Linear Transfer Function Models

Consider an input series $\{X_t\}$ to a system and the corresponding output series $\{Y_t\}$ from it.

Hold the input at a fixed level X and let the system reach its equilibrium, one obtains the *steady-state* output $Y_\infty(X)$. For a linear system, $Y_\infty = \mu_y + gX$, where g is the *steady-state gain*. Without loss of generality, assume $\mu_y = 0$.

The inertia of a linear system may be represented by a linear filter,

$$Y_t = v_0 X_t + v_1 X_{t-1} + v_2 X_{t-2} + \cdots = v(B) X_t,$$

where $v(B)$ is the *transfer function* of the filter.

The general linear process discussed earlier in the course is seen to be the output from a linear filter with white noise input,

$$z_t = \psi_0 a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots = \psi(B) a_t.$$

Difference Equation, Stability

Many continuous time dynamic systems follow certain differential equations. On replacing the differential operators by difference operators, one gets discrete models of the form

$$(1+\xi_1\nabla+\cdots+\xi_r\nabla^r)Y_t=(1+\eta_1\nabla+\cdots+\eta_s\nabla^s)X_{t-b},$$

where $\nabla=1-B$. This leads to models similar to the ARMA models for stationary processes,

$$(1-\delta_1B-\cdots-\delta_rB^r)Y_t=(\omega_0-\omega_1B-\cdots-\omega_sB^s)X_{t-b},$$

or $\delta(B)Y_t=\omega(B)X_{t-b}=B^b\omega(B)X_t=\Omega(B)X_t$.

The filter is *stable* if $v(B)=\delta^{-1}(B)\Omega(B)$ is convergent for all $|B|\le 1$. For a stable filter, $g=\sum_{j=0}^{\infty}v_j$. The stability of the filter is governed by the roots of $\delta(B)$.

Impulse and Step Responses, Added Noise

Responding to an *impulse input*, $X_t = I_{[t=0]}$, one has $Y_t = v_t$, $t = 0, 1, \ldots$, so the weights v_j gives the *impulse response*. When the response is delayed by b lags, one has $v_0 = \cdots = v_{b-1} = 0$.

Responding to a *step input*, $X_t = I_{[t \geq 0]}$, one has $Y_t = V_t = \sum_{j=0}^{t} v_j$, $t = 0, 1, \ldots$, which gives the *step response*.

Consider a filter $\delta(B)Y_t = \omega(B)X_{t-b}$. The impulse response follows the difference equation $\delta(B)v_j = 0$ for $j > s + b$.

A system may be subject to various kinds of disturbance, and it is convenient to model the effect of the disturbance in the form of added noise in the output,

$$Y_t = \delta^{-1}(B)\omega(B)X_{t-b} + N_t,$$

where N_t may follow an ARIMA model, $\varphi(B)N_t = \theta(B)a_t$.

Cross Correlation Function

Consider a stationary bivariate stochastic process (x_t, y_t) with means μ_x, μ_y and variances σ_x^2, σ_y^2. The *cross covariance function* between x and y at lag k is defined by

$$\gamma_{xy}(k) = E[(x_t - \mu_x)(y_{t+k} - \mu_y)], \quad k = 0, 1, 2, \ldots.$$

Note that $\gamma_{xy}(k) = \gamma_{yx}(-k)$, but in general $\gamma_{xy}(k) \neq \gamma_{xy}(-k)$. The *cross correlation function* is defined by

$$\rho_{xy}(k) = \gamma_{xy}(k)/\sigma_x\sigma_y, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Observing (x_t, y_t), $t = 1, \ldots, n$, one estimates $\gamma_{xy}(k)$ by

$$c_{xy}(k) = \begin{cases} \frac{1}{n}\sum_{t=1}^{n-k}(x_t - \bar{x})(y_{t+k} - \bar{y}), & k \geq 0, \\ \frac{1}{n}\sum_{t=1}^{n-k}(y_t - \bar{y})(x_{t-k} - \bar{x}), & k \leq 0, \end{cases}$$

and estimates $\rho_{xy}(k)$ by $r_{xy}(k) = c_{xy}(k)/s_x s_y$, where $s_x = \sqrt{c_{xx}(0)}$, $s_y = \sqrt{c_{yy}(0)}$.

Model Identification

Write $y_t = v(B)x_t + n_t$ and assume $\mu_x = 0$, $\mu_n = 0$, and the independence of x_t and n_t. Taking expectation of

$$x_{t-k}y_t = v_0 x_{t-k}x_t + v_1 x_{t-k}x_{t-1} + v_2 x_{t-k}x_{t-2} + \cdots + x_{t-k}n_t,$$

one has, for $k \geq 0$,

$$\gamma_{xy}(k) = v_0\gamma_{xx}(k) + v_1\gamma_{xx}(k-1) + v_2\gamma_{xx}(k-2) + \cdots.$$

When x_t are white noise, one has $\gamma_{xy}(k) = v_k\sigma_x^2$.

Model identification consists of the following steps.

1. Prewhitening x_t: Obtain α_t from $\phi_x(B)x_t = \theta_x(B)\alpha_t$.
2. Estimating v_k: Obtain $\beta_t = \theta_x^{-1}(B)\phi_x(B)y_t$, calculate $\hat{v}_k = r_{\alpha\beta}(k)s_\beta/s_\alpha$, and guess r, s, b.

Note that $\beta_t = v(B)\alpha_t + \epsilon_t$, where $\epsilon_t = \theta_x^{-1}(B)\phi_x(B)n_t$.

Model Identification: Example

Consider the gas furnace data (series J) in BJR.

```
series.J<-ts(matrix(scan("gas-furnace"),ncol=2,byrow=T))
x<-series.J[,1]; y<-series.J[,2]; acf(x); pacf(x); PP.test(x)
fit.x<-arima(x,c(3,0,0)); fit.x
acf(fit.x$res); Box.test(fit.x$res,10,,4)
alpha<-filter(x,c(1,-fit.x$coef[1:3]),sides=1)[-(1:3)]
beta<-filter(y,c(1,-fit.x$coef[1:3]),sides=1)[-(1:3)]
jk<-ccf(ts(beta),ts(alpha)); abline(v=0,lty=2,col=3)
```

The orders are seen to be $(r, s, b) = (1, 2, 3)$ or $(r, s, b) = (2, 2, 3)$.

```
v<-jk$acf[jk$lag%in%(3:7),1,1]*sqrt(var(beta))/sqrt(var(alpha))
v[4]/v[3]; v[5]/v[4]; sqrt(v[5]/v[3])
```

Take $r = 1$ and set $\delta = .61$, one solves for $\omega_0 = v_3 = -.55$, $\omega_1 = -(v_4 - \delta v_3) = .31$, and $\omega_2 = -(v_5 - \delta v_4) = .49$.

Identification of Noise Model, Estimation

Given the preliminary estimates of $\delta(B)$, $\omega(B)$, and b, one may calculate $\hat{n}_t = y_t - \delta^{-1}(B)\omega(B)x_{t-b}$, then apply standard techniques to identify the order of $\phi(B)n_t = \theta(B)a_t$.

```
wk<-filter(x,c(0,0,0,-.55,-.31,-.49),sides=1)[-(1:5)]
nwk<-(y[-(1:5)]-filter(wk,.61,"recursive"))
acf(nwk); pacf(nwk); acf(arima(nwk,c(2,0,0))$res)
```

This suggest AR(2) for n_t.

Putting things together, one may estimate the whole model using `arima` as follows.

```
xy<-cbind(y,lag(y,-1),lag(x,-3),lag(x,-4),lag(x,-5))
ind<-is.na(apply(xy,1,sum)); xy<-xy[!ind,]
fit.y<-arima(xy[,1],c(2,0,2),xreg=xy[,-1])
fit.y; acf(fit.y$res)
```

Miscellaneous

If x_t or y_t are nonstationary, one may difference (both series) to stationarity with $v(B)$ intact.

Given a *known* transfer function model,

$$\varphi(B)\delta(B)y_t = \varphi(B)\omega(B)x_{t-b} + \delta(B)\theta(B)a_t,$$

where $\varphi_x(B)x_t = \theta_x\alpha_t$, one may conveniently calculate the forecast $\hat{y}_t(l)$ via the difference equation, with any a_{t+k}, $k > 0$ replaced by 0 and x_{t+k}, $k > 0$ replaced by $\hat{x}_t(k)$.

Let $\nu(B) = \sum_{j=0}^{\infty} \nu_j B^j$ satisfy $\delta(B)\varphi_x(B)\nu(B) = \omega(B)B^b\theta_x(B)$ and $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ satisfy $\varphi(B)\psi(B) = \theta(B)$, the variance of the forecast error $y_{t+l} - \hat{y}_t(l)$ is seen to be

$$V(l) = \sum_{j=0}^{l-1}(\nu_j^2\sigma_\alpha^2 + \psi_j^2\sigma_a^2).$$