1. For an AR(1) process $z_t = \phi z_{t-1} + a_t$, $|\phi| < 1$, its spectral density can be shown to be $f(\omega) = (\sigma_2^2/\gamma_0)|1 - \phi e^{-i2\pi\omega}|^{-2}$. Verify that $f(\omega) = (1-\phi^2)/(1 - 2\phi\cos 2\pi\omega + \phi^2)$.

   Solution: Note that $\gamma_0 = \sigma_a^2/(1-\phi^2)$, and

$$\begin{aligned}|1 - \phi e^{-i2\pi\omega}|^2 &= 1 + \phi^2 - \phi e^{-i2\pi\omega} - \phi e^{i2\pi\omega} \\ &= 1 + \phi^2 - 2\phi\cos 2\pi\omega.\end{aligned}$$

2. For an MA(1) process $z_t = a_t - \theta a_{t-1}$, its spectral density can be shown to be $f(\omega) = (\sigma_2^2/\gamma_0)|1 - \theta e^{-i2\pi\omega}|^2$. Verify that $f(\omega) = 1 - 2\theta\cos 2\pi\omega/(1+\theta^2)$.

   Solution: Note that $\gamma_0 = \sigma_a^2(1+\theta^2)$, and

$$\begin{aligned}|1 - \theta e^{-i2\pi\omega}|^2 &= 1 + \theta^2 - \theta e^{-i2\pi\omega} - \theta e^{i2\pi\omega} \\ &= 1 + \theta^2 - 2\theta\cos 2\pi\omega.\end{aligned}$$

3. Let $\{y_t\}$ be a stationary process with the power spectrum $p_y(\omega)$. Define $z_t = y_t - y_{t-1}$. Obtain the power spectrum of z_t in terms of $p_y(\omega)$.

   Solution:

$$\begin{aligned}p_z(\omega) &= \sum_k \gamma_z(k) e^{-i2\pi k\omega} \\ &= \sum_k (2\gamma_y(k) - \gamma_y(k-1) - \gamma_y(k+1)) e^{-i2\pi k\omega} \\ &= p_y(\omega)(2 - e^{-i2\pi\omega} - e^{i2\pi\omega}) \\ &= p_y(\omega)|1 - e^{-i2\pi\omega}|^2\end{aligned}$$

4. Let $\{y_t\}_{-\infty}^{\infty}$ and $\{z_t\}_{-\infty}^{\infty}$ be two stationary processes, independent of each other, with power spectrums $p_y(\omega)$ and $p_z(\omega)$. Find the power spectrum of the stationary process $w_t = ay_t + bz_t$ in terms of $p_y(\omega)$ and $p_z(\omega)$, where a and b are constants.

   Solution: Since $\gamma_w(k) = a^2\gamma_y(k) + b^2\gamma_z(k)$, $p_w(\omega) = a^2 p_y(\omega) + b^2 p_z(\omega)$.

5. Let a_i, b_i be independent r.v.'s with $E[a_i] = E[b_i] = 0$ and $\text{var}[a_i] = \text{var}[b_i] = \sigma_i^2$. Find the spectral distribution of the stationary process $z_t = \sum_{i=1}^m (a_i \cos 2\pi\omega_i t + b_i \sin 2\pi\omega_i t)$. Note that the spectral density does not exist since the spectral distribution is discrete.

   Solution: Since $\gamma_k = \sum_{i=1}^m \sigma_i^2 \cos 2\pi\omega_i k$, $F(\omega)$ has jumps of size $\sigma_i^2/2\sum_{j=1}^m \sigma_j^2$ at $\omega = \pm\omega_i$.

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