1. Consider the ARIMA model $(1-1.4B+.49B^2)(1-B)z_t=(1-.5B)a_t$.
   (a) Find the coefficients $\psi_1,\ldots,\psi_6$ in the MA form of z_t.
   (b) Give a general expression of ψ_j for $j>1$.
   (c) Express z_t in a truncated MA form with respect to the time origin $t=3$.
   (d) Find the π weights in the infinite order AR form, $(1-\sum_{j=1}^{\infty}\pi_jB^j)z_t=a_t$, and verify that $\sum_{j=1}^{\infty}\pi_j=1$.
   (e) Find the variance and autocorrelation of $w_t=z_t-z_{t-1}$.

   Solution:

   (a) $(\psi_1,\ldots,\psi_6)=(1.9,2.67,3.307,3.8215,4.22967,4.549003)$.
   (b) Since $1-1.4x+.49x^2=(1-.7x)^2$, $\psi_j=A_1+A_2(.7)^j+A_3j(.7)^j$, $j>1$. From $A_1+.7A_2+.7A_3=\psi_1=1.9$, $A_1+A_2=\psi_0=1$, and $A_1+A_2/.7-A_3/.7=\psi_{-1}=0$, one solves for $A_1=50/9$, $A_2=-41/9$, $A_3=-2/3$.
   (c) One has for $t>4$

$$z_t=\sum_{j=0}^{t-4}(A_1+A_2(.7)^j+A_3j(.7)^j)a_{t-j}+b_0+b_1(.7)^{t-3}+b_2(t-3)(.7)^{t-3},$$

   where (A_1,A_2,A_3) are as given in (b) and (b_0,b_1,b_2) satisfy

$$\begin{aligned}a_4+b_0+.7b_1+.7b_2&=z_4=2.4z_3-1.89z_2+.49z_1+a_4-.5a_3,\\ b_0+b_1&=z_3,\\ b_0+b_1/.7-b_2/.7&=z_2,\end{aligned}$$

   yielding

$$\begin{aligned}b_0&=(z_3-1.4z_2+.49z_1-.5a_3)/.09,\\ b_1&=z_3-b_0,\\ b_2&=z_3-.7z_2-.3b_0.\end{aligned}$$

   (d) Since

$$\pi(B)=\frac{1-2.4B+1.89B^2-.49B^3}{1-.5B}=(1-2.4B+1.89B^2-.49B^3)(1+\sum_{j=1}^{\infty}(.5)^jB^j),$$

   one has

$$\begin{aligned}\pi_1&=1.9,\\ \pi_2&=-.94,\\ \pi_j&=.02(.5)^{j-3},\quad j>2.\end{aligned}$$

   $\sum_{j=1}^{\infty}\pi_j=1.9-.94+.02/(1-.5)=1.$

(e) For $w_t = (1-B)z_t$ an ARMA(2,1) process, one has

$$\gamma_0 = \phi_1^2\gamma_0 + \phi_2^2\gamma_0 + 2\phi_1\phi_2\gamma_1 - 2\phi_1\theta\sigma_a^2 + \sigma_a^2 + \theta^2\sigma_a^2,$$
$$\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1 - \theta\sigma_a^2.$$

Plugging in $\phi_1 = 1.4$, $\phi_2 = -.49$, and $\theta = .5$, one has

$$\begin{pmatrix} -1.2001 & 1.372 \\ -1.4 & 1.49 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -0.15 \\ -0.5 \end{pmatrix},$$

or $\gamma_0 = 3.486593$ and $\gamma_1 = 2.940423$. Thus $\rho_1 = \gamma_1/\gamma_0 = 0.8433514$. For $k > 1$, $\rho_k = 1.4\rho_{k-1} - .49\rho_{k-2}$, so $\rho_k = A_1(.7)^k + A_2 k(.7)^k$, with $A_1 = 1$ and $A_2 = 0.2047877$ determined by $\rho_0 = 1$ and $\rho_1 = 0.8433514$.

2. Consider $Z_t = z_t + b_t$, where $z_t = z_{t-1} + a_t$ with a_t a white noise, $\sigma_a^2 = 1$, and b_t is another white noise independent of a_t, $\sigma_b^2 = 2$. Show that Z_t is an IMA(0,1,1) process, and specify all its parameters.

   Solution: Consider $w_t = (1-B)Z_t = a_t + b_t - b_{t-1}$. One has $\gamma_w(0) = \sigma_a^2 + 2\sigma_b^2 = 5$, $\gamma_w(1) = -\sigma_b^2 = -2$, $\gamma_k = 0$, $k > 1$. So Z_t is IMA(0,1,1).

   Write $w_t = u_t - \Theta u_{t-1}$. From $\gamma_0 = 5 = \sigma_u^2(1+\Theta^2)$ and $\gamma_1 = -2 = -\Theta\sigma_u^2$, one solves for $\Theta = 0.5$ and $\sigma_u^2 = 4$.

3. Consider $Z_t = z_t + b_t$, where $z_t = \phi z_{t-1} + a_t$ is stationary with a_t a white noise, and b_t is another white noise independent of a_t.

   (a) Show that Z_t is an ARMA process and identify its orders.

   (b) Express the parameters of Z_t in terms of ϕ, σ_a^2, and σ_b^2.

   Solution:

   (a) $(1-\phi B)Z_t = a_t + b_t - \phi b_{t-1}$, so Z_t is ARMA(1,1).

   (b) Write $w_t = (1-\phi B)Z_t = u_t - \theta u_{t-1}$. From $\gamma_w(0) = \sigma_u^2(1+\theta^2) = \sigma_a^2 + \sigma_b^2(1+\phi^2)$ and $\gamma_w(1) = -\theta\sigma_u^2 = -\phi\sigma_b^2$, one solves

$$\theta = \left(r + 1 + \phi^2 - \sqrt{r^2 + 2r(1+\phi^2) + (1-\phi^2)^2}\right)/2\phi,$$
$$\sigma_u^2 = \sigma_b^2(\phi/\theta),$$

   where $r = \sigma_a^2/\sigma_b^2$. The ϕ parameter remains the same.