Function Space and Montel’s Theorem

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Abstract
This theorem touches the final topic required by the comprehensive exam in complex.

1 Notations

$\mathbb{C}$ the complex plane. $\Omega$ a domain in $\mathbb{C}$. $\mathcal{F}$ a family of functions $f$. $(S,d)$ a metric space where $f$ assumes value. $C(U)$ the set of all continuous function defined on the open set $U$. For sequence of functions $\Rightarrow$ means uniform convergence on the specified set. $\Delta$ always denotes closed disk

2 Arzelà-Ascoli Theorem and Montel’s Theorem

Definition 2.1 The functions in a family $\mathcal{F}$ are said to be equicontinuous on a set $E \subseteq \Omega$ iff for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(f(z), f(z_0)) < \varepsilon$ whenever $|z - z_0| < \delta$ and $z, z_0 \in E$, simultaneously for all functions $f \in \mathcal{F}$.

Definition 2.2 A family $\mathcal{F}$ is said to be normal in $\Omega$ if every sequence $\{f_n\}$ of functions $f_n \in \mathcal{F}$ has a subsequence $\{f_{n_k}\}$ which either converges uniformly or tends uniformly to $\infty$ on every compact subset of $\Omega$.

For the purpose of this note, the most significant feature of equicontinuity is that it bridges the gap between pointwise convergence and normal convergence.
Lemma 2.3 Let \((S,d)\) be a complete metric space, \(U\) be an open subset of \(S\), \(K\) a compact subset of \(S\) contained in \(U\). Then there is some \(\rho > 0\) such that for all \(z \in K\), \(B(z, \rho) \subseteq U\).

**Proof.** The hypothesis implies that
\[
d(K, \partial U) = r > 0
\]

else, \(K \cap \partial U \neq \emptyset\) and \(K \cap (S \sim U) \neq \emptyset\). To be more precise, suppose the contrary. Then for each \(\rho_n = 1/n\) with \(n \in \mathbb{N}\), there exists \(z_n \in K\) such that \(B(z_n, \rho_n) \cap (S \sim U) = \emptyset\). Since \(K\) is compact, \((z_n)\) must have a subsequence \((z_{n_j})\) such that
\[
\lim_{j \to \infty} z_{n_j} = z_0 \in K \subseteq U
\]
which further means there is some \(\delta > 0\) such that \(B(z_0, \delta) \subseteq U\). But for this \(\delta > 0\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n_j > n_0\),
\[
z_{n_j} \in B(z_0, \delta)
\]
and hence for each \(n_j > n_0\), there is some \(\delta_{n_j} > 0\) such that
\[
B(z_{n_j}, \delta_{n_j}) \subseteq B(z_0, \delta)
\]
Since \(\lim_{n_j \to \infty} \rho_{n_j} = 0\), then for sufficiently large \(n_j\), it’s clear that \(\rho_{n_j} < \delta_{n_j}\) and
\[
B(z_{n_j}, \rho_{n_j}) \subseteq B(z_{n_j}, \delta_{n_j}) \subseteq B(z_0, \delta)
\]
which is a contradiction. 


\[\square\]

Lemma 2.4 Let \((f_n)\) be a sequence from an equicontinuous subfamily \(\mathcal{F}\) of \(C(U)\). Suppose that this sequence converges pointwise in \(U\). Then it converges normally in \(U\).

**Proof.** (Contrapositive) Let \(f\) be the pointwise limit function of \((f_n)\) and \(K\) be any arbitrary compact subset of \(U\). It suffices to show that,

(i) for any \(\varepsilon > 0\), there exists \(n_0 = n(\varepsilon) \in \mathbb{N}\), such that for any \(n, m > n_0\) and for all \(z \in K\),
\[
|f_n(z) - f_m(z)| < \varepsilon
\]
Suppose (i) does not hold, then there is some \(\varepsilon' > 0\) such that for each \(n \in \mathbb{N}\), there are \(n_k > m_k > k\) with \(\lim_{k \to \infty} m_k = \infty\) and some \(z_k \in K\) such that
\[
|f_{n_k}(z_k) - f_{m_k}(z_k)| > \varepsilon'
\]
Since \(K\) is compact, \(\{z_k\}\) has a convergent subsequence \(\{z_{k_l}\}\) such that
\[
\lim_{k \to \infty} z_{k_l} = z_0 \in K
\]
For $\varepsilon/3$, the equicontinuity of $\mathcal{F}$ ensures that there is some $\delta > 0$ such that whenever $|z_k - z_0| < \delta$

$$|f_n(z_k) - f_n(z_0)| < \varepsilon/3$$

for all $n$. Thus

$$\lim_{k \to \infty} |f_{n_k}(z_0) - f_{m_k}(z_0)| = 0$$

and for all $k > k_0$

$$|f_{n_k}(z_0) - f_{m_k}(z_0)| < \varepsilon/3$$

Moreover, for this $\delta$, it’s true $|z_k - z_0| < \delta$ for all $k > k_0$. Hence

$$\varepsilon' < |f_{n_k}(z_k) - f_{m_k}(z_k)| \leq |f_{n_k}(z_k) - f_{m_k}(z_0)| +$$

$$+ |f_{n_k}(z_0) - f_{m_k}(z_0)| + |f_{n_k}(z_k) - f_{n_k}(z_0)|$$

$$< \varepsilon'$$

which is a contradiction. □

**Proof.** (Direct proof) Target: To show

$$f_n \Rightarrow g$$

Let $K$ be any compact subset of $U$, then there is some $r > 0$ such that for any $z \in K$,

$$B(z, r) \subseteq U$$

Obviously, $\mathcal{O} = \{B(z, r) : z \in K, r > 0\}$ forms an open cover of $K$ and the compactness of $K$ implies there exist some $m \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{i=1}^{m} \{B(z_i, r) : B(z_i, r) \in \mathcal{O}\}$$

By the equicontinuity of $\mathcal{F}$, for any given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z'| < \delta$

$$|f_{n}(z) - f_{m}(z')| < \varepsilon$$

for all $m, n \in \mathbb{N}$. (Correction by Dr. Bleecker: it should be $|f_{n}(z) - f_{m}(z')| < \varepsilon$ for all $n \in \mathbb{N}$) Let $\rho = \min\{r, \delta\}$. Then it’s clear that $B(z, \rho) \subseteq U$ for all $z \in K$ and $\mathcal{O}_1 = \{B(z, \rho) : z \in K\}$ also forms an open cover of $K$ and hence $K \subseteq \bigcup_{i=1}^{m_1} \{B(z_i, \rho) : B(z_i, \rho) \in \mathcal{O}_1\}$ for some $m_1 \in \mathbb{N}$. Since $(f_n)$ converges pointwise to $g$, there exist $n_0 \in \mathbb{N}$ such that for all $z_i, i = 1, \ldots, m$,

$$|f_{n}(z_i) - g(z_i)| < \varepsilon$$

whenever $n > n_0$. Finally, for any $z \in K$, it’s obvious that $z \in B(z_{i_0}, \rho)$ for some $i_0$ with $1 \leq i_0 \leq m_1$ and whenever $n > n_0$,

$$|f_{n}(z) - g(z)| = |f_{n}(z) - f_{n}(z_{i_0}) - f_{n}(z) + f_{n}(z_{i_0}) + f_{n}(z) - g(z)|$$

$$\leq |f_{n}(z) - f_{n}(z_{i_0})| + |f_{n}(z_{i_0}) - f_{n}(z)| + |f_{n}(z) - g(z)| < 3\varepsilon$$

which means $(f_n)$ converges uniformly to $g$ on $K$. □
Lemma 2.5 A normal family $\mathcal{F}$ of $C(U)$ is locally bounded in $U$.

**Proof.** Let $K$ be any compact subset of $U$. Then by (2.3), there exists $z_i \in K, i = 1, \ldots, m$ and $r > 0$ such that

$$K \subseteq \cup_{i=1}^{m} \{ \Delta(z_i, r) : z_i \in K, r > 0 \} \subseteq U$$

Since every $f \in C(U)$ and $K$ is compact, then $E_f = f(K) \subseteq \mathbb{C}$ is compact, $|f|$ is continuous on $U$ and $G_f = |f|(K) \subseteq \mathbb{R}$. The inequality

$$||a| - |b|| < |a - b|$$

for all $a, b \in \mathbb{C}$ implies that $\{|f| : f \in \mathcal{F}\}$ is also a normal family on $U$. Let

$$\Lambda = \left\{ \alpha_f = \max_{z \in K} |f(z)| : f \in \mathcal{F} \right\}$$

Suppose $\Lambda$ is not compact. Then there exist $O = \{ B(z, r) : z \in \mathbb{C}, r > 0 \}$ but for any $m \in \mathbb{N},$

$$K \not\subseteq \cup_{i=1}^{m} \{ B(z_i, r) : B(z_i, r) \in O \}$$

Specifically, since $\mathcal{F}$ is normal, ■

**Theorem 2.1 (Arzela-Ascoli)** A subfamily $\mathcal{F} \subseteq C(U)$ is normal iff it is both equicontinuous and pointwise bounded.

**Theorem 2.2 (Montel)** A subfamily $\mathcal{F} \subseteq C(U)$ is normal iff it is locally bounded on $U$.

**Proof.** Use Arzela-Ascoli theorem and Cauchy integral formula. ■

**Lemma 2.6** If $G$ is open in $\mathbb{C}$ then there is a sequence $(K_n)$ of compact subsets of $G$ such that

1. $G = \cup_{n=1}^{\infty} K_n$
2. $K \subseteq G$ and $K$ compact implies $K \subseteq K_n$ for some $n$
3. Every component of $T_n = \mathbb{C} \sim K_n$ contains a component of $T = \mathbb{C} \sim G$ (Here $\mathbb{C}_\infty = \hat{\mathbb{C}}$ is the Riemann Sphere), that is,

let $\mathcal{C}_1, \mathcal{C}_2$ be the set of components of $T_n$ and $T$, then $\mathcal{C}_2 \preceq \mathcal{C}_1$, i.e., $\mathcal{C}_2$ is finer than $\mathcal{C}_1$

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Proof. Apply the basic trick by for each \( n \in \mathbb{N} \) define

\[
K_n = \{ z \in \mathbb{C} : |z| \leq n \} \cap \left\{ z \in \mathbb{C} : d(z, \mathbb{C} \sim G) \geq \frac{1}{n} \right\}
\]

Then \( K_n \) is compact. For

\[
H_n = \{ z \in \mathbb{C} : |z| < n + 1 \} \cap \left\{ z \in \mathbb{C} : d(z, \mathbb{C} \sim G) > \frac{1}{n + 1} \right\}
\]

it’s clear that \( H_n \) is open and

\[
K_n \subseteq H_n \subseteq K_{n+1}
\]

Then

\[
G = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} \text{Int} \ K_n
\]

since

\[
z \in \mathbb{C} : d(z, \mathbb{C} \sim G) \geq \frac{1}{n} \iff \exists \rho > 0 \text{ s.t. } B(z, \rho) \subseteq G
\]

Further, for any compact \( K \) with \( K \subseteq G \), it’s clear that \( K \subseteq \bigcup_{n\in\mathbb{N}} \text{Int} \ K_n \) and \( K \subseteq K_{n_0} \) for some \( n_0 \in \mathbb{N} \).

To see (3), Let \( E \) be the unbounded component of \( \mathbb{C} \sim K_n \subseteq \mathbb{C} \sim G \) and let \( F \) be the unbounded component of \( \mathbb{C} \sim G \). Then \( E \supseteq \{ z \in \mathbb{C} : |z| > n \} \), \( \infty \in E \) and \( E \supseteq F \) since \( \mathbb{C} \sim K_n \subseteq \mathbb{C} \sim G \). So if \( D \) is a bounded component of \( \mathbb{C} \sim K_n \) it contains some \( z \) with \( d(z, \mathbb{C} \sim G) < \frac{1}{n} \). But, then there is some \( w \in \mathbb{C} \sim G \) with \( |z - w| < \frac{1}{n} \) and \( z \in B(w; \frac{1}{n}) \subseteq \mathbb{C} \sim K_n \). Since disks are connected and \( z \in D \) with \( D \subseteq \mathbb{C} \sim K_n \) being a component, then \( B(w; \frac{1}{n}) \subseteq D \). If \( D_1 \) is the component of \( \mathbb{C} \sim G \) that contains \( w \) it follows that \( D_1 \subseteq D \). \( \blacksquare \)

Remark 3 This works for any complete metric space.

4 Metrization of \( \mathcal{H}(U) \)

Let \( G \) and \( K_n \) be as given (2.6) and \( f, g \in C(G, \Omega) \). Define

\[
\rho_n(f, g) = \sup \{ d(f(z_0, g(z)) : z \in K_n \}
\]

and

\[
\rho(f, g) = \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \frac{\rho_n(f, g)}{1 + \rho_n(f, g)}
\]

Theorem 4.1 \((C(G, \Omega), \rho)\) is a complete, locally convex metric space

Remark 5 Consult Kosako Yoshida for Locally convex metric space