1. Let $\{a_t\}_{-\infty}^{\infty}$ be a white noise process with $E[a_t] = 0$ and $\text{var}[a_t] = \sigma_a^2$. Define $z_t = \mu + a_t - .5a_{t-1}$. Find the mean, autocovariance, and autocorrelation of z_t, and verify that $\{z_t\}_{-\infty}^{\infty}$ is stationary.

   Solution:

   $E[z_t] = \mu$. $\text{var}[z_t] = 1.25\sigma_a^2$, $\text{cov}[z_t, z_{t-1}] = -.5\sigma_a^2$, $\text{cov}[z_t, z_{t-k}] = 0$, $k > 1$. $\rho_1 = -.4$, $\rho_k = 0, k > 1$. All are independent of t.

2. Let $\{y_t\}$ be a stationary process with mean μ_y and autocovariance $\gamma_y(s) = \text{cov}[y_t, y_{t-s}]$. Define $z_t = y_t - y_{t-1}$. Obtain the mean and autocovariance of $\{z_t\}_{-\infty}^{\infty}$ in terms of those of y_t and verify that it is stationary.

   Solution:

   $E[z_t] = 0$. $\text{cov}[z_t, z_{t-k}] = \text{cov}[y_t - y_{t-1}, y_{t-k} - y_{t-k-1}] = 2\gamma_y(k) - \gamma_y(k-1) - \gamma_y(k+1)$. All are independent of t.

3. Let $\{y_t\}_{-\infty}^{\infty}$ and $\{z_t\}_{-\infty}^{\infty}$ be two stationary processes with means μ_y and μ_z and autocovariances $\gamma_y(s)$ and $\gamma_z(s)$, independent of each other. Find the mean and autocovariance of $w_t = ay_t + bz_t$, where a and b are constants, and show that $\{w_t\}_{-\infty}^{\infty}$ is stationary.

   Solution:

   $E[w_t] = a\mu_y + b\mu_z$. $\text{cov}[w_t, w_{t-k}] = a^2\gamma_y(k) + b^2\gamma_z(k)$. All are independent of t.

4. Let a_i, b_i be independent r.v.'s with $E[a_i] = E[b_i] = 0$ and $\text{var}[a_i] = \text{var}[b_i] = \sigma_i^2$. Compute the mean and autocovariance of $z_t = \sum_{i=1}^{m}(a_i \cos 2\pi\omega_i t + b_i \sin 2\pi\omega_i t)$ and show that it is stationary. [Hint: you may want to use the trigonometric identity $\cos x \cos y + \sin x \sin y = \cos(x - y)$.]

   Solution:

   $E[z_t] = 0$. $\text{cov}[z_t, z_{t-k}] = \sum_{i=1}^{m} \sigma_i^2 \cos 2\pi\omega_i k$. All are independent of t.

5. Let $\{a_t\}_1^{\infty}$ be a white noise process with mean 0 and variance σ_a^2. Define $z_0 = 0$, $z_t = \phi z_{t-1} + a_t$, $t = 1, 2, \ldots$, where $|\phi| < 1$.

   (a) Express z_t explicitly in terms of a_t.

   (b) Calculate the autocovariance $\text{cov}[z_t, z_{t+s}]$ for $s > 0$, and show that for large t, z_t is approximately stationary.

   Solution:

   (a) $z_t = \sum_{i=0}^{t-1} \phi^i a_{t-i}$.

   (b) $\text{cov}[z_t, z_{t+s}] = \sum_{i=0}^{t-1} \phi^{2i+s}\sigma_a^2 = \sigma_a^2\phi^s(1 - \phi^{2t})/(1 - \phi^2) \to \sigma_a^2\phi^s/(1 - \phi^2)$

6. Observing $z_1, \ldots, z_N$ from a stationary process with autocovariance γ_k and autocorrelation $\rho_k = \gamma_k/\gamma_0$. It is known that $\text{var}[\bar{z}] = (\gamma_0/N)[1 + 2\sum_{k=1}^{N-1}(1 - k/N)\rho_k]$.

   (a) If $\rho_k \to 0$ as $k \to \infty$, show that $\text{var}[\bar{z}] \to 0$ as $N \to \infty$.

(b) Compare var$[\bar{z}]$ with the following autocorrelations: (i) $\rho_k = 0$, $k \neq 0$; (ii) $\rho_1 = .8$, $\rho_2 = .55$, $\rho_k = 0$, $k > 2$.

Solution:

(a) Only need to prove $\sum_{k=1}^{N-1}(1-k/N)\rho_k/N \to 0$. For any $\delta > 0$, there exists M such that $|\rho_k| < \delta$, $k > M$. Let $K > M/\delta$. For $N > K$, one has

$$|\sum_{k=1}^{N-1}(1-k/N)\rho_k/N| \leq \sum_{k\leq M} N^{-1} + \sum_{k>M} \delta/N < 2\delta.$$

(b) For (i), $\text{var}[\bar{z}] = \gamma_0/N$. For (ii),

$$\text{var}[\bar{z}] = (\gamma_0/N)(1 + 2(1-1/N)(.8) + 2(1-2/N)(.55)) = (\gamma_0/N)(3.7 - 3.8/N).$$

7. Problem 2.1 in the text (p. 569 in 3rd ed; p.701 in 4th ed.), plus

(d) After inspecting the graphs in (a)-(c), do you think the series is stationary?

(e) Calculate and plot the sample ACF for lags up to 6.

(f) Assume $\rho_k = 0$, $k > 2$. Obtain approximate standard errors for r_1, r_2, and r_k, $k > 2$.

(g) Assume $\rho_k = 0$, $k > 2$. Obtain approximate correlation between r_4 and r_5.

Solution:

Read the data into `x` in R.

(a) `plot(ts(x))`.

(b) `plot(x[-36],x[-1])`.

(c) `plot(x[-(35:36)],x[-(1:2)])`.

(d) It appears to be stationary as there is no obvious pattern suggesting otherwise.

(e) `x.acf<-acf(x,lag.max=6); x.acf` gives

$$(r_1, \ldots, r_6) = (0.4910, 0.1639, -0.0486, -0.1729, -0.2921, -0.5113).$$

(f) Substituting r_1, r_2 for ρ_1, ρ_2 in the approximate formula for $\text{var}[r_k]$, one has

$$\text{s.e.}[r_1] \approx \sqrt{[(1+2r_1^2)(1+2r_1^2+2r_2^2) + 2r_2 + r_1^2 - 8r_1^2(1+r_2)]/N} = 0.1292,$$
$$\text{s.e.}[r_2] \approx \sqrt{[(1+2r_2^2)(1+2r_1^2+2r_2^2) + r_2^2 - 4r_2(2r_2+r_1^2)]/N} = 0.1880,$$
$$\text{s.e.}[r_k] \approx \sqrt{(1+2r_1^2+2r_2^2)/N} = 0.2066, k > 2.$$

(g) $\text{cov}[r_4, r_5] \approx 2(r_1r_2 + r_1)/N = 0.0317$. $\text{corr}[r_4, r_5] \approx 0.03175/0.2066^2 = 0.744$.