1. Specify the exact likelihood of a Gaussian AR(2) process.

   Solution: The likelihood is seen to be

$$f(\mathbf{z}|\boldsymbol{\phi},\sigma_a^2)=f(\mathbf{z}_{(2)}|\mathbf{z}_2,\boldsymbol{\phi},\sigma_a^2)f(\mathbf{z}_2|\boldsymbol{\phi},\sigma_a^2),$$

   where $\mathbf{z}_2=(z_1,z_2)^T$ and $\mathbf{z}_{(2)}=(z_3,\dots,z_n)^T$. One has

$$f(\mathbf{z}_{(2)}|\mathbf{z}_2,\boldsymbol{\phi},\sigma_a^2)\propto(\sigma_a^2)^{-(n-2)/2}\exp(-\sum_{t=3}^{n}a_t^2/2\sigma_a^2),$$

   where $a_t=z_t-\phi_1z_{t-1}-\phi_2z_{t-2}$, and

$$f(\mathbf{z}_2|\boldsymbol{\phi},\sigma_a^2)\propto(\sigma_a^2)^{-1}|\mathbf{M}_2|^{1/2}\exp(-\mathbf{z}_2^T\mathbf{M}_2\mathbf{z}_2/2\sigma_a^2),$$

   where

$$\mathbf{M}_2=(1-\phi_2^2)\begin{pmatrix}1 & -\phi_1/(1-\phi_2)\\ -\phi_1/(1-\phi_2) & 1\end{pmatrix}=\begin{pmatrix}1-\phi_2^2 & -\phi_1(1+\phi_2)\\ -\phi_1(1+\phi_2) & 1-\phi_2^2\end{pmatrix}.$$

   The log likelihood is thus

$$-\frac{n}{2}\log\sigma_a^2+\frac{1}{2}\log\{(1+\phi_2)^2((1-\phi_2)^2-\phi_1^2)\}$$
$$-\frac{1}{2\sigma_a^2}\{(1-\phi_2^2)(z_1^2+z_2^2)-2\phi_1(1+\phi_2)z_1z_2+\textstyle\sum_{t=3}^{n}(z_t-\phi_1z_{t-1}-\phi_2z_{t-2})^2\}.$$

2. Specify the innovations coefficients $\theta_{t,j}$ and variance v_t for an ARMA(2,1) process. To simplify the notation, you may choose to express things also in terms of γ_0, γ_1, γ_2, etc. instead of exclusively in terms of ϕ_1, ϕ_2, θ_1, and σ_a^2.

   Solution: For ARMA(2,1), one has $u_t=z_tI_{[t\le 2]}+(a_t-\theta a_{t-1})I_{[t>2]}$. $E[u_tu_s]=\kappa_{t,s}$ for $s\le t$ are given by

$$\kappa_{t,s}=\begin{cases}\gamma_{t-s}, & t\le 2,\\ (1+\theta^2)\sigma_a^2, & s=t>2,\\ -\theta\sigma_a^2, & s+1=t>2,\\ 0, & \text{otherwise.}\end{cases}$$

The recursive algorithm yields

$$\begin{aligned}
v_0 &= \kappa_{1,1} = \gamma_0, \\
\theta_{1,1} &= \kappa_{2,1}/v_0 = \gamma_1/\gamma_0 = \rho_1, \\
v_1 &= \kappa_{2,2} - \theta_{1,1}^2 v_0 = (1-\rho_1^2)\gamma_0, \\
\theta_{2,2} &= \kappa_{3,1}/v_0 = 0, \\
\theta_{2,1} &= \kappa_{3,2}/v_1 = -\theta\sigma_a^2/v_1 = -\theta/\tilde{v}_1, \\
v_2 &= \kappa_{3,3} - \theta_{2,1}^2 v_1 = \sigma_a^2(1+\theta^2-\theta^2/\tilde{v}_1) = \sigma_a^2(1+\theta^2(\tilde{v}_1-1)/\tilde{v}_1) = \sigma_a^2\tilde{v}_2, \\
\theta_{3,3} &= \theta_{3,2} = 0, \\
\theta_{3,1} &= \kappa_{4,3}/v_2 = -\theta\sigma_a^2/v_2 = -\theta/\tilde{v}_2, \\
v_3 &= \kappa_{4,4} - \theta_{3,1}^2 v_2 = \sigma_a^2(1+\theta^2(\tilde{v}_2-1)/\tilde{v}_2) = \sigma_a^2\tilde{v}_3, \\
&\ldots \\
\theta_{t,j} &= 0, \quad j>1, \\
\theta_{t,1} &= \kappa_{t+1,t}/v_{t-1} = -\theta\sigma_a^2/v_{t-1} = -\theta/\tilde{v}_{t-1}, \\
v_t &= \sigma_a^2(1+\theta^2(\tilde{v}_{t-1}-1)/\tilde{v}_{t-1}) = \sigma_a^2\tilde{v}_t, \\
&\ldots
\end{aligned}$$

3. For an ARMA(2,1) process, specify equations one may solve to obtain $\tilde{\gamma}_0$, $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ in terms of ϕ_1, ϕ_2, θ, where $\tilde{\gamma}_k = \gamma_k/\sigma_a^2$.

   Solution: One may solve from

$$\begin{pmatrix} 1 & -\phi_1 & -\phi_2 \\ -\phi_1 & 1-\phi_2 & 0 \\ -\phi_2 & -\phi_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\gamma}_0 \\ \tilde{\gamma}_1 \\ \tilde{\gamma}_2 \end{pmatrix} = -\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix},$$

   where $b_0 = \theta_0\psi_0 + \theta_1\psi_1 = -1 + \theta(\phi_1 - \theta)$, $b_1 = \theta_0\psi_{-1} + \theta_1\psi_0 = \theta$, and $b_2 = \theta_0\psi_{-2} + \theta_1\psi_{-1} = 0$.