1. Express an MA(2) process as a state space model and derive the corresponding Kalman filter.

   Solution: Following Slide 2, one has the state equation

$$\mathbf{Y}_t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{Y}_{t-1} + \begin{pmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{pmatrix} a_t,$$

   where $y_t^{(1)} = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} = z_t$, $y_t^{(2)} = -\theta_1 a_t - \theta_2 a_{t-1}$, and $y_t^{(3)} = -\theta_2 a_t$. The observation equation is simply

$$z_t = (1, 0, 0)\mathbf{Y}_t.$$

   The stationary distribution of $\mathbf{Y}_t$ has mean $\mathbf{y}_0 = \mathbf{0}$ and

$$\begin{aligned} V_0 &= \sigma_a^2 \begin{pmatrix} 1+\theta_1^2+\theta_2^2 & -\theta_1+\theta_1\theta_2 & -\theta_2 \\ -\theta_1+\theta_1\theta_2 & \theta_1^2+\theta_2^2 & \theta_1\theta_2 \\ -\theta_2 & \theta_1\theta_2 & \theta_2^2 \end{pmatrix} \\ &= \sigma_a^2 \left[\begin{pmatrix} -\theta_2 \\ 0 \\ 0 \end{pmatrix} (-\theta_2, 0, 0) + \begin{pmatrix} -\theta_1 \\ -\theta_2 \\ 0 \end{pmatrix} (-\theta_1, -\theta_2, 0) + \begin{pmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{pmatrix} (1, -\theta_1, -\theta_2) \right] \\ &= \sigma_a^2 \left[\begin{pmatrix} \tilde{V}_0 & \mathbf{0} \\ \mathbf{0^T} & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{pmatrix} (1, -\theta_1, -\theta_2) \right]. \end{aligned}$$

   Write $\Phi = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $A = \sigma_a^2 \left(\begin{smallmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{smallmatrix}\right)(1, -\theta_1, -\theta_2)$. One has $\mathbf{y}_{1|0} = \Phi\mathbf{y_0} = \mathbf{0}$ and $V_{1|0} = \Phi V_0 \Phi^T + A = V_0$. Observing z_1, the conditional distribution of $\mathbf{Y}_1|z_1$ has mean

$$\mathbf{y}_1 = \mathbf{y}_{1|0} + K_1 z_1 = \begin{pmatrix} z_1 \\ \rho_1 z_1 \\ \rho_2 z_1 \end{pmatrix} = \begin{pmatrix} z_1 \\ -\theta_1 \tilde{a}_{1|1} - \theta_2 \tilde{a}_{0|1} \\ -\theta_2 \tilde{a}_{1|1} \end{pmatrix},$$

   where $\tilde{a}_{1|1} = E[a_1|z_1]$ and $\tilde{a}_{0|1} = E[a_0|z_1]$, and covariance

$$V_1 = (I - K_1(1,0,0))V_{1|0} = \sigma_a^2 \begin{pmatrix} 0 & \mathbf{0^T} \\ \mathbf{0} & \tilde{V}_1 \end{pmatrix};$$

   the Kalman gain matrix K_1 is given by

$$K_1 = V_{1|0} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sigma_a^2(1+\theta_1^2+\theta_2^2)} = \mathbf{v}_{1|0}/v_{1|0} = \begin{pmatrix} 1 \\ \rho_1 \\ \rho_2 \end{pmatrix},$$

   where $\mathbf{v_{1|0}}$ is the first column of $V_{1|0}/\sigma_a^2$, $v_{1|0}$ is the $(1,1)$th entry of $V_{1|0}/\sigma_a^2$, $\rho_1 = (-\theta_1 + \theta_1\theta_2)/(1+\theta_1^2+\theta_2^2)$ and $\rho_2 = (-\theta_2)/(1+\theta_1^2+\theta_2^2)$.

With $\mathbf{y}_1$ and V_1 given above, one has

$$\mathbf{y}_{2|1} = \Phi \mathbf{y}_1 = \begin{pmatrix} -\theta_1 \tilde{a}_{1|1} - \theta_2 \tilde{a}_{0|1} \\ -\theta_2 \tilde{a}_{1|1} \\ 0 \end{pmatrix},$$

$$V_{2|1} = \Phi V_1 \Phi^T + A = \sigma_a^2 \left[\begin{pmatrix} \tilde{V}_1 & \mathbf{0} \\ \mathbf{0^T} & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -\theta_1 \\ -\theta_2 \end{pmatrix} (1, -\theta_1, -\theta_2) \right].$$

Observing z_2, the conditional distribution of $\mathbf{Y}_2|(z_1, z_2)$ has mean and covariance

$$\mathbf{y}_2 = \mathbf{y}_{2|1} + K_2 e_2 = \begin{pmatrix} z_2 \\ -\theta_1 \tilde{a}_{2|2} - \theta_2 \tilde{a}_{1|2} \\ -\theta_2 \tilde{a}_{2|2} \end{pmatrix},$$

$$V_2 = (I - K_2(1,0,0)) V_{2|1} = \sigma_a^2 \begin{pmatrix} 0 & \mathbf{0^T} \\ \mathbf{0} & \tilde{V}_2 \end{pmatrix};$$

where $e_2 = z_2 - \hat{z}_{2|1} = z_2 + \theta_1 \tilde{a}_{1|1} + \theta_2 \tilde{a}_{0|1}$, $\tilde{a}_{2|2} = E[a_2|z_1, z_2]$, $\tilde{a}_{1|2} = E[a_1|z_1, z_2]$, and the Kalman gain matrix K_2 is given by

$$K_2 = V_{2|1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{1}{\sigma_a^2 v_{2|1}} = \mathbf{v}_{2|1} / v_{2|1},$$

where $\mathbf{v_{2|1}}$ is the first column of $V_{2|1}/\sigma_a^2$ and $v_{2|1}$ is the $(1,1)$th entry of $V_{2|1}/\sigma_a^2$. Replacing 1 by $t-1$ and 2 by t, one has the general formula.

2. Repeat the above problem for ARMA(2,1). Do specify V_0 in terms of γ_0, γ_1, σ_a^2, ϕ_1, ϕ_2, and θ as necessary, and do verify $\Phi V_0 \Phi^T + A = V_0$.

Solution: Similar to ARMA(1,1), one has the state equation

$$\mathbf{Y}_t = \begin{pmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{pmatrix} \mathbf{Y}_{t-1} + \begin{pmatrix} 1 \\ -\theta \end{pmatrix} a_t,$$

where $y_t^{(1)} = \phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t - \theta a_{t-1} = z_t$ and $y_t^{(2)} = \phi_2 z_{t-1} - \theta a_t$. The observation equation is simply

$$z_t = (1, 0) \mathbf{Y}_t.$$

The stationary distribution of $\mathbf{Y_t}$ has mean $\mathbf{y_0} = \mathbf{0}$ and covariance

$$V_0 = \begin{pmatrix} \gamma_0 & \phi_2 \gamma_1 - \theta \sigma_a^2 \\ \phi_2 \gamma_1 - \theta \sigma_a^2 & \phi_2^2 \gamma_0 + \theta^2 \sigma_a^2 \end{pmatrix}.$$

To verify $V_0 = \Phi V_0 \Phi^T + A = V_0$, where $\Phi = \left(\begin{smallmatrix} \phi_1 & 1 \\ \phi_2 & 0 \end{smallmatrix} \right)$ and $A = \sigma_a^2 \left(\begin{smallmatrix} 1 & -\theta \\ -\theta & \theta^2 \end{smallmatrix} \right)$, note that

$$\begin{aligned} \gamma_0 &= E[\phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t - \theta a_{t-1}]^2 \\ &= (\phi_1^2 + \phi_2^2)\gamma_0 + (1 + \theta^2)\sigma_a^2 - 2\phi_1 \phi_2 \gamma_1 - 2\phi_1 \theta \sigma_a^2, \\ \gamma_1 &= E z_{t-1}[\phi_1 z_{t-1} + \phi_2 z_{t-2} + a_t - \theta a_{t-1}] \\ &= \phi_1 \gamma_0 + \phi_2 \gamma_1 - \theta \sigma_a^2 \end{aligned}$$

Given $\mathbf{y}_0$ and V_0, one has $\mathbf{y}_{1|0} = \Phi\mathbf{y}_0 = \mathbf{0}$ and $V_{1|0} = \Phi V_0 \Phi^T + A$ $(= V_0)$. Observing z_1, the conditional distribution of $\mathbf{Y}_1|z_1$ has mean

$$\mathbf{y}_1 = \mathbf{y}_{1|0} + K_1 z_1 = \begin{pmatrix} z_1 \\ \phi_2\tilde{z}_0 - \theta\tilde{a}_1 \end{pmatrix},$$

where $\tilde{z}_0 = E[z_0|z_1]$ and $\tilde{a}_1 = E[a_1|z_1]$, and covariance

$$V_1 = (I - K_1(1,0))V_{1|0} = \begin{pmatrix} 0 & 0 \\ 0 & v_1 \end{pmatrix};$$

the Kalman gain matrix K_1 is given by

$$K_1 = V_{1|0}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{v_{1|0}} = \mathbf{v}_{1|0}/v_{1|0} = \begin{pmatrix} 1 \\ \kappa_1 \end{pmatrix},$$

where $\mathbf{v_{1|0}}$ is the first column of $V_{1|0}$ and $v_{1|0}$ is the $(1,1)$th entry of $V_{1|0}$.

With $\mathbf{y_1}$ and V_1 given above, one has

$$\mathbf{y}_{2|1} = \Phi\mathbf{y}_1 = \begin{pmatrix} \phi_1 z_1 + \phi_2\tilde{z}_0 - \theta\tilde{a}_1 \\ \phi_2 z_1 \end{pmatrix},$$
$$V_{2|1} = \Phi V_1 \Phi^T + A = \begin{pmatrix} v_1 & 0 \\ 0 & 0 \end{pmatrix} + \sigma_a^2\begin{pmatrix} 1 & -\theta \\ -\theta & \theta^2 \end{pmatrix}.$$

Observing z_2, the conditional distribution of $\mathbf{Y}_2|(z_1, z_2)$ has mean and covariance

$$\mathbf{y}_2 = \mathbf{y}_{2|1} + K_2 e_2 = \begin{pmatrix} z_2 \\ \phi_2 z_1 - \theta\tilde{a}_2 \end{pmatrix},$$
$$V_2 = (I - K_2(1,0))V_{2|1} = \begin{pmatrix} 0 & 0 \\ 0 & v_2 \end{pmatrix};$$

where $e_2 = z_2 - \hat{z}_{2|1} = z_2 - (\phi_1 z_1 + \phi_2\tilde{z}_0 - \theta\tilde{a}_1)$, $\tilde{a}_2 = E[a_2|z_1, z_2]$, and the Kalman gain matrix K_2 is given by

$$K_2 = V_{2|1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{v_{2|1}} = \mathbf{v}_{2|1}/v_{2|1} = \begin{pmatrix} 1 \\ \kappa_2 \end{pmatrix},$$

where $\mathbf{v_{2|1}}$ is the first column of $V_{2|1}$ and $v_{2|1}$ is the $(1,1)$th entry of $V_{2|1}$. Replacing 1 by $t-1$ and 2 by t, one has the general formula.