Periodogram of Time Series

Consider the DFT of $(z_1, \ldots, z_N)$,

$$\zeta_v = \tfrac{1}{\sqrt{N}} \textstyle\sum_{t=1}^{N} z_t e^{-i2\pi tv/N}, \qquad v = 1, \ldots, N.$$

Write $\omega_j = j/N$. The *periodogram* at ω_j is given by $I(\omega_j) = |\zeta_j|^2$. As angular frequencies with a 2π multiple, $\omega_{N-j} = -\omega_j$. We consider $\omega_j \in (-\frac{1}{2}, \frac{1}{2}]$. At $\omega_0 = 0$, $I(\omega_0) = N|\bar{z}|^2$. At $\omega_j \neq 0$,

$$\begin{aligned} I(\omega_j) &= \frac{1}{N} \sum_{t=1}^{N} \sum_{s=1}^{N} (z_t - \bar{z})(z_s - \bar{z}) e^{i2\pi(s-t)\omega_j} \\ &= \frac{1}{N} \Big[\sum_{t=1}^{N} (z_t - \bar{z})^2 + \sum_{k=1}^{N-1} (e^{i2\pi k\omega_j} + e^{-i2\pi k\omega_j}) \sum_{t=1}^{N-k} (z_t - \bar{z})(z_{t+k} - \bar{z}) \Big] \\ &= c_0 + 2 \sum_{k=1}^{N-1} c_k \cos 2\pi k\omega_j . \end{aligned}$$

Hence, $I(\omega_j)$ is the sample version of $\gamma_0 f(\omega_j)$, the *power spectrum*.

Properties of Periodogram – I

The Fourier matrix Γ with the (t,v)th entry $e^{i2\pi tv/N}/\sqrt{N}$ is orthogonal ($\Gamma^H\Gamma = I$), so $\sum_{t=1}^N |z_t|^2 = \sum_{v=1}^N |\zeta_v|^2$.

For z_t real, note that $I(\omega_0) = N\bar{z}^2$ and $I(-\omega_j) = I(\omega_j)$, one has

$$\sum_{t=1}^N (z_t - \bar{z})^2 = 2\sum_{j\in(0,N/2)} I(\omega_j) + I(\omega_{N/2}),$$

where the last term is 0 for N odd.

In terms of sines and cosines,

$$I(\omega_j) = \tfrac{1}{N}(\textstyle\sum_{t=1}^N z_t \cos 2\pi t\omega_j)^2 + \tfrac{1}{N}(\textstyle\sum_{t=1}^N z_t \sin 2\pi t\omega_j)^2.$$

It is known that

$$\{\sqrt{\tfrac{1}{N}}, \sqrt{\tfrac{2}{N}}\cos 2\pi t\omega_j, \sqrt{\tfrac{2}{N}}\sin 2\pi t\omega_j, j \in (0, \tfrac{N}{2}), \sqrt{\tfrac{1}{N}}\cos \pi t\}$$

form an orthonormal basis in R^N, where the last term disappears for N odd.

Properties of Periodogram – II

For Gaussian white noise series z_t with $\text{var}[z_t] = \sigma^2$, it is easy to see that $I(\omega_j)$'s are independent, $2I(\omega_j)/\sigma^2 \sim \chi^2_2$, $j \in (0, N/2)$, and $I(\omega_{N/2})/\sigma^2 \sim \chi^2_1$ for N even. Note that χ^2_2 is the exponential dist.

For general stationary series with N large, it can be shown that

$$E[I(0)] - N\mu^2 \approx \sigma^2(1 + 2\sum_{k=1}^{\infty} \rho_k) = \sigma^2 f(0),$$
$$E[I(\omega_j)] \approx \sigma^2 f(\omega_j), \quad j \neq 0, N/2.$$

In fact, $I(\omega_j)$'s are asymptotically independent exponential r.v.'s for $j \in (0, N/2)$.

$I(\omega_j)$'s are nearly the raw data, so can not be used as reliable estimates of $\gamma_0 f(\omega_j)$. Assuming a smooth spectral density, better estimates of $f(\omega)$ can be obtained through moving averages.

Fourier Analysis of Convolution and Product

Extend $f(\omega)$, $g(\omega)$ beyond $(-1/2, 1/2)$ by periodicity and consider their convolution $h(\omega) = \int_{-1/2}^{1/2} f(x)g(\omega - x)dx$. The Fourier coefficients of $h(\omega)$ is seen to be the product of those of $f(\omega)$ and $g(\omega)$,

$$\begin{aligned} h_v &= \int_{-1/2}^{1/2} e^{i2\pi vx}dx \int_{-1/2}^{1/2} f(y)g(x-y)dy \\ &= \int_{-1/2}^{1/2} f(y)e^{i2\pi vy}dy \int_{-1/2}^{1/2} g(s)e^{i2\pi vs}ds = f_v g_v. \end{aligned}$$

Similarly, the Fourier coefficients of the product $h(\omega) = f(\omega)g(\omega)$ are the convolution of f_v and g_v, $h_v = \sum_u f_u g_{v-u}$.

$$\begin{aligned} h(x) &= \sum_{u=-\infty}^{\infty} f_u e^{-i2\pi ux} \sum_{s=-\infty}^{\infty} g_s e^{-i2\pi sx} \\ &= \sum_{v=-\infty}^{\infty} (\sum_{u=-\infty}^{\infty} f_u g_{v-u})e^{-i2\pi vx} = \sum_{v=-\infty}^{\infty} h_v e^{-i2\pi vx}. \end{aligned}$$

Lag-Window Estimates of Spectrum

A *lag-window estimate* of $p(\omega) = \gamma_0 f(\omega)$ is of the form

$$\hat{p}(\omega) = c_0 + 2\sum_{k=1}^{M} w(\tfrac{k}{M}) c_k \cos 2\pi k\omega = \sum_{|k|\leq M} w(\tfrac{k}{M}) c_k e^{-i2\pi k\omega},$$

where the *lag window* $w(x)$ is symmetric, $|w(x)| \leq 1 = w(0)$.

Write $W_M(\omega) = \sum_v w(\frac{v}{M}) e^{i2\pi v\omega}$ the *spectral window*. It follows from the Fourier analysis of convolution that

$$\hat{p}(\omega) = \int_{-1/2}^{1/2} \tilde{I}(x) W_M(\omega - x) dx,$$

where $\tilde{I}(\omega) = c_0 + 2\sum_{k=1}^{N-1} c_k \cos 2\pi k\omega$ coincides with the periodogram at $\omega_j \neq 0$. Approximating the integral by a Riemann sum, one has a weighted moving average (*i.e.*, kernel estimate),

$$\hat{p}(\omega) \approx \frac{1}{N} \sum_{j\in(-N/2,N/2]} W_M(\omega - \omega_j) \tilde{I}(\omega_j).$$

Examples of Lag and Spectral Windows

Here are some examples of lag and spectral windows.

Truncated $w(x) = I_{[|x|\leq 1]}$; $W_M(\omega) = \frac{\sin \pi(2M+1)\omega}{\sin \pi\omega} = D_M(\omega)$. $D_M(\omega)$ is known as the Dirichlet kernel.

Bartlett $w(x) = (1-|x|)I_{[|x|\leq 1]}$; $W_M(\omega) = \frac{1}{M}(\frac{\sin \pi M\omega}{\sin \pi\omega})^2 = F_M(\omega)$. $F_M(\omega)$ is known as the Fejer kernel of order M.

Daniell $w(x) = \frac{\sin \pi x}{\pi x}$; $W_M(\omega) = MI_{[|\omega|\leq 1/2M]}$.

Blackman-Tukey $w(x) = (1-2a+2a\cos x)I_{[|x|\leq 1]}$; $W_M(\omega) = aD_M(\omega - \frac{1}{2M}) + (1-2a)D_M(\omega) + aD_M(\omega + \frac{1}{2M})$.

Parzen $w(x) = (1-6|x|^2+6|x|^3)I_{[|x|\leq 1/2]} + 2(1-|x|)^3 I_{[1/2<|x|\leq 1]}$; $W_M(\omega) = \frac{12}{M^3}(\frac{\sin(\pi M\omega/2)}{\sin \pi\omega})^4(1-\frac{2}{3}\sin^2 \pi\omega)$.

The Bartlett, Daniell and Parzen windows assure nonnegative $\hat{p}(\omega)$.

Properties of Lag-Window Estimates

Generally speaking, the lag-window estimate $\hat{p}(\omega)$ converges to $p(\omega)$ in probability as $M \to \infty$ and $N/M \to \infty$. $W_M(\omega)$ gets spikier as M increases. A large M reduces bias whereas a large N/M keeps variance under control.

In practice, it is convenient to smooth the periodogram directly through $\tilde{p}(\omega_j) = \sum_{|v|\leq m} \tilde{w}_v \tilde{I}(\omega_{j+v})$, with $\tilde{w}_v \propto W_M(\omega_v)$ truncated for $|v| > m$ and scaled to $\sum_{|v|\leq m} \tilde{w}_v = 1$. Asymptotics plus moment matching approximation suggest that $\nu\tilde{p}(\omega_j)/p(\omega_j) \sim \chi^2_\nu$, where $\nu = 2/\sum_{|v|\leq m} \tilde{w}_v^2$. It then follows that

$$\log \tilde{p}(\omega_j) + \log \nu - (\log \chi^2_{.975,\nu}, \log \chi^2_{.025,\nu})$$

forms an approximate 95% CI for $p(\omega_j)$.

Spectral Analysis in R

The `spec.pgram` function in the R library `ts` can be used to calculate and plot the raw and smoothed periodogram. For the raw periodogram, simply use

```
spec.pgram(x,detrend=F,demean=T,taper=0,fast=F)
```

where the options are specified to override defaults that alter the raw data. To calculate $\tilde{p}(\omega_j) = \frac{1}{9}\sum_{|v|\leq 4}\tilde{I}(\omega_{j+v})$, add the option `kernel=kernel("daniell",4)`. The option `spans=9` invokes a 9-point smoother with half weights at the edges, which can also be specified via `kernel=kernel("modified.daniell",4)`. Other kernels implemented are the Dirichlet and Fejer kernels.

By default, the vertical axis is on the log scale (`log="yes"`); other choices are `log="no"` or `log="dB"` for $10\log_{10}\tilde{p}(\omega_j)$.

Spectral Analysis in R: Convoluted Kernels

The kernel used in `spec.pgram` can be in convoluted form, as the examples below demonstrate.

♣ `kernel("daniell",c(2,1))`; `kernel("daniell",c(1,2))`.
It is the convolution of $(1,1,1,1,1)/5$ and $(1,1,1)/3$.

♣ `kernel("modified.daniell",c(2,3))`.
It is the convolution of $(.5,1,1,1,.5)/4$ and $(.5,1,1,1,1,1,.5)/6$.

To use the kernel specified in the second example, one may simply specify `spec.pgram(...,spans=c(5,7))`.

The scheme extends recursively to more layers of convolution, say `kernel("modified.daniell",c(1,2,3))`, which is equivalent to `spans=c(3,5,7)`.

Cumulative Periodogram

Given the periodogram of $z_1, \dots, z_N$ from a stationary process,

$$I(\omega_j) = \frac{1}{N}\left(\sum_{t=1}^N z_t \cos 2\pi t\omega_j\right)^2 + \frac{1}{N}\left(\sum_{t=1}^N z_t \sin 2\pi t\omega_j\right)^2,$$

where $\omega_j = j/N$, the *cumulative periodogram* on $(0, 1/2)$,

$$C(\omega) = \sum_{0<\omega_j\leq\omega} I(\omega_j) \Big/ \sum_{0<\omega_j\leq 1/2} I(\omega_j),$$

is the empirical version of $P(\omega) = \int_{-\omega}^{\omega} p(\lambda)d\lambda = 2\int_0^{\omega} p(\lambda)d\lambda$, where $p(\lambda)$ is the spectral density. To test the hypothesis that the spectral distribution is given by some known $P(\omega)$, one may use the Kolmogorov-Smirnov statistic, $\sup|C(\omega) - P(\omega)|$.

For white noise, the spectral density $p(\omega) = 1$ is uniform, and $P(\omega) = 2\omega$. Tolerance band for $C(\omega)$ under the null can be constructed from the Kolmogorov-Smirnov distribution. The cumulative periodogram check is implemented in `cpgram`.