1. Consider the IMA(0,1,1) model, $(1-B)z_t = (1-\theta B)a_t$, $t = 1, 2, \ldots$ with $z_0 = a_0 = 0$, where a_t are *i.i.d.* $N(0,1)$.

(6 pts.) (a) Verify that $z_t = a_t + (1-\theta)\sum_{j=1}^{t-1} a_{t-j} = a_t + (1-\theta)(a_{t-1} + \cdots + a_1)$.

(8 pts.) (b) For a sample $z_1, \ldots, z_n$, define $\tilde{c}_k = n^{-1}\sum_{t=1}^{n-k} z_t z_{t+k}$. Find an explicit expression for $E[\tilde{c}_k]$, and hence for $E[\tilde{c}_k]/E[\tilde{c}_0]$.

(8 pts.) (c) Investigate the behavior of $E[\tilde{c}_k]/E[\tilde{c}_0]$ for $\theta = .8$ and $n = 100$ by calculating the value for the first few lags, and verify that it dies out more linearly than exponentially as k increases. ($E[\tilde{c}_k]/E[\tilde{c}_0]$ is the main term of $E[\tilde{r}_k] = E[\tilde{c}_k/\tilde{c}_0]$.)

2. Consider $z_t = x_t + y_t$, where $\{x_t\}$ represents a "trend component" satisfying the model $(1-B)x_t = (1-\alpha B)a_t$ with a_t *i.i.d.* $N(0,\sigma_a^2)$, and $\{y_t\}$ is stationary satisfying $(1-\beta B)y_t = b_t$ with b_t *i.i.d.* $N(0,\sigma_b^2)$; $\{a_t\}$ and $\{b_t\}$ are independent.

(8 pts.) (a) Show that $\{z_t\}$ follows an ARIMA(p,d,q) model, and determine the values of p, d, q.

(10 pts.) (b) Exhibit the basic relations (but do not attempt to solve explicitly) between the parameters of the ARIMA model and the parameters α, β, σ_a^2, and σ_b^2.

(6 pts.) (c) What will the orders p, d, q be if $\beta = 0, 1$?

3. A time series of $n = 100$ observations gave the following sample ACF and PACF,

Lag k	1	2	3	4	5	6	7	8
r_k	.50	−.20	−.48	−.22	.09	.25	.15	−.05
ϕ_{kk}	.50	−.60	−.04	.12	−.14	.17	−.07	−.05

with sample mean $\bar{z} = 15.0$ and sample variance $c_0 = 4$.

An AR(2) model, $(1-\phi_1 B - \phi_2 B^2)z_t = \delta + a_t$, appears to be appropriate for the data

(10 pts.) (a) Obtain the Yule-Walker estimates $\hat{\phi}_1$ and $\hat{\phi}_2$. Also obtain estimates for δ and σ_a^2.

(10 pts.) (b) The Yule-Walker estimates $\hat{\boldsymbol{\phi}}$ is asymptotically unbiased with var$[\hat{\boldsymbol{\phi}}] \approx (\sigma_a^2/n)\Gamma^{-1}$, where $\Gamma = \begin{pmatrix} \gamma_0 & \gamma_1 \\ \gamma_1 & \gamma_0 \end{pmatrix}$. Estimate the asymptotic standard deviations of $\hat{\phi}_1$ and $\hat{\phi}_2$.

(8 pts.) (c) Using $\hat{\phi}_1$ and $\hat{\phi}_2$ as the "true" values of ϕ_1 and ϕ_2, calculate ρ_k, $k = 1, 2, 3, 4$.

4. Consider an ARIMA(1,1,1) model $(1-.7B)(1-B)z_t = (1+.9B)a_t$.

(6 pts.) (a) Find the first two ψ weights, ψ_0 and ψ_1.

(10 pts.) (b) Find a general expression for the ψ weights, ψ_k, $k > 1$. Calculate ψ_5.

(10 pts.) (c) Find the π weights π_k, $k = 1, 2$.