1. Consider the MA(2) process $z_t = a_t - a_{t-1} + \frac{2}{9}a_{t-2}$, $\text{var}[a_t] = \sigma_a^2 = 3$.

   (a) Find the variance and autocorrelation function of $\{z_t\}$.

   (b) Determine if the process is invertible, and if so express the process in the infinite autoregressive form, $z_t = \sum_{j=1}^{\infty} \pi_j z_{t-j} + a_t$, providing numerical values for $\pi_1, \ldots, \pi_5$.

   (c) Show that $\pi_j - \theta_1\pi_{j-1} - \theta_2\pi_{j-2} = 0$ for $j > 0$, where $\pi_{-1} = 0$. Find a general expression for π_j.

   Solution:

   (a) $\gamma_0 = 3(1 + 1 + (2/9)^2) = 166/27$, $\gamma_1 = 3(-1 - 2/9) = -11/3$, $\gamma_2 = 3(2/9) = 2/3$, $\gamma_k = 0$, $k > 2$. $\rho_1 = -99/166$, $\rho_2 = 9/83$, $\rho_k = 0$, $k > 2$.

   (b) Since $1 - x + (2/9)x^2 = (1 - (1/3)x)(1 - (2/3)x)$, so the process is invertible. Since

   $$\pi(B) = \frac{1}{\theta(B)} = \frac{2}{1-(2/3)B} - \frac{1}{1-(1/3)B}$$
   $$= 1 + \sum_{j=1}^{\infty}[2(2/3)^j - (1/3)^j]B^j,$$

   $\pi_1 = -1$, $\pi_2 = -7/9$, $\pi_3 = -5/9$, $\pi_4 = -31/81$, $\pi_5 = -7/27$.

   (c) One has $-(1 - \theta_1 B - \theta_2 B^2)(\sum_{j=-\infty}^{\infty} \pi_j B^j) = 1$, where $\pi_0 = -1$ and $\pi_j = 0$, $j < 0$. The coefficient of B^j is seen to be

   $$\pi_j - \theta_1\pi_{j-1} - \theta_2\pi_{j-2} = 0,$$

   $j > 0$. Using the difference equation, $\pi_j = A_1(2/3)^j + A_2(1/3)^j$, $j > 0$. Solving from $\pi_{-1} = 0 = A_1(3/2) + A_2 3$ and $\pi_0 = -1 = A_1 + A_2$, one gets $A_1 = -2$, and $A_2 = 1$, so $\pi_j = -2(2/3)^j + (1/3)^j$.

2. Consider the AR(2) process $z_t = 1.6z_{t-1} - .89z_{t-2} + a_t$, $\sigma_a^2 = 2$.

   (a) Verify that $\{z_t\}$ is stationary, and find the variance γ_0.

   (b) Solve the first 2 Yule-Walker equations for ρ_1 and ρ_2, and then compute ρ_k for $k = 3, \ldots, 8$ using the recursive equation $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2}$.

   (c) Give a general formula for ρ_k as the solution of the linear difference equation, $\rho_k - 1.6\rho_{k-1} + .89\rho_{k-2} = 0$, $k > 0$. Do express your result in terms of real numbers and real functions.

   (d) Consider the infinite moving average form of the process, $z_t = a_t + \sum_{j=1}^{\infty} \psi_j a_{t-j}$. Show that $\psi_j - 1.6\psi_{j-1} + .89\psi_{j-2} = 0$, $j > 0$, where $\psi_{-1} = 0$, and find a general expression for ψ_j.

   Solution:

(a) The roots of $1 - 1.6x + .89x^2$, $1.059998e^{\pm 0.5585993i}$, are out of the unit circle, so the process is stationary.

$$\gamma_0 = \frac{1-(-.89)}{1+(-.89)} \frac{2}{(1-(-.89))^2 - 1.6^2} = 33.95281.$$

(b) From $\phi_1 + \rho_1\phi_2 = \rho_1$, one has $\rho_1 = \phi_1/(1-\phi_2) = 1.6/1.89 = 0.84656085$. From $\rho_1\phi_1 + \phi_2 = \rho_2$, one has $\rho_2 = 0.46449735$. ρ_k for $k = 3, \ldots, 8$ are given by $(-0.01024339, -0.42979206, -0.67855069, -0.70316616, -0.52115575, -0.20803131)$.

(c) $\rho_k = (0.9433981)^k(a\cos(0.5585993k) + b\sin(0.5585993k))$. Solving from $\rho_{-1} = 0.84656085 = (0.9433981)^{-1}(a0.8479983 - b0.5299989)$ and $\rho_0 = 1 = a$, one has $a = 1$, $b = 0.0931217$

(d) $\psi_k = (0.9433981)^k(a\cos(0.5585993k) + b\sin(0.5585993k))$. Solving from $\psi_{-1} = 0 = (0.9433981)^{-1}(a0.8479983 - b0.5299989)$ and $\psi_0 = 1 = a$, one has $a = 1$ and $b = 1.6$.

3. Assume that the stationary process $\{z_t\}$ is generated by a low order AR model with $p \leq 4$, and that $\{z_t\}$ has the following variance and autocorrelation function,

$$\gamma_0 = 12, \quad \rho_1 = .8, \quad \rho_2 = .46, \quad \rho_3 = .152, \quad \rho_4 = -.0476.$$

(a) For each of the possible orders 1, 2, 3, solve the first sets of Yule-Walker equations to obtain the corresponding ϕ_j's and σ_a^2.

(b) What is the actual AR order p of this process?

Solution:

(a) For order 1, $\phi_1 = \rho_1 = .8$. $\sigma_a^2 = \gamma_0(1 - \phi_1\rho_1) = 4.32$.
For order 2,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 1 & .8 \\ .8 & 1 \end{pmatrix}^{-1} \begin{pmatrix} .8 \\ .46 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.5 \end{pmatrix}.$$

$\sigma_a^2 = \gamma_0(1 - \phi_1\rho_1 - \phi_2\rho_2) = 3.24$.
For order 3,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 1 & .8 & .46 \\ .8 & 1 & .8 \\ .46 & .8 & 1 \end{pmatrix}^{-1} \begin{pmatrix} .8 \\ .46 \\ .152 \end{pmatrix} = \begin{pmatrix} 1.2 \\ -0.5 \\ 0 \end{pmatrix}$$

$\sigma_a^2 = \gamma_0(1 - \phi_1\rho_1 - \phi_2\rho_2 - \phi_3\rho_3) = 3.24$.

(b) From (a), it appears that the true order is $p = 2$. To confirm, double check that $1.2\rho_3 - .5\rho_2 = -.0476 = \rho_4$, or solve the order 4 Y-W equation.

4. Suppose a sequence of real numbers $\{x_t\}_0^\infty$ satisfy the linear difference equation,

$$x_t + ax_{t-1} + bx_{t-2} = 0, \quad t > 1.$$

(a) Find the coefficients a, b, such that x_t is sinusoidal with period 12,

$$x_t = a_1 \cos(2\pi t/12) + a_2 \sin(2\pi t/12).$$

(b) Find the coefficients a, b, such that x_t is damped sinusoidal with damping factor .8 and period 12,

$$x_t = (.8)^t \{a_1 \cos(2\pi t/12) + a_2 \sin(2\pi t/12)\}.$$

Solution:

(a) The roots of $1 + ax + bx^2$ should be $e^{\pm 2\pi/12i}$, so

$$1 + ax + bx^2 = (1 - e^{2\pi/12i}x)(1 - e^{-2\pi/12i}x) = 1 - \sqrt{3}x + x^2,$$

or $a = -\sqrt{3}$ and $b = 1$.

(b) The roots of $1 + ax + bx^2$ should be $(.8)^{-1}e^{\pm 2\pi/12i}$, so

$$1 + ax + bx^2 = (1 - .8e^{2\pi/12i}x)(1 - .8e^{-2\pi/12i}x) = 1 - .8\sqrt{3}x + .64x^2,$$

or $a = -.8\sqrt{3}$ and $b = .64$.