

## Control Charts for Process Monitoring

Industrialized manufacturing processes achieve great product uniformity through controlled environments such as machine calibration, worker training, etc., but natural variation inevitably remains. Properly designed and calibrated processes keep product characteristics within a tolerable range.

Control charts are used to monitor the processes to see if they are “out-of-tune.” Samples of products are taken at regular time points and product characteristics of the samples are charted against pre-specified *control limits*. When the samples raise yellow flags, *assignable causes* will be sought and corrective actions will be taken, if necessary, to restore the processes back “in-tune.”

## $\bar{X}$ -Charts for Process Location

Three specimens of motor oil are randomly selected each day and viscosity is determined. When the process is in control, it is known that  $\mu = 10.5$  and  $\sigma = .18$  by past experience. For the control chart, one has

$$\text{LCL} = 10.5 - 3 \frac{.18}{\sqrt{3}} = 10.188,$$

$$\text{UCL} = 10.5 + 3 \frac{.18}{\sqrt{3}} = 10.812.$$

As long as  $10.188 < \bar{x} < 10.812$ , one leaves the process alone.

At regular inspection points, one takes a sample of size  $n$  and calculate  $\bar{X}$ . When  $\bar{X}$  is within a pair of control limits,

$$\text{LCL} = \mu - 3\sigma/\sqrt{n},$$

$$\text{UCL} = \mu + 3\sigma/\sqrt{n},$$

the process is considered to be in control.

- The 3- $\sigma$  rule is the industry standard.
- The probability of false alarm is  $P(|Z| > 3) = .0027$ .

## $\bar{X}$ -Charts for Process Location

For motor oil viscosity, 25 days of data were collected: (Table 16.1)

$x$	$\bar{x}$	$s$	$r$
10.37, 10.19, 10.36 .....	10.307	.101	.18

which yield

$$\bar{\bar{x}} = \sum_{i=1}^{25} \bar{x}_i / 25 = 10.476,$$

$$\bar{s} = \sum_{i=1}^{25} s_i / 25 = .153,$$

$$\bar{r} = \sum_{i=1}^{25} r_i / 25 = .292.$$

One has  $\hat{\mu} = 10.476$ ,  $\hat{\sigma} = \bar{s}/a_3 = .153/.886 = .1727$ , or  $\hat{\sigma} = \bar{r}/b_3 = .292/1.693 = .1725$ , where  $\mu_s = a_n\sigma$  and  $\mu_r = b_n\sigma$ .

When  $\mu$  and  $\sigma$  are unknown, one collect “training” samples with the process in control, and estimate the parameters to use in the  $\bar{x}$ -chart.

To estimate  $\mu$ , one uses the sample mean of the “training”  $\bar{x}$ 's,  $\bar{\bar{x}}$ .

To estimate  $\sigma$ , one may use the sample mean of the “training”  $s$ 's, or the sample mean of the “training” ranges,  $r_i = \max_j x_{ij} - \min_j x_{ij}$ .

## *S/R*-Charts for Process Variation

For motor oil viscosity,  $\bar{s} = .153$ ,  $a_3 = .886$ , so the  $r$ -chart is given by the control limits  $.153 \pm 3(.153)\sqrt{1 - .886^2}/.886 = (0, 0.393)$ , where the LCL is truncated. For  $\sigma = .18$  known, the limits would be  $.886(.18) \pm 3(.18)\sqrt{1 - .886^2} = (0, 0.410)$ .

With  $\bar{r} = .292$ ,  $b_3 = 1.693$ , and  $c_3 = .888$ , the  $r$ -chart limits are  $.292 \pm 3(.292)(.888)/1.693 = (0, 0.751)$ . For  $\sigma = .18$  known, the limits would be  $1.693(.18) \pm 3(.18)(.888) = (0, 0.784)$ .

To monitor the process variation, one may use the  $s$ -chart or the  $r$ -chart.

Since  $\sigma_s = \sigma\sqrt{1 - a_n^2}$ , the  $3\text{-}\sigma$  control limits for the  $s$ -chart are

$$\bar{s} \pm 3\bar{s}\sqrt{1 - a_n^2}/a_n$$

for estimated  $\sigma$ ; for known  $\sigma$ , the limits are  $a_n\sigma \pm 3\sigma\sqrt{1 - a_n^2}$ .

As  $\sigma_r = c_n\sigma$ , the  $3\text{-}\sigma$  control limits for the  $r$ -chart are

$$\bar{r} \pm 3\bar{r}c_n/b_n$$

for estimated  $\sigma$ , or  $b_n\sigma \pm 3c_n\sigma$  for known  $\sigma$ .

## Charts for Defectives or Defects

When items are either usable or not, such as circuit chips, one needs to monitor the fraction of defectives. The number of defectives in a sample of size  $n$  follows  $\text{Bin}(n, p)$ .

The  $p$ -chart for the fraction of defectives has limits

$$\text{LCL} = p - 3\sqrt{p(1-p)/n},$$

$$\text{UCL} = p + 3\sqrt{p(1-p)/n}.$$

For  $p$  unknown, take  $k$  samples and estimate by  $\bar{p} = \sum_{i=1}^k \hat{p}_i/k$ .

When the value of products is affected by defects, such as blemishes on car paint, one needs to monitor the number of defects. The number of defects in a sample of fixed volume follows  $\text{Poisson}(\lambda)$ .

The  $c$ -chart for the number of defects has limits

$$\text{LCL} = \lambda - 3\sqrt{\lambda},$$

$$\text{UCL} = \lambda + 3\sqrt{\lambda}.$$

For  $\lambda$  unknown, take  $k$  samples and estimate by  $\bar{x} = \sum_{i=1}^k x_i/k$ .

## Acceptance Sampling

When manufactured products change hands, the consumer will typically inspect the goods before taking position from the producer. When items are categorized as either defective or non-defective, acceptance sampling is often used to determine whether to accept a delivery.

From a lot of  $N$  items, a random sample of  $n$  items are selected. If the number of defectives  $x$  in the sample is no more than a pre-specified number  $c$ , the lot will be accepted.

Suppose the fraction of defectives in the lot is  $p$ , then with  $d = Np$ ,  $P(x) = C_x^d C_{n-x}^{N-d} / C_n^N$ . For  $n/N \approx 0$ ,  $P(x) \approx C_x^n p^x (1-p)^{n-x}$ .

## Operating Characteristics (OC) Curve

For an acceptance sampling plan, the probability of acceptance is a function of the true fraction of defectives. The curve of  $P(A)$  versus  $p$  is called the OC curve of the plan. This is similar to the power curve or the  $\beta$ -risk of statistical tests.

Consider  $N = 1000$ ,  $n = 50$ , and  $c = 2$ . The OC curve is easy to obtain using the `phyper` and `pbinom` functions in R.

```
d <- 0:200; p <- d/1000
plot(p,phyper(2,d,1000-d,50),type="l",ylab="P(A)")
lines(p,pbinom(2,50,p),col=3) # binomial approximation
lines(c(0,.2),c(0,0),col=5) # base line
```

## Designing a Sampling Plan

To design a plan  $(n, c)$  to satisfy both parties, one needs the *acceptable quality level* (AQL)  $p_1$  and the *lot tolerance percent defective* (LTPD)  $p_2$ . For  $\alpha, \beta$  small, the producer wants  $P(A|p_1) \geq 1 - \alpha$ , and the consumer wants  $P(A|p_2) \leq \beta$ .

Consider  $p_1 = .01$ ,  $p_2 = .04$ , and  $\alpha = \beta = .05$ . One may design a plan by trial-and-error using `pbinom` in R.

```
pbinom(1:10, 200, .01); pbinom(5, 200, .04) # no good  
pbinom(1:10, 300, .01); pbinom(6, 300, .04) # good  
pbinom(1:10, 295, .01); pbinom(6, 295, .04) # good  
pbinom(1:10, 290, .01); pbinom(6, 290, .04) # no good
```

So both  $(295, 6)$  and  $(300, 6)$  would do, assuming  $N$  much larger.