Autoregressive Conditional Heteroskedasticity

The *autoregressive conditional heteroskedasticity models*, or ARCH models, were developed by econometricians to model variance heteroskedasticity in financial data.

Consider $u_t = \sqrt{h_t} v_t$, where v_t is *i.i.d.* with $E[v_t] = 0$ and $E[v_t^2] = 1$, independent of h_t. The process is said to follow an ARCH(m) model if h_t is determined through

$$h_t = \zeta + \alpha_1 u_{t-1}^2 + \cdots + \alpha_m u_{t-m}^2.$$

Clearly, h_t has to be positive, and a sufficient condition is that $\zeta > 0$, $\alpha_j > 0$, $j = 1, \ldots, m$. Write

$$u_t^2 = h_t v_t^2 = h_t + h_t(v_t^2 - 1) = h_t + w_t,$$

it is seen that u_t^2 follows an AR(m) model with non-normal shocks $w_t = h_t(v_t^2 - 1)$. $E[w_t] = 0$, $E[w_t^2] = E[h_t^2]E[(v_t^2 - 1)^2]$.

Stationarity of ARCH(m)

With $\alpha_j \geq 0$ and $\sum_{j=1}^{m} \alpha_j < 1$, the roots of $1 - \alpha_1 B - \cdots - \alpha_m B^m$ are outside the unit circle. For u_t^2 to be stationary, however, one also needs $E[w_t^2]$ to be a constant.

Consider ARCH(1), $h_t = \zeta + \alpha u_{t-1}^2$, and assume $E[w_t^2] = \lambda^2$. It follows that $E[u_t^2] = \zeta/(1-\alpha)$ and $\text{var}[u_t^2] = \lambda^2/(1-\alpha^2)$. One has

$$\begin{aligned} E[h_t^2] &= \zeta^2 + 2\zeta\alpha E[u_{t-1}^2] + \alpha^2 E[u_{t-1}^4] \\ &= \zeta^2 + 2\alpha\zeta^2/(1-\alpha) + \alpha^2\{\zeta^2/(1-\alpha)^2 + \lambda^2/(1-\alpha^2)\} \\ &= \zeta^2/(1-\alpha)^2 + \lambda^2\alpha^2/(1-\alpha^2) \end{aligned}$$

Take $E[(v_t^2 - 1)^2] = 2$ for v_t normal so $E[h_t^2] = \lambda^2/2$, one has

$$\lambda^2(1-3\alpha^2)/(2(1-\alpha^2)) = \zeta^2/(1-\alpha)^2.$$

Hence, one needs $\alpha^2 < 1/3$ for u_t^2 to be stationary.

GARCH Models

Consider $u_t = \sqrt{h_t} v_t$, which is said to follow GARCH(r,m) if

$$h_t = \kappa + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} + \alpha_1 u_{t-1}^2 + \cdots + \alpha_m u_{t-m}^2.$$

It can be seen that $h_t = \zeta + (\sum_{j=1}^{\infty} \pi_j B^j) u_t^2$, where

$$(\textstyle\sum_{j=1}^{\infty} \pi_j B^j) = (\sum_{j=1}^{m} \alpha_j B^j)/(1 - \sum_{j=1}^{r} \delta_j B^j),$$

and $\zeta = \kappa/(1 - \delta_1 - \cdots - \delta_r)$. Remember that $w_t = u_t^2 - h_t$,

$$\begin{aligned} u_t^2 = h_t + w_t = \kappa + w_t - \delta_1 w_{t-1} - \cdots - \delta_r w_{r-1} \\ + (\alpha_1 + \delta_1) u_{t-1}^2 + \cdots + (\alpha_p + \delta_p) u_{t-p}^2, \end{aligned}$$

where $p = \max(r, m)$. Hence, u_t^2 follows an ARMA(p,r) model.

The conditions $\kappa > 0$ and $\alpha_j, \delta_j \geq 0$ assure h_t to be positive, and a further condition $\sum_{j=1}^{p} (\alpha_j + \delta_j) < 1$ puts the roots of $1 - (\alpha_1 + \delta_1) B - \cdots - (\alpha_p + \delta_p) B^p$ outside the unit circle.

Maximum Likelihood Estimation

For v_t normal, the recursive evaluation of the likelihood function is straightforward given initial values, and the maximization can be done through numerical optimization. Extra regression terms pose no further difficulty.

In R, GARCH models (without regression terms) can be fitted using `garch` in the `tseries` package.

```
library(tseries); diff.B<-diff(series.B)
plot(series.B); plot(diff.B)
acf(series.B); acf(diff.B); acf(diff.B^2)
fit.B<-garch(diff.B,c(0,2),trace=FALSE)
predict(fit.B); resid(fit.B)
plot(fit.B); acf(rnorm(400)^2)
plot(diff.B); lines(2:369,predict(fit.B)[,1],col=2)
```