\ge e^{-x}/2$ for $x\ge 0$. The second step is to check that $f(x)/\overline{F}(x)=1/(1+e^{-x})\to 1$ as $x\to\infty$. Applying Theorem 4 with $\beta=1$, we conclude that Y_n is M-indet for $n\ge 2$. Thus the proof is complete. □

Remark 7. We have seen that if ξ is a logistic random variable, then $\mathbf{P}[\xi=0]=0$, and $X_2=\xi^2$ is M-det on $\mathbb{R}$, however $Y_2=\xi_1\xi_2$ is M-indet on $\mathbb{R}$. Consequently, X_2 and Y_2 do not share the same moment determinacy property. Further, for odd $n\geq 3$, both X_n and Y_n are M-indet. Note that for even $n\geq 4$, the power X_n has values in $\mathbb{R}_+=[0,\infty)$ (Stieltjes case), while Y_n is with values in $\mathbb{R}$ (Hamburger case). However, if we still wish to compare X_n and Y_n for even $n\geq 4$, the conclusion is that they are both M-indet on $\mathbb{R}$.

Again, though the next result is known, we find it useful to apply our Theorem 3 and provide a new proof. The proof is of its own interest, however the same idea and steps can be applied to many other distributions on $\mathbb{R}$.

Theorem LH (Lin and Huang, 1997). *Let ξ have the standard logistic density. Then the power $X_n=\xi^n$ is M-det for $n=1,2$, and it is M-indet for any odd integer $n\geq 3$.*

New Proof. First, for any real $s>0$, we find explicitly the density h_s of $|\xi|^s$:

$$h_s(z)=\frac{2}{s}z^{1/s-1}\frac{e^{-z^{1/s}}}{(1+e^{-z^{1/s}})^2},\quad z>0.$$

Since $\frac{1}{4}\leq(1+e^{-u})^{-2}\leq 1$ for any $u\geq 0$, we estimate the moments of $|\xi|^s$:

$$\frac{1}{2}\Gamma(ks+1)\leq\mathbf{E}\left[(|\xi|^s)^k\right]\leq\int_0^\infty\frac{2}{s}z^{k+1/s-1}e^{-z^{1/s}}dz=2\Gamma(ks+1).$$

Therefore

$$\mathbf{E}\left[(|\xi|^s)^{k+1}\right]\Big/\mathbf{E}\left[(|\xi|^s)^k\right]\leq 4\cdot\frac{\Gamma((k+1)s+1)}{\Gamma(ks+1)}\simeq 4s^s(k+1)^s\quad\text{as } k\to\infty.$$

On the other hand, we have

$$\frac{\mathbf{E}[(|\xi|^s)^{k+1}]}{\mathbf{E}[(|\xi|^s)^k]}\geq\frac{1}{4}\cdot\frac{\Gamma((k+1)s+1)}{\Gamma(ks+1)}\simeq\frac{1}{4}s^s(k+1)^s\quad\text{as } k\to\infty.$$

If we now take $X_1=\xi$, where ξ is the original logistic random variable, we see that the ratio of the even order moments of X_1 is

$$m_{2(k+1)}/m_{2k}=\mathcal{O}((k+1)^2)\quad\text{as } k\to\infty.$$

It follows by Theorem 1 that ξ itself is M-det. For $n=2$ we have $X_2=\xi^2=|\xi|^2$ and the ratio of moments is $\mathbf{E}[X_2^{k+1}]/\mathbf{E}[X_2^k]=\mathcal{O}((k+1)^2)$ as $k\to\infty$. Therefore, X_2 is M-det by Theorem 1 of Lin and Stoyanov (2013) (noting that ξ is a continuous random variable). It remains to show that $X_n=\xi^n$ is M-indet for $n=3,5,7,\ldots$. Recall that n is fixed and the ratio of even order moments of ξ^n is

$$\frac{\mathbf{E}[(\xi^n)^{2k+2}]}{\mathbf{E}[(\xi^n)^{2k}]}=\frac{\mathbf{E}[\xi^{2nk+2n}]}{\mathbf{E}[\xi^{2nk}]}\geq\frac{1}{4}\cdot\frac{\Gamma(2n(k+1)+1)}{\Gamma(2nk+1)}\simeq\frac{1}{4}(2n)^{2n}(k+1)^{2n}\quad\text{as } k\to\infty.$$

Thus the moment condition in Theorem 3 is satisfied for X_n if $n=3,5,7,\ldots$. We need to show that the density g_n of X_n satisfies the condition (3). Indeed, we easily find g_n and check that

$$\begin{aligned}L_{g_n}(y)&:=-\frac{yg_n'(y)}{g_n(y)}\\&=1-\frac{1}{n}+\frac{1}{n}y^{1/n}-\frac{2}{n}y^{1/n}\frac{e^{-y^{1/n}}}{1+e^{-y^{1/n}}}\nearrow\infty\text{ ultimately as } y\to\infty.\end{aligned}$$

Therefore, by Theorem 3, $X_n=\xi^n$ is M-indet for $n=3,5,7,\ldots$. The proof is complete. □

4. Random powers and random products of normal random variables

We have studied above the moment determinacy of powers ξ^n and products $\xi_1\cdots\xi_n$, where n is a fixed positive integer number. Instead of fixed n, let us consider a positive integer-valued random variable N. In order to avoid trivialities, assume that N takes with positive probabilities at least two different values. In such a case ξ^N is called a 'random power' and $\xi_1\cdots\xi_N$ a 'random product' and we want to study their moment (in)determinacy.

The approach followed in this section can be applied to many symmetric distributions, but we deal only with normal distributions here.

We formulate and prove separately two results: Theorem 7 is for random products, Theorem 8 is for random powers. While intuitively the statements are seemingly expected, it is good to have proofs written down.

Theorem 7. *Let $\{\xi_n,\ n=1,2,\ldots\}$ be an infinite sequence of independent random variables distributed $\mathcal{N}(0,1)$, and let N, independent of $\{\xi_n\}$, be a positive random variable taking values in the set $\mathbb{N}=\{1,2,\ldots\}$ or some subset of $\mathbb{N}$. Suppose further that $\mathbf{P}[N=n_*]:=p_{n_*}>0$ for some $n_*\geq 3$. Then the random product $\tilde{Y}=\xi_1\cdots\xi_N$ is M-indet.*

Proof. Let $\tilde{g}$ and g_j be the density functions of $\tilde{Y}$ and Y_j, respectively. Then both $\tilde{g}$ and g_j are symmetric about zero and

$$\tilde{g}(x) = \sum_{j=1}^{\infty} p_j g_j(x) \ge p_{n_*} g_{n_*}(x), \quad x \in \mathbb{R}.$$

The Krein quantity for $\tilde{g}$ is

$$\mathrm{K}[\tilde{g}] \le 2\int_0^{\infty} \frac{-\log p_{n_*}}{1+x^2}\,dx + \mathrm{K}[g_{n_*}].$$

Since $\int_0^{\infty}(1+x^2)^{-1}dx = \pi/2$ and $\mathrm{K}[g_{n^*}] < \infty$ by Theorem 4, we see that $\mathrm{K}[\tilde{g}] < \infty$ and hence $\tilde{Y}$ is M-indet. □

Now we look at random powers of a normal random variable. Recall that if $\xi \sim \mathcal{N}(0,1)$, and $X_n = \xi^n$, then $X_1 = \xi,\ X_2 = \xi^2$ and $X_4 = \xi^4$ are all M-det, while all others, $X_3, X_5, X_6, \ldots$ are M-det (see, e.g., Berg, 1988). Let us note that a very short proof of the M-det property of $X_4 = \xi^4$ based on Hardy's criterion is given in Stoyanov and Lin (2012).

For random powers of ξ, we have the following result.

Theorem 8. *Let $\xi \sim \mathcal{N}(0,1)$ and N, independent of ξ, be a positive random variable taking values in the set $\mathbb{N}$ or in some its subset. Let $\tilde{X} = \xi^N$ and $p_n = \mathbf{P}[N=n]$. Consider two cases*:

Case (i): *$p_{n_*} > 0$ for some odd $n_* \ge 3$.*

Case (ii): *$p_{k_*} > 0$ for some even $k_* \ge 6$ and $p_{n_*} = 0$ for each odd n_*.*

Then, in each of these cases the random power $\tilde{X} = \xi^N$ is M-indet.

Proof. Let $\tilde{f}$ and f_n be the density functions of $\tilde{X}$ and X_n, respectively. Note that $\tilde{f}$ is not symmetric about zero if $p_j > 0$ for some even j, and that $\tilde{X}$ takes all values in $\mathbb{R}$ if $p_i > 0$ for some odd i. Then we have, for each n with $p_n > 0$,

$$\tilde{f}(x) = \sum_{j=1}^{\infty} p_j f_j(x) \ge p_n f_n(x), \quad x \in \mathbb{R}.$$

(i) If $p_{n_*} > 0$ for some odd $n_* \ge 3$, then the Krein quantity for $\tilde{f}$ is

$$\mathrm{K}\left[\tilde{f}\right] = \int_{-\infty}^{\infty} \frac{-\log \tilde{f}(x)}{1+x^2}\,dx \le \int_{-\infty}^{\infty} \frac{-\log p_{n^*}}{1+x^2}\,dx + \int_{-\infty}^{\infty} \frac{-\log f_{n^*}(x)}{1+x^2}\,dx.$$

Denote, for odd n

$$\mathrm{K}[f_n] = \int_{-\infty}^{\infty} \frac{-\log f_n(x)}{1+x^2}\,dx.$$

It can be shown that $\mathrm{K}[f_n] < \infty$ for odd $n \ge 3$. Therefore $\mathrm{K}[\tilde{f}] < \infty$, hence in case (i), $\tilde{X} = \xi^N$ is M-indet.

(ii) For even m, ξ^m takes nonnegative values, hence the Krein quantity for f_m is

$$\mathrm{K}[f_m] = \int_0^{\infty} \frac{-\log f_m(x^2)}{1+x^2}\,dx.$$

It can be shown that $\mathrm{K}[f_m] < \infty$ for any even $m \ge 6$. Therefore $\mathrm{K}[\tilde{f}] < \infty$ in case (ii), and hence $\tilde{X} = \xi^N$ is M-indet. This completes the proof. □

Remark 8. The M-indet property of ξ^k for even $k \ge 6$ is a part of the statement that the product of six or more independent half-normal random variables is M-indet, see Lin and Stoyanov (2013).

Remark 9. If $\mathbf{P}[N=0] > 0$, we define, e.g., $\xi_0 = 1$ and $\xi^0 = 1$. Then Theorems 7 and 8 remain valid as soon as there are at least two other values which N takes with positive probabilities. Here is a particular case. If $\xi \sim \mathcal{N}(0,1)$ and N is a Poisson random variable independent of ξ, then the random power ξ^N is M-indet. Moreover, the random product $\xi_1 \cdots \xi_N$ is also M-indet. In words, a Poisson power of a normal and a product of normals of Poisson length are both M-indet.

Similarly, we can take $N \sim Ge(p)$, geometrically distributed random variable with parameter $p \in (0,1)$. Then the random power ξ^N and the random product $\xi_1 \cdots \xi_N$ are both M-indet for any $p \in (0,1)$. Another case is to consider $N \sim Bin(n,p)$, binomially distributed random variable. For any integer $n \ge 3$ and any $p \in (0,1)$, the random power ξ^N and the random product $\xi_1 \cdots \xi_N$ are both M-indet. It is easy to see that these two random quantities are M-det for n=1, 2.

Instead of normally distributed random variables, we can take variables from the class $DGG(\alpha,\beta,\gamma)$ and N being Poisson, geometric or binomial random variable. We easily find values of the parameters involved when random powers or random products are M-det or M-indet. We do not give details.

5. Determinacy of products of differently distributed random variables

It is interesting to consider products of independent random variables which may not have the same distribution. We have the following general result.

Please cite this article as: Stoyanov, J., et al., Hamburger moment problem for powers and products of random variables. Journal of Statistical Planning and Inference (2013), http://dx.doi.org/10.1016/j.jspi.2013.11.002

Theorem 9. *Suppose that ξ and η are two independent random variables with symmetric densities, f and g, respectively. Both f and g are assumed to be strictly positive on $\mathbb{R}$. With ξ_1 and ξ_2 being independent copies of ξ, assume that the Krein quantity for the random variable $Y_2 = \xi_1\xi_2$ is finite (hence Y_2 is M-indet). Assume further that the ratio $g(t)/f(t)$ is bounded away from zero in the sense that $\inf\{g(t)/f(t) : t \geq 0\} > 0$. Then the product $Z = \xi\eta$ is M-indet.*

Proof. Note that the density g_2 of Y_2 is symmetric about zero and we have

$$g_2(x) = 2\int_0^\infty t^{-1} f(t) f(x/t)\, dt, \quad x > 0.$$

The density h of Z is also symmetric about zero and is given by

$$h(x) = 2\int_0^\infty t^{-1} g(t) f(x/t)\, dt, \quad x > 0.$$

Since the ratio $g(t)/f(t)$ is bounded away from zero, there exists a number $c > 0$ such that $g(t) \geq cf(t)$ for all $t \geq 0$, and hence

$$h(x) \geq cg_2(x), \quad x > 0.$$

Therefore, the Krein quantity for h is as follows:

$$\mathrm{K}[h] = 2\int_0^\infty \frac{-\log h(x)}{1+x^2}\, dx \leq 2\int_0^\infty \frac{-\log c}{1+x^2}\, dx + \mathrm{K}[g_2] < \infty.$$

This implies that Z is M-indet. □

Corollary 5. *Suppose that ξ and η are two independent random variables obeying the standard Laplace and the standard logistic distribution, respectively. Then the product $Z = \xi\eta$ is M-indet.*

Proof. Let $f(x) = \frac{1}{2} e^{-|x|}$, $x \in \mathbb{R}$, and $g(x) = 1/(2+e^x+e^{-x})$, $x \in \mathbb{R}$, be the standard Laplace and standard logistic density, respectively. Let ξ_1 and ξ_2 be independent copies of ξ, so each has density f. The first step is to use Theorem 4 and conclude that the density g_2 of the product $Y_2 = \xi_1\xi_2$ has a finite Krein quantity, namely, $\mathrm{K}[g_2] < \infty$. The second step is to examine the ratio $g(t)/f(t)$. Since $g(t)/f(t) = 2(1+e^{-t})^{-2} \geq \frac{1}{2}$, $t \geq 0$, the ratio $g(t)/f(t)$ is bounded away from zero. Thus, by Theorem 9, the product $Z = \xi\eta$ is indeed M-indet. □

Finally, we extend the result given in Example 2.

Theorem 10. *Let $U_1, \ldots, U_n$ be n independent random variables, and let $U_i \sim DGG(\alpha_i, \beta_i, \gamma_i)$, where $\alpha_i, \beta_i, \gamma_i > 0$, $i = 1, 2, \ldots, n$. Then the product $T_n = U_1 \cdots U_n$ is M-det if $\sum_{i=1}^n \beta_i^{-1} \leq 1$.*

Proof. As in Example 2, the ratio

$$\frac{\mathbf{E}[T_n^{2(k+1)}]}{\mathbf{E}[T_n^{2k}]} = \prod_{i=1}^n \frac{\mathbf{E}[U_i^{2(k+1)}]}{\mathbf{E}[U_i^{2k}]} \simeq C(k+1)^{2b_n} \quad \text{as } k \to \infty,$$

where $C = \prod_{i=1}^n [2/(\alpha_i\beta_i)]^{2/\beta_i}$ and $b_n := \sum_{i=1}^n \beta_i^{-1}$. Hence, by Theorem 1, we conclude that U_n is M-det if $\sum_{i=1}^n \beta_i^{-1} \leq 1$. The proof is complete. □

Corollary 6. *Let $\xi_1, \ldots, \xi_n$ be independent random variables distributed as ξ, where $\xi \sim DGG(\alpha, \beta, \gamma)$. For some positive integers p_i and q_i with q_i odd, $i = 1, 2, \ldots, n$, define the numbers $r_i = p_i/q_i$, $i = 1, \ldots, n$. Then the product/power $Z_n = \xi_1^{r_1} \cdots \xi_n^{r_n}$ is a well-defined random variable which is M-det if $\sum_{i=1}^n r_i \leq \beta$.*

Remark 10. In Theorem 9 we studied products of two random variables both with values in $\mathbb{R}$. Suppose now that we multiply two independent random variables, X and Y, where X, with values in $\mathbb{R}$, is symmetric, while $Y > 0$. Then the product $Z = XY$ is also symmetric on $\mathbb{R}$ and if both X and Y have finite all moments, then so does the product Z. If at least one of X and Y is M-indet, then the product $Z = XY$ is also M-indet. It turns out that even if both X and Y are M-det, then the product may 'inherit' the M-det property, however, it may become M-indet. Everything depends on the behavior of the tails of the distributions of X and Y. One way to express this is to use the growth rate of the moments of the two factors. Indeed, suppose that the moments of X and Y satisfy the relations: as $k \to \infty$,

$$\mathbf{E}[X^{2(k+1)}]/\mathbf{E}[X^{2k}] \simeq c_1(k+1)^a \quad \text{and} \quad \mathbf{E}[Y^{k+1}]/\mathbf{E}[Y^k] \simeq c_2(k+1)^b,$$

for some positive constants a, b, c_1 and c_2. Then, if $a + 2b \leq 2$, the product $Z = XY$ is M-det. If, however, $a + 2b > 2$, the moments of $Z = XY$ grow quite 'fast', and under an additional condition, the product $Z = XY$ will be M-indet.

The claim follows basically from Theorem 1. Here are a couple of particular but interesting cases. If X is a normal random variable and Y is half-normal, then the product XY is M-det. However, if X is logistic and $Y \sim Exp$, then the product XY is M-indet. Another case is when X is normal and $Y \sim \chi^2$. Then the product XY is M-indet. The reader can easily compose his/her own cases of products of random variables which are either M-det or M-indet.

6. Concluding remarks

For a given random variable ξ with support $\mathbb{R}$, one might expect that the power $X_n = \xi^n$ and the (i.i.d.) product $Y_n = \xi_1 \cdots \xi_n$ become M-indet for large enough n, as shown in the examples above. However, this is not in general true. Let us consider the random variable ξ with density $f(x) = c\,\exp(-e^{|x|})$, $x \in \mathbb{R}$, where c is a norming constant. Then any power ξ^n, $n = 1, 2, \ldots$, is M-det. Moreover, if $\xi_1, \xi_2, \ldots$ are independent copies of ξ, then the product of any number n of them, $\xi_1 \cdots \xi_n$, is M-det. We can also consider the following discrete centered random variable Y which is obtained from ξ: $Y = \lfloor \xi \rfloor$ if $\xi \geq 0$, and $Y = \lfloor \xi \rfloor + 1$ if $\xi < 0$. It can be shown that any power of Y is M-det because of the relation $\mathbf{P}[|Y| > \varepsilon] \leq \mathbf{P}[|\xi| > \varepsilon]$ for all $\varepsilon > 0$. The reason for these properties is that the distribution of ξ has the tails so 'light' that any power still remains with light tails. We do not further pursue this here.

Acknowledgments

We dedicate this work to our deeply cherished memories of W.J. Studden (1935–2013), an exemplary scholar and gentleman. In his books with Samuel Karlin and Holger Dette (see Karlin and Studden (1966) and Dette and Studden (1997)), and in numerous papers on optimal design, matrix measures and random walks, Bill Studden made innumerable contributions to the study and application of the theory of moments. In a totally understated way, he was a master. It is an honor to write for such a teacher and friend, one that gave us so much without ever expecting anything in return.

The work of JS was partly supported by the 'Emeritus Fellowship' provided by the Leverhulme Trust (UK), that of GDL by the National Science Council of ROC (Taiwan), under Grant NSC 102-2118-M-001-008-MY2, and that of ADG by a contract from Elsevier.

We thank the Referee and the Associate Editor for their useful comments and suggestions.

References

Akhiezer, N.I., 1965. The Classical Moment Problem and Some Related Questions in Analysis. Oliver & Boyd, Edinburgh (Russian edition: Nauka, Moscow, 1961).
Berg, C., 1988. The cube of a normal distribution is indeterminate. Ann. Probab. 16, 910–913.
Berg, C., 2005. On powers of Stieltjes moment sequences I. J. Theor. Probab. 18, 871–889.
Carmona, R., Molchanov, S.A., 1993. Intermittency and phase transitions for some particle systems in random media. In: Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals (Sanda/Kyoto, 1990). Pitman Research Notes in Mathematics Series (Longman Science & Technology), vol. 283, Harlow, pp. 15–36.
DasGupta, A., 1997. On the Determinacy of Powers, Products and Convolutions. Technical Report #97–21. University of California, San Diego. Web. ⟨http://www.stat.purdue.edu.research/technical_reports/1997/tr97-21.pdf⟩.
DasGupta, A., 2008. Asymptotic Theory of Statistics and Probability. Springer, New York.
De Abreu, G.T.F., 2010. On the moment-determinance and random mixture of Nakagami-m variates. IEEE Trans. Commun. 58, 2561–2575.
Dette, H., Studden, W.J., 1997. The Theory of Canonical Moments with Applications in Statistics, Probability and Analysis. John Wiley & Sons, New York.
Diaconis, P., 1987. Application of the method of moments in probability and statistics. In: AMS Proceedings Symposia in Applied Mathematics, vol. 37, pp. 125–142.
Frisch, U., Sornette, D., 1997. Extreme deviations and applications. J. Phys. Ser. I 7, 1155–1171.
Galambos, J., Simonelli, I., 2004. Products of Random Variables. Applications to Problems of Physics and to Arithmetical Functions. Marcel Dekker New York.
Glasser, L., Kohl, K.T., Koutschan, C., Moll, V.H., Straub, A., 2012. The integrals in Gradshteyn and Ryzhik. Part 22. Bessel-K functions. SCIENTIA Ser. A: Math. Sci. 22, 129–151.
Hamedani, G.G., Walter, G.G., 1985. On the product of symmetric random variables. Stat. Probab. Lett. 3, 251–253. (Erratum: Stat. Probab. Lett. 4 (1986) 51.)
Karlin, S., Studden, W.J., 1966. Tchebycheff Systems: With Applications to Analysis and Statistics. John Wiley & Sons, New York (Russian translation: Nauka, Moscow, 1976).
Lin, G.D., 1997. On the moment problems. Stat. Probab. Lett. 35, 85–90. (Erratum: Stat. Probab. Lett. 50 (2000) 205.)
Lin, G.D., Huang, J.S., 1997. The cube of a logistic distribution is indeterminate. Aust. J. Stat. 39, 247–252.
Lin, G.D., Stoyanov, J., 2013. Moment determinacy of powers, products and other transforms of nonnegative random variables, submitted for publication.
Malham, S.J.A., 2005. An Introduction to Asymptotic Analysis. Lecture Notes. Department of Mathematics, Heriot-Watt University, UK. Website ⟨http://www.ma.hw.ac.uk/~simonm/ae.pdf⟩.
Pakes, A.G., 2001. Remarks on converse Carleman and Krein criteria for the classical moment problem. J. Aust. Math. Soc. 71, 81–104.
Pedersen, H.L., 1998. On Krein's theorem for indeterminacy of the classical moment problem. J. Approximation Theory 95, 90–100.
Shiryaev, A.N., 1996. Probability, 2nd ed. Springer, Berlin.
Shohat, J., Tamarkin, J.D., 1943. The Problem of Moments. AMS, Providence, RI (Revised edition 1950).
Simon, B., 1998. The classical moment problem as a self-adjoint finite difference operator. Adv. Math. 137, 82–203.
Slud, E.V., 1993. The moment problem for polynomial forms of normal random variables. Ann. Probab. 21, 2200–2214.
Stoyanov, J.M., 2013. Counterexamples in Probability, 3rd ed. Dover Publications, New York.
Stoyanov, J., Lin, G.D., 2012. Hardy's condition in the moment problem for probability distributions. Theory Probab. Appl. 57, 811–820. (SIAM edition: 57 (2013), 699–708.)
Whittaker, E.T., Watson, G.N., 1927. A Course of Modern Analysis, 4th ed. Cambridge University Press, Cambridge.