Lecture 7: Random Effects in CRD

Design of Experiments - Montgomery

Section 12-1

Random Effects vs Fixed Effects

- Consider factor with numerous possible levels
- Want to draw inference **on population of levels**
- Not concerned with any specific levels
- Example of difference (1=fixed, 2=random)
  1. Compare reading ability of 10 2nd grade classes in NY
  2. Compare variability **among all** 2nd grade classes in NY

  1. Select $a = 10$ specific classes of interest. Randomly choose n students from each classroom. Want to compare τ_i (class-specific effects).

  2. **Randomly choose** $a = 10$ classes from large number of classes. Randomly choose n students from each classroom. Want to assess σ_τ^2 (class to class variability).

- Inference broader in random effects case: iference on population with randomly chosen levels

Random Effects Model (CRD)

- Same model as in the fixed case

$$y_{ij} = \mu + \tau_i + \epsilon_{ij} \quad \begin{cases} i = 1, 2 \ldots a \\ j = 1, 2, \ldots n_i \end{cases}$$

  μ - grand mean

  τ_i - ith treatment effect

  $\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

  But view number of treatment levels as infinite

- Instead of $\sum \tau_i = 0$, assume

  $\tau_i \sim \mathrm{N}(0, \sigma_\tau^2)$

  $\{\tau_i\}$ and $\{\epsilon_{ij}\}$ independent

Random Effects Model vs Fixed Effect Model

- Fixed effect model:

$$\text{Var}(y_{ij}) = \sigma^2; \text{Cov}(y_{ij}, y_{ks}) = 0;$$

- Random effect model:

$$\text{Var}(y_{ij}|\tau) = \sigma^2, \quad \text{Var}(y_{ij}) = \sigma_\tau^2 + \sigma^2;$$

$$\text{Cov}(y_{ij}, y_{ks}|\tau) = 0, \quad \text{Cov}(y_{ij}, y_{ks}) = \begin{cases} 0, & \text{if } i \neq k; \\ \sigma_\tau^2, & \text{if } i = k; \end{cases}$$

Random Effects Model

- The hypotheses are:

$$H_0 \quad : \quad \sigma_\tau^2 = 0$$
$$H_1 \quad : \quad \sigma_\tau^2 > 0$$

- Partitioning of Total Sum of Squares identical

  E(MS_{E})=σ^2;

  E($MS_{\mathrm{Treatment}}$)=$\sigma^2 + n_0\sigma_\tau^2$,
  where $n_0 = ((\sum n_i)^2 - \sum n_i^2)/((a-1)\sum n_i)$

- Under H_0, $F_0 \sim F_{a-1,N-a}$
  Under H_1 and balanced case, $F_0 \sim (1 + n\sigma_\tau^2/\sigma^2)F_{a-1,N-a}$

- Same test as before

- Direct comparison of variabilities (between vs within)

- Conclusions, however, pertain to entire population

Model Estimates

- Usually interested in estimating variances
- Use mean squares (known as ANOVA method)

$$\hat{\sigma}^2 = \mathrm{MS_E}$$

$$\hat{\sigma}_\tau^2 = (\mathrm{MS_{Treatment}} - \mathrm{MS_E})/n_0$$

- Estimate of σ_τ^2 can be negative
  - Supports H_0? Use zero as estimate?
  - Validity of model? Nonlinear? Outlier?
  - MLE, RMLE, or Bayesian approach (nonnegative prior)

Confidence intervals

- σ^2: Since

$$\frac{(N-a)\mathrm{MS_E}}{\sigma^2} \sim \chi^2_{N-a},$$

$$Pr\left(\frac{(N-a)\hat{\sigma}^2}{\sigma^2} \in (\chi^2_{1-\alpha/2,N-a}, \chi^2_{\alpha/2,N-a})\right) = 1-\alpha,$$

we obtain the following $1-\alpha$ confidence interval:

$$\frac{(N-a)\hat{\sigma}^2}{\chi^2_{\alpha/2,N-a}} = \frac{(N-a)\mathrm{MS_E}}{\chi^2_{\alpha/2,N-a}} \leq \sigma^2 \leq \frac{(N-a)\mathrm{MS_E}}{\chi^2_{1-\alpha/2,N-a}} = \frac{(N-a)\hat{\sigma}^2}{\chi^2_{1-\alpha/2,N-a}}$$

- σ^2_τ (balanced design): Linear combination of χ^2

$$\frac{(a-1)\mathrm{MS_{Trt}}}{\sigma^2+n\sigma^2_\tau} \sim \chi^2_{a-1}, \; \hat{\sigma}^2_\tau = \frac{\sigma^2+n\sigma^2_\tau}{n(a-1)}\chi^2_{a-1} - \frac{\sigma^2}{n(N-a)}\chi^2_{N-a}$$

No closed form expression for this distribution, approx. available (Sec. 13-7)

- Proportion of σ_τ^2 in Var(y_{ij}), i.e., Intraclass Correlation Coefficient

  Common estimate if goal is to reduce variance

  Uses the fact that $F_0 \sim (1 + n\sigma_\tau^2/\sigma^2)F_{a-1,N-a}$, we can derive

$$\frac{\mathrm{L}}{\mathrm{L}+1} \le \frac{\sigma_\tau^2}{\sigma^2+\sigma_\tau^2} \le \frac{\mathrm{U}}{\mathrm{U}+1}$$

$$\mathrm{L} = \frac{1}{n}\left(\frac{\mathrm{MS_{Trt}}}{\mathrm{MS_E} \quad F_{\alpha/2,a-1,N-a}} - 1\right), \ \mathrm{U} = \frac{1}{n}\left(\frac{\mathrm{MS_{Trt}}}{\mathrm{MS_E} \quad F_{1-\alpha/2,a-1,N-a}} - 1\right)$$

  Or, equivalently

$$\frac{F_0 - F_{\alpha/2,a-1,N-a}}{F_0 + (n-1)F_{\alpha/2,a-1,N-a}} \le \frac{\sigma_\tau^2}{\sigma^2+\sigma_\tau^2} \le \frac{F_0 - F_{1-\alpha/2,a-1,N-a}}{F_0 + (n-1)F_{1-\alpha/2,a-1,N-a}}$$

- Grand mean -μ

  Example: Average reading ability of 2nd grade class. Unit is the class, selection of students is subsampling.

  $$\overline{y}_{..} = \frac{1}{N}(y_{11} + ... + y_{an})$$

  or

  $$\overline{y}_{..} = \frac{1}{a}(\overline{y}_{1.} + \overline{y}_{2.} + ... + \overline{y}_{a.})$$

  y_{ij} are not iid Normal, but $\overline{y}_{i.}$ iid Normal.

  CI for μ : $\overline{y}_{..} \pm t_{1-\alpha/2,t-1}\sqrt{\text{MS}_{\text{Trt}}/(an)}$

Example

A supplier delivers several hundred batches of raw material to a company each year. The company is interested in a high yield from each batch of raw material (percent usable). Therefore, to investigate the consistency of this supplier, an experiment is done where five batches were selected at random and three yield determinations were made on each batch.

Batch				
1	2	3	4	5
74	68	75	72	79
76	71	77	74	81
75	72	77	73	79

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between	147.73	4	36.93	20.5
Within	18.00	10	1.80	
Total	165.73	14		

Highly significant result ($F_{.05,4,10} = 3.48$)

$\hat{\sigma}_\tau^2 = (36.93 - 1.80)/3 = 11.71$

86.7% (=11.71/(11.71+1.80)) is attributable to batch differences

Time to improve consistency of the batches

Example

Confidence Intervals

- 95% CI for σ^2

$$\frac{\mathrm{SS_E}}{\chi^2_{.025,10}} \le \sigma^2 \le \frac{\mathrm{SS_E}}{\chi^2_{.975,10}} = (18.00/20.48, 18.00/3.25)$$
$$= (0.879, 5.538)$$

- 95% CI for Intraclass Correlation

$$\left(\frac{20.52-4.47}{20.52+(3-1)4.47}, \frac{20.52-(1/8.84)}{20.52+(3-1)(1/8.84)}\right)$$

$$(0.545, 0.984)$$

using property that $F_{1-\alpha/2,v_1,v_2} = 1/F_{\alpha/2,v_2,v_1}$

Using SAS

```
options nocenter ps=35 ls=72;
data example;
    input batch percent;
    cards;
     1 74 1 76 1 75 2 68 ... 5 79
  ;

proc glm;
  class batch;
  model percent=batch;
  random batch;
  output out=diag r=res p=pred;
  proc plot;
  plot res*pred;

proc varcomp method = type1;
  class batch;
  model percent = batch;
```

```
proc mixed cl;
 class batch;
 model percent = ;
 random batch;
run;
```

- GLM : Uses least squares for estimation

  –Not designed for random effects....Must compute variance estimates / CIs by hand

- MIXED: Uses restricted maximum likelihood (REML)

  –Often preferred to ML because it produces unbiased estimates of covariance parameters by taking into account the loss of degrees of freedom in estimating fixed effects.

  –Usually residual variance profiled out of the likelihood

Dependent Variable: PERCENT

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	4	147.73333	36.93333	20.52	0.0001
Error	10	18.00000	1.80000		
Corrected Total	14	165.73333			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
BATCH	4	147.73333	36.93333	20.52	0.0001

Source	Type III Expected Mean Square
BATCH	Var(Error) + 3 Var(BATCH)

Variance Components Estimation Procedure

Dependent Variable: PERCENT

Source	DF	Type I SS	Type I MS
BATCH	4	147.73333333	36.93333333
Error	10	18.00000000	1.80000000
Corrected Total	14	165.73333333	

Source	Expected Mean Square
BATCH	Var(Error) + 3 Var(BATCH)
Error	Var(Error)

Variance Component	Estimate
Var(BATCH)	11.71111111
Var(Error)	1.80000000

```
                    The MIXED Procedure

                Class     Levels  Values
                BATCH          5  1 2 3 4 5

             REML Estimation Iteration History

     Iteration  Evaluations     Objective     Criterion
             0            1   51.30656858
             1            1   37.02237479    0.00000000
Convergence criteria met.

           Covariance Parameter Estimates (REML)

     Cov Parm       Estimate  Alpha     Lower     Upper
     BATCH       11.71111111   0.05    4.0450  114.2090
     Residual     1.80000000   0.05    0.8788    5.5436

           Model Fitting Information for PERCENT

          Description                        Value
          Observations                     15.0000
          Res Log Likelihood              -31.3763
          Akaike's Information Criterion  -33.3763
          Schwarz's Bayesian Criterion    -34.0154
          -2 Res Log Likelihood            62.7527
```

Negative σ_τ^2 Estimate Example

```
options nocenter ps=39 ls=64;
data new;
input class subj score @@;
cards;
1 1  74.62 1 2  73.90 1 3  72.27 1 4  71.60 1 5  73.80
1 6  77.42 1 7  72.16 1 8  76.69 1 9  75.84 1 10 70.35
2 1  72.55 2 2  71.44 2 3  72.67 2 4  72.59 2 5  71.25
2 6  68.99 2 7  69.61 2 8  77.44 2 9  73.99 2 10 73.90
3 1  76.66 3 2  74.76 3 3  70.47 3 4  75.38 3 5  68.32
3 6  76.69 3 7  73.34 3 8  68.24 3 9  69.33 3 10 78.22
;

proc glm;
 class class;
 model score = class;
 random class / test;

proc varcomp method = type1;
 class class;
 model score = class;

proc varcomp method = reml;
 class class;
 model score = class;

proc mixed cl;
 class class;
 model score = ;
 random class;
run;
```

```
General Linear Models Procedure

Dependent Variable: SCORE
Source                 DF    Sum of Squares  F Value   Pr > F
Model                   2       10.11154667     0.60   0.5557
Error                  27      227.34895000
Corrected Total        29      237.46049667

                 R-Square              C.V.        SCORE Mean
                 0.042582          3.966909        73.1496667


Source                 DF         Type I SS  F Value   Pr > F
CLASS                   2       10.11154667     0.60   0.5557

Source                 DF       Type III SS  F Value   Pr > F
CLASS                   2       10.11154667     0.60   0.5557

General Linear Models Procedure

Source      Type III Expected Mean Square
CLASS       Var(Error) + 10 Var(CLASS)

Tests of Hypotheses for Random Model Analysis of Variance

Source: CLASS
Error: MS(Error)
                         Denominator    Denominator
 DF    Type III MS            DF               MS     F Value  Pr > F
 2   5.0557733333           27   8.4203314815      0.6004   0.5557
```

```
Variance Components Estimation Procedure

Class    Levels    Values
CLASS         3    1 2 3

Number of observations in data set = 30

Variance Components Estimation Procedure
Dependent Variable: SCORE
Source                   DF         Type I SS         Type I MS
CLASS                     2       10.11154667        5.05577333
Error                    27      227.34895000        8.42033148
Corrected Total          29      237.46049667

Source                   Expected Mean Square
CLASS                    Var(Error) + 10 Var(CLASS)
Error                    Var(Error)

Variance Component                    Estimate
Var(CLASS)                         -0.33645581
Var(Error)                          8.42033148
```

```
                        REML Procedure
Dependent Variable: SCORE

Iteration        Objective       Var(CLASS)       Var(Error)
    0          60.97845805                0       8.18829299
    1          60.97845805                0       8.18829299

Convergence criteria met.
```

Asymptotic Covariance Matrix of Estimates

	Var(CLASS)	Var(Error)
Var(CLASS)	0	0
Var(Error)	0	4.6240097976

```
                 The MIXED Procedure

               Class Level Information
               Class     Levels  Values
               CLASS          3  1 2 3

          REML Estimation Iteration History

    Iteration  Evaluations     Objective     Criterion
            0            1   93.37965543
            1            1   93.37965543    0.00000000
Convergence criteria met.


        Covariance Parameter Estimates (REML)

    Cov Parm       Estimate  Alpha     Lower     Upper
    CLASS        0.00000000      .         .         .
    Residual     8.18829299   0.05    5.1935   14.7977


         Model Fitting Information for SCORE

         Description                        Value
         Observations                     30.0000
         Res Log Likelihood              -73.3390
         Akaike's Information Criterion  -75.3390
         Schwarz's Bayesian Criterion    -76.7063
         -2 Res Log Likelihood           146.6781
```

Choice of Sample Size

Fixed Effects (Review)

- Must now consider all treatments ($a \geq 2$)
- Calculate power of overall F-test
- For simplicity, assume n_i's constant

  Type II error: $\beta =$ Pr($F_0 < F_{\alpha,a-1,N-a} | H_0$ false)

  Need to know distribution of F_0 when H_0 false

  Can show $F_0 \sim F_{a-1,N-a}(\delta)$

  where $\delta = n \sum \tau_i^2 / \sigma^2$, non-centrality parameter

  Recall E(MS$_{\text{Trt}}$)=$\sigma^2 + n \sum \tau_i^2 / (a-1)$

- OCC given in Chart V

  Plots β vs Φ

  $\Phi^2 = n \sum \tau_i^2 / a\sigma^2 = \delta / a$

- Use SAS function probf: power=1-probf($F_{\alpha,a-1,N-a}$,$a-1$,$N-a$,δ)

Determination of τ_i's or Φ^2

1. Choose treatment means $(\mu + \tau_i)$
   - Solve for $\tau_i' s$ and compute Φ^2 or δ
   - Difficult to select group of treatment means
2. Tolerance for pairwise treatment difference
   - Rejection if any $|\tau_i - \tau_j| > D$
   - Use $\Phi^2 = nD^2/2a\sigma^2$ (i.e., $\{\tau_i\} = \{-D/2, 0, \ldots, 0, D/2\}$)
   - Power of test at least $1 - \beta$

Example - Batch Example

- Consider new experiment fixed effect problem
  - Assume five new batches are of interest
  - Use model estimates to determine sample size

    Want to detect D=3 (percent) with 80% power

    Use $\hat{\sigma}^2 = 1.8$ and $\alpha = .05$

  - Using Table V : $\Phi^2 = 9n/2(5)(1.8) = .5n$

n	4	5	6
Φ	$\sqrt{2}$	$\sqrt{2.5}$	$\sqrt{3}$
df_{E}	15	20	25
β	45%	31%	18%

- Using SAS : $\delta = a\Phi^2$

n	4	5	6
δ	10.0	12.5	15.0
df_{E}	15	20	25
β	43.5%	28.9%	18.2%

- Appears $n = 6$ is proper choice

```
options nocenter ps=35 ls=72;

data params;
  input a alpha d var;
cards;
  5 .05 3.0 1.8
;

data new;
 set params;
 do n=2 to 10;
    df = a*(n-1);
    nc = n*d*d/(2*var);
    fcut  = finv(1-alpha,a-1,df);
    beta=probf(fcut,a-1,df,nc);
    output;
 end;

proc print;
var n nc beta;
run;
```

OBS	N	NC	BETA
1	2	5.0	0.81008
2	3	7.5	0.61721
3	4	10.0	0.43549
4	5	12.5	0.28897
5	6	15.0	0.18227
6	7	17.5	0.11017
7	8	20.0	0.06421
8	9	22.5	0.03626
9	10	25.0	0.01992

SAS PROC POWER

- PROC POWER calculate $\sum_{i=1}^{a} \tau_i^2$ using pre-specified treatment means τ_i

Example:

```
proc power;
onewayanova test=overall power=.80 npergroup=. stddev=18.27
groupmeans = -15|0|0|0|15;
run; quit
```

```
   Overall F Test for One-Way ANOVA
       Fixed Scenario Elements
Method                              Exact
Group Means                  -15 0 0 0 15
Standard Deviation                  18.27
Nominal Power                         0.8
Alpha                                0.05

Computed N Per Group
Actual Power                  N Per Group
0.808                                  10
```

Choice of Sample Size

Random Effects

- Can use central F distribution

  $(N-a)\text{MS}_{\text{E}}/\sigma^2 \sim \chi^2_{N-a}$

  $(a-1)\text{MS}_{\text{Trt}}/(\sigma^2 + n\sigma^2_\tau) \sim \chi^2_{a-1}$

  Thus $F_0/(1 + n\sigma^2_\tau/\sigma^2) \sim F_{a-1,N-a}$

  Can specify ratio of σ^2_τ/σ^2

- OCC given in Chart VI

  Plots β vs λ

  $\lambda^2 = 1 + n\sigma^2_\tau/\sigma^2$

- Use SAS function probf

  power = 1-probf($F_{\alpha,a-1,N-a}/\lambda^2$,$a-1$,$N-a$)

Example - Batch Example on Slide 7-7

- Consider new experiment a random effects problem
  - The variance estimate is $\sigma^2 = 1.8$
  - Desire to detect situation when $\sigma_\tau^2 \geq 3.6 = 2\sigma^2$
  - Set power at 80% and $\alpha = .05$
  - Using Table VI : $\lambda = \sqrt{1 + 2n}$

n	3	4	5
λ	$\sqrt{7}$	$\sqrt{9}$	$\sqrt{11}$
df_{E}	10	15	20
β	28%	18%	15%

  - Using SAS : use λ^2

n	3	4	5
λ^2	7	9	11
df_{E}	10	15	20
β	26.1%	15.3%	10.0%

- Appears $n = 4$ gives appropriate power

```
options nocenter ps=35 ls=72;

data params;
  input a alpha ratiovar;
cards;
  5 .05 2.0
;

data new;
 set params;
 do n=2 to 10;
    df = a*(n-1);
    lambdasq = 1+ratiovar*n;
    fcut  = finv(1-alpha,a-1,df);
    beta=probf(fcut/lambdasq,a-1,df);
    output;
 end;

proc print;
var n beta;
run;
```

OBS	N	BETA
1	2	0.52933
2	3	0.26112
3	4	0.15292
4	5	0.10027
5	6	0.07081
6	7	0.05267
7	8	0.04072
8	9	0.03242
9	10	0.02643

Random effect in RCBD: case one

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

μ - grand mean

τ_i - ith treatment effect

β_j - jth random block effect, $\beta_j \sim$ N$(0, \sigma_b^2)$

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

Can show:

E(MS$_{\mathrm{E}}$)=σ^2

E(MS$_{\mathrm{Treatment}}$) = $\sigma^2 + b\sum_{i=1}^{a} \tau_i^2/(a-1)$

E(MS$_{\mathrm{Block}}$) = $\sigma^2 + a\sigma_b^2$

Inference for Random effect in RCBD: case one

- Still using the same test for $\tau_i = 0$;
- $F_0 =$MS$_{\mathrm{Trt}}/$MS$_{\mathrm{E}}$;
- The distribution of MS$_{\mathrm{Trt}}$ and MS$_{\mathrm{E}}$ are the same as in fixed block effect model (both Under H_0 and H_1)
- Power calculation keeps the same.

- Inference for τ_i is somehow different
- Var$(\hat{\tau}_i) =$ Var$(\bar{y}_{i.}) = \sigma^2/b + \sigma_b^2/b$;
- $SE(\hat{\tau}_i) =$? with $df =$?
- Var$(\hat{\tau}_i - \hat{\tau}_j) =$ Var$(\bar{y}_{i.} - \bar{y}_{j.}) = \sigma^2(1/b + 1/b)$
- $SE(\hat{\tau}_i - \hat{\tau}_j) = \sqrt{\text{MS}_\text{E}}\sqrt{2/b}$ with $df = (a-1)(b-1)$

Random effect in RCBD: case two

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

μ - grand mean

τ_i - ith treatment effect, $\tau_j \sim$ N$(0, \sigma_\tau^2)$

β_j - jth random block effect

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

Can show:

E(MS$_{\mathrm{E}}$)=σ^2

E(MS$_{\mathrm{Treatment}}$) = $\sigma^2 + b\sigma_\tau^2$

E(MS$_{\mathrm{Block}}$) = $\sigma^2 + a\sum_{j=1}^{b} \beta_j^2/(b-1)$

Inference for Random effect in RCBD: case two

- Still using the same test for $\sigma_\tau^2 = 0$;
- $F_0 =$MS$_{\mathrm{Trt}}/$MS$_{\mathrm{E}}$;
- The distribution of MS$_{\mathrm{Trt}}$ and MS$_{\mathrm{E}}$ are the same as in random effect CRD model (both Under H_0 and H_1)
- Power calculation keeps the same.
- Estimation of σ^2, σ_τ^2, and $\sigma_\tau/(\sigma^2 + \sigma_\tau^2)$ can be the same.

Random effect in RCBD: case three

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

μ - grand mean

τ_i - ith treatment effect, $\tau_j \sim$ N$(0, \sigma_\tau^2)$

β_j - jth random block effect, $\beta_j \sim$ N$(0, \sigma_b^2)$

$\epsilon_{ij} \sim \mathrm{N}(0, \sigma^2)$

Can show:

E(MS$_{\mathrm{E}}$)=σ^2

E(MS$_{\mathrm{Treatment}}$) = $\sigma^2 + b\sigma_\tau^2$

E(MS$_{\mathrm{Block}}$) = $\sigma^2 + a\sigma_b^2$

Inference for Random effect in RCBD: case three

- Still using the same test for $\sigma_\tau^2 = 0$;
- $F_0 =$MS$_{\mathrm{Trt}}/$MS$_{\mathrm{E}}$;
- The distribution of MS$_{\mathrm{Trt}}$ and MS$_{\mathrm{E}}$ are the same as in random effect CRD model (both Under H_0 and H_1)
- Power calculation keeps the same.
- Estimation of σ^2 and σ_τ^2 can be the same.
- Estimation of Intraclass correlation $\sigma_\tau/(\sigma^2 + \sigma_b^2 + \sigma_\tau^2)$ is different.