Lecture 8: Latin Squares and Related Designs

Montgomery: Section 4.2 and 4.3

Automobile Emission Experiment

Four cars and **four drivers** are employed in a study of **four gasoline additives**(A, B, C, D). Even though cars can be identical models, systematic differences are likely to occur in their performance, and even though each driver may do his best to drive the car in the manner required by the test, systematic differences can occur from driver to driver. It would be desirable to eliminate both the car-to-car and driver-to-driver variations when comparing the additives.

	cars			
drivers	1	2	3	4
1	A=24	B=26	D=20	C=25
2	D=23	C=26	A=20	B=27
3	B=15	D=13	C=16	A=16
4	C=17	A=15	B=20	D=20

Design Matrix and Orthogonality

drivers	cars	additives
1	1	A
1	2	B
1	3	D
1	4	C
2	1	D
2	2	C
2	3	A
2	4	B
3	1	B
3	2	D
3	3	C
3	4	A
4	1	C
4	2	A
4	3	B
4	4	D

Orthogonality: for any two columns, all possible combinations appear and appear only once.

Latin Square Design

- Design is represented in $p \times p$ grid, rows and columns are blocks and Latin letters are treatments.
  - Every row contains all the Latin letters and every column contains all the Latin letters.
- Standard Latin Square: letters in first row and first column are in alphabetic order.
- Examples

C	B	A
B	A	C
A	C	B

B	A	C
A	C	B
C	B	A

```
PROC PLAN SEED=5140514;
    FACTORS drivers=4 ORDERED cars=4 ORDERED;
    TREATMENTS additives=4 CYCLIC;
    OUTPUT OUT=ls4additives
             drivers NVALS=(1 2 3 4) RANDOM
             cars NVALS=(1 2 3 4) RANDOM
             additives CVALS=('A' 'B' 'C' 'D') RANDOM;
QUIT;

PROC SORT DATA=ls4additives OUT=mydesign;
    BY  drivers cars;
PROC TRANSPOSE DATA=mydesign(RENAME=(cars=_NAME_))
               OUT=tmydesign(drop=_NAME_);
      BY drivers;  VAR additives;
PROC PRINT DATA=tmydesign NOOBS;
RUN;
```

drivers	_1	_2	_3	_4
1	A	B	D	C
2	D	A	C	B
3	C	D	B	A
4	B	C	A	D

Latin Square Design

- Properties:
  - Block on two nuisance factors
  - Two restrictions on randomization
- Model and Assumptions

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk}, \quad i, j, k = 1, 2, \ldots, p$$

μ - grand mean

α_i - ith block 1 effect (ith row effect); $\sum_{i=1}^{p} \alpha_i = 0$.

τ_j - jth treatment effect; $\sum_{i=1}^{p} \tau_j = 0$

β_k - kth block 2 effect (kth column effect); $\sum_{k=1}^{p} \beta_k = 0$

$\epsilon_{ijk} \sim \mathrm{N}(0, \sigma^2)$; (Normality, Independence, Constant Variance).

Completely additive model (no interaction)

Estimation and Basic Hypotheses Testing

- Rewrite observation y_{ijk} as:

$$\begin{aligned} y_{ijk} &= \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{..k} - \bar{y}_{...}) + (y_{ijk} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k} + 2\bar{y}_{...}) \\ &= \hat{\mu} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\beta}_k + \hat{\epsilon}_{ijk} \end{aligned}$$

- Partition $\mathrm{SS_T} = \sum\sum(y_{ijk} - \bar{y}_{...})^2$ into

$$\begin{array}{lllll} & p\sum(\bar{y}_{i..} - \bar{y}_{...})^2 & +p\sum(\bar{y}_{.j.} - \bar{y}_{...})^2 & + p\sum(\bar{y}_{..k} - \bar{y}_{...})^2 & + \sum\sum\hat{\epsilon}_{ijk}^2 \\ = & \mathrm{SS_{Row}} & +\mathrm{SS_{Treatment}} & + \mathrm{SS_{Col}} & + \mathrm{SS_E} \\ \mathrm{df}: & (p-1) & (p-1) & (p-1) & (p-1)(p-2) \end{array}$$

  – Dividing SS by df gives MS: $\mathrm{MS_{Treatment}}$, $\mathrm{MS_{Row}}$, $\mathrm{MS_{Col}}$ and $\mathrm{MS_E}$

- Basic hypotheses: $H_0 : \tau_1 = \tau_2 = \cdots = \tau_p = 0$ vs H_1: at least one is not
- Test Statistic: $F_0 = \mathrm{MS_{Treatment}}/\mathrm{MS_E} \sim F_{p-1,(p-1)(p-2)}$ under H_0.
- Caution testing row effects ($\alpha_i = 0$) and column effects ($\beta_k = 0$).

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Rows	SS_{Row}	$p-1$	MS_{Row}	
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Column	$\text{SS}_{\text{Column}}$	$p-1$	$\text{MS}_{\text{Column}}$	
Error	SS_{E}	$(p-2)(p-1)$	MS_{E}	
Total	SS_{T}	p^2-1		

$$\text{SS}_{\text{T}} = \sum\sum\sum y_{ijk}^2 - y_{...}^2/p^2; \quad \text{SS}_{\text{Row}} = \frac{1}{p}\sum y_{i..}^2 - y_{...}^2/p^2$$

$$\text{SS}_{\text{Treatment}} = \frac{1}{p}\sum y_{.j.}^2 - y_{...}^2/p^2 \quad \text{SS}_{\text{Column}} = \frac{1}{p}\sum y_{..k}^2 - y_{...}^2/p^2$$

$$\text{SS}_{\text{Error}} = \text{SS}_{\text{T}} - \text{SS}_{\text{Row}} - \text{SS}_{\text{Treatment}} - \text{SS}_{\text{Col}}$$

- Decision Rule:

  If $F_0 > F_{\alpha,p-1,(p-2)(p-1)}$ then reject H_0.

SAS Code

Consider an experiment to investigate the effect of 4 diets on milk production. There are 4 cows. Each lactation period the cows receive a different diet. Assume there is a washout period so previous diet does not affect future results. Will block on lactation period and cow. A 4 by 4 Latin square is used.

	Cows			
Periods	1	2	3	4
1	A=38	B=39	C=45	D=41
2	B=32	C=37	D=38	A=30
3	C=35	D=36	A=37	B=32
4	D=33	A=30	B=35	C=33

```
options nocenter ls=75;goptions colors=(none);
data new;
 input cow period trt resp @@;
datalines;
```

```
 1 1 1 38 1 2 2 32 1 3 3 35 1 4 4 33
 2 1 2 39 2 2 3 37 2 3 4 36 2 4 1 30
 3 1 3 45 3 2 4 38 3 3 1 37 3 4 2 35
 4 1 4 41 4 2 1 30 4 3 2 32 4 4 3 33
;

proc glm;
class cow trt period;
model resp=trt period cow;
means trt/ lines tukey;
means period cow;
output out=new1 r=res p=pred;

symbol1 v=circle;
proc gplot; plot res*pred;

proc univariate noprint normal;
histogram res / normal (L=1 mu=0 sigma=est) kernel (L=2);
qqplot res/normal (L=1 mu=0 sigma=est);
run;
```

Output

Dependent Variable: resp

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	9	242.5625000	26.9513889	33.17	0.0002
Error	6	4.8750000	0.8125000		
Corrected Total	15	247.4375000			

Source	DF	Type I SS	Mean Square	F Value	Pr > F
trt	3	40.6875000	13.5625000	16.69	0.0026
period	3	147.1875000	49.0625000	60.38	<.0001
cow	3	54.6875000	18.2291667	22.44	0.0012

Source	DF	Type III SS	Mean Square	F Value	Pr > F
trt	3	40.6875000	13.5625000	16.69	0.0026
period	3	147.1875000	49.0625000	60.38	<.0001
cow	3	54.6875000	18.2291667	22.44	0.0012

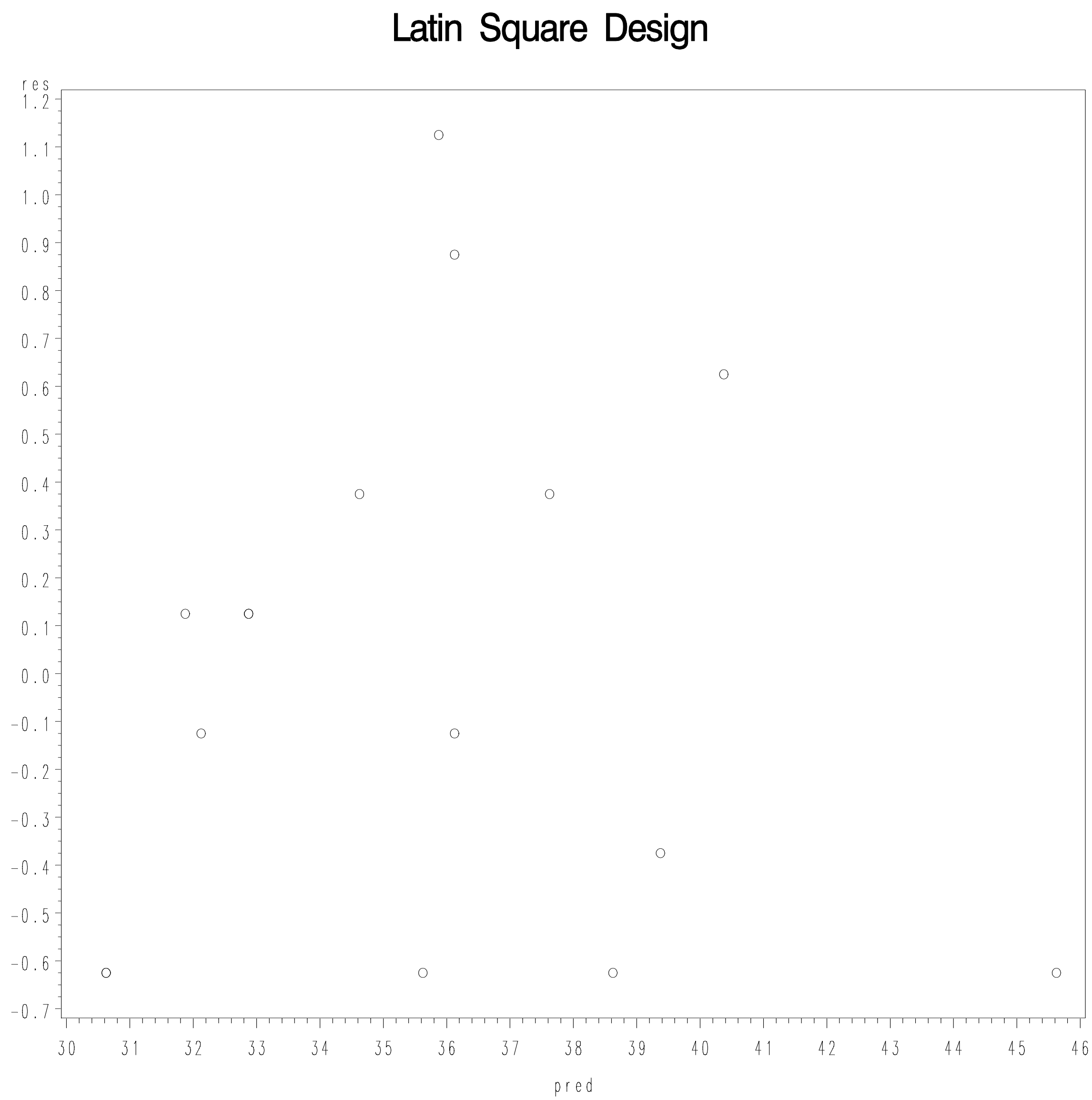
Output (continued)

```
Tukey's Studentized Range (HSD) Test for resp

Alpha                                        0.05
Error Degrees of Freedom                        6
Error Mean Square                          0.8125
Critical Value of Studentized Range       4.89559
Minimum Significant Difference             2.2064

                 Mean      N    trt
A             37.5000      4    3
A             37.0000      4    4

B             34.5000      4    2
B             33.7500      4    1
```


Latin Square Design
res
pred

Replicating Latin Squares

- Latin Squares result in small degree of freedom for $\mathrm{SS_E}$:
  $\mathrm{df} = (p-1)(p-2)$.
  - If 3 treatments: $\mathrm{df_E} = 2$
  - If 4 treatments $\mathrm{df_E} = 6$
  - If 5 treatments $\mathrm{df_E} = 12$

  Use replication to increase $\mathrm{df_E}$

- Different ways for replicating Latin squares:
  1. Same rows and same columns
  2. New rows and same columns
  3. Same rows and new columns
  4. New rows and new columns

  Degree of freedom for $\mathrm{SS_E}$ depends on which method is used;

  Often need to include an additional blocking factor for ”replicate” effect.

Method 1: same rows and same columns in additional squares

```
Example:
  1  2  3         data          replication
1 A  B  C       7.0 8.0 9.0
2 B  C  A       4.0 5.0 6.0          1
3 C  A  B       6.0 3.0 4.0

  1  2  3          data
1 C  B  A        8.0 4.0 7.0
2 B  A  C        6.0 3.0 6.0         2
3 A  C  B        5.0 8.0 7.0

  1  2  3          data
1 B  A  C         9.0 6.0 8.0
2 A  C  B         5.0 7.0 6.0        3
3 C  B  A         9.0 3.0 7.0
```

Model and ANOVA Table for Method 1

Usually includes replicate (e.g., time) effects $(\delta_1, \ldots, \delta_n)$

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$p-1$		
Columns	$\text{SS}_{\text{Column}}$	$p-1$		
Replicate	$\text{SS}_{\text{Replicate}}$	$n-1$		
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(n(p+1)-3)$	MS_{E}	
Total	SS_{T}	np^2-1		

data input and SAS code for Method 1

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0

2 1 1 3 8.0
2 1 2 2 4.0
2 1 3 1 7.0
2 2 1 2 6.0
2 2 2 1 3.0
2 2 3 3 6.0
```

```
2 3 1 1 5.0
2 3 2 3 8.0
2 3 3 2 7.0

3 1 1 2 9.0
3 1 2 1 6.0
3 1 3 3 8.0
3 2 1 1 5.0
3 2 2 3 7.0
3 2 3 2 6.0
3 3 1 3 9.0
3 3 2 2 3.0
3 3 3 1 7.0
;
```

```
proc glm data=new;
class rep row col trt;
model resp=rep row col trt;
run;
quit;
```

SAS Output for Method 1

```
Dependent Variable: resp
Sum of
Source          DF      Squares     Mean Square    F Value    Pr > F
Model            8  61.18518519      7.64814815       4.21    0.005
Error           18  32.66666667      1.81481481
Corrected Total 26  93.85185185
           R-Square     Coeff Var      Root MSE     resp Mean
           0.651934      22.17870      1.347151      6.074074

Source         DF     Type I SS     Mean Square    F Value    Pr > F
rep            2      5.62962963     2.81481481       1.55    0.2391
row            2     23.40740741    11.70370370       6.45    0.0077
col            2      9.85185185     4.92592593       2.71    0.0933
trt            2     22.29629630    11.14814815       6.14    0.0093
```

Method 2: New (different) rows and same columns

```
Example:
  1  2  3        data         replication
1 A  B  C      7.0 8.0 9.0
2 B  C  A      4.0 5.0 6.0          1
3 C  A  B      6.0 3.0 4.0

  1  2  3        data
4 C  B  A      8.0 4.0 7.0
5 B  A  C      6.0 3.0 6.0          2
6 A  C  B      5.0 8.0 7.0

  1  2  3        data
7 B  A  C      9.0 6.0 8.0
8 A  C  B      5.0 7.0 6.0          3
9 C  B  A      9.0 3.0 7.0
```

Model and ANOVA Table for Method 2

- Row effects are **nested** within square
- α_i can be different for different squares, so they are denoted $\alpha_{i(l)}$ for $i = 1, \ldots, p$ and $l = 1, \ldots, n$, and satisfy by $\sum_{i=1}^{p} \alpha_{i(l)} = 0$ for any fixed l.

$$y_{ijkl} = \mu + \alpha_{i(l)} + \tau_j + \beta_k + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$n(p-1)$		
Columns	$\text{SS}_{\text{Column}}$	$p-1$		
Replicate	$\text{SS}_{\text{Replicate}}$	$n-1$		
Treatment	$\text{SS}_{\text{Treatment}}$	$p-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(np-2)$	MS_{E}	
Total	SS_{T}	np^2-1		

data input and SAS code for Method 2

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0

2 4 1 3 8.0
2 4 2 2 4.0
2 4 3 1 7.0
2 5 1 2 6.0
2 5 2 1 3.0
2 5 3 3 6.0
```

```
2 6 1 1 5.0
2 6 2 3 8.0
2 6 3 2 7.0

3 7 1 2 9.0
3 7 2 1 6.0
3 7 3 3 8.0
3 8 1 1 5.0
3 8 2 3 7.0
3 8 3 2 6.0
3 9 1 3 9.0
3 9 2 2 3.0
3 9 3 1 7.0
;
```

```
proc glm data=new;
class rep row col trt;
model resp=rep row(rep) col trt;
run;
quit;
```

SAS ouput

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	12	74.00000000	6.16666667	4.35	0.0054
Error	14	19.85185185	1.41798942		
C.Total	26	93.85185185			

R-Square	Coeff Var	Root MSE	resp Mean
0.788477	19.60453	1.190794	6.074074

Source	DF	Type I SS	Mean Square	F Value	Pr > F
rep	2	5.62962963	2.81481481	1.99	0.1742
row(rep)	6	36.22222222	6.03703704	4.26	0.0120
col	2	9.85185185	4.92592593	3.47	0.0596
trt	2	22.29629630	11.14814815	7.86	0.0051

- Q: Should we reanalyze the data without “replicate effects”? How?

Latin Rectangle

- If there is no specific interest on "replicate effects", we can pool the replicate-grouped rows, so the replicated Latin squares form a Latin Rectangle with np separate rows.
- Row effects are $\alpha_1, \alpha_2, \ldots, \alpha_{np}$ satisfying $\sum_{i=1}^{np} \alpha_i = 0$.
- Model:

$$y_{ijk} = \mu + \alpha_i + \tau_j + \beta_k + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, np \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

- ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS_{Row}	$np - 1$		
Columns	$\text{SS}_{\text{Column}}$	$p - 1$		
Treatment	$\text{SS}_{\text{Treatment}}$	$p - 1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$(p-1)(np-2)$	MS_{E}	
Total	SS_{T}	$np^2 - 1$		

Method 3: Same rows and new (different) columns, is similar to Method 2. Details are omitted

Method 4: New (different) rows and new columns in additional squares

```
Example:
  1  2  3        data         replication
1 A  B  C      7.0 8.0 9.0
2 B  C  A      4.0 5.0 6.0          1
3 C  A  B      6.0 3.0 4.0

  4  5  6         data
4 C  B  A      8.0 4.0 7.0
5 B  A  C      6.0 3.0 6.0          2
6 A  C  B      5.0 8.0 7.0

  7  8  9         data
7 B  A  C      9.0 6.0 8.0
8 A  C  B      5.0 7.0 6.0          3
9 C  B  A      9.0 3.0 7.0
```

Model and ANOVA Table for Method 4

- Usually both row effects ($\alpha_{i(l)}$) and column effects ($\beta_{k(l)}$) are **nested** with squares.

$$y_{ijkl} = \mu + \alpha_{i(l)} + \tau_j + \beta_{k(l)} + \delta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \\ l = 1, 2, \ldots, n \end{cases}$$

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Rows	SS$_{\text{Row}}$	$n(p-1)$		
Columns	SS$_{\text{Column}}$	$n(p-1)$		
Replicate	SS$_{\text{Replicate}}$	$n-1$		
Treatment	SS$_{\text{Treatment}}$	$p-1$	MS$_{\text{Treatment}}$	F_0
Error	SS$_{\text{E}}$	$(p-1)(n(p-1)-1)$	MS$_{\text{E}}$	
Total	SS$_{\text{T}}$	np^2-1		

Data input and SAS code

```
data new;
input rep row col trt resp;
datalines;
1 1 1 1 7.0
1 1 2 2 8.0
1 1 3 3 9.0
1 2 1 2 4.0
1 2 2 3 5.0
1 2 3 1 4.0
1 3 1 3 6.0
1 3 2 1 3.0
1 3 3 2 4.0

2 4 4 3 8.0
2 4 5 2 4.0
2 4 6 1 7.0
2 5 4 2 6.0
2 5 5 1 3.0
2 5 6 3 6.0
```

```
2 6 4 1 5.0
2 6 5 3 8.0
2 6 6 2 7.0

3 7 7 2 9.0
3 7 8 1 6.0
3 7 9 3 8.0
3 8 7 1 5.0
3 8 8 3 7.0
3 8 9 2 6.0
3 9 7 3 9.0
3 9 8 2 3.0
3 9 9 1 7.0
;
```

```
proc glm data=new;
class rep row col trt;
model resp=rep row(rep) col(rep) trt;
run;
quit;
```

SAS output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	16	77.70370370	4.85648148	3.01	0.0411
Error	10	16.14814815	1.61481481		
C.Total	26	93.85185185			

R-Square	Coeff Var	Root MSE	resp Mean
0.827940	20.92094	1.270754	6.074074

Source	DF	Type I SS	Mean Square	F Value	Pr > F
rep	2	5.62962963	2.81481481	1.74	0.2242
row(rep)	6	36.22222222	6.03703704	3.74	0.0324
col(rep)	6	13.55555556	2.25925926	1.40	0.3042
trt	2	22.29629630	11.14814815	6.90	0.0131

Crossover (Changeover) Design Consider an experiment for investigating the effects of 3 diets (A, B, C) on milk production. Suppose the experiment involves 3 lactation periods. Cows take different diets in different periods, that is, a cow will not take the same diet more than once (crossover).

- Case 1: 3 cows are used.

	cow 1	cow 2	cow 3
period 1	A	B	C
period 2	B	C	A
period 3	C	A	B

- Case 2: 6 cows are employed:

	cow 1	cow 2	cow 3	cow 4	cow 5	cow 6
pd 1	A	B	C	A	B	C
pd 2	B	C	A	C	A	B
pd 3	C	A	B	B	C	A

Crossover (Changeover) Design

- In general, there are p treatments, np experimental units, and p periods. n Latin squares ($p \times p$) are needed to form a rectangle ($p \times np$). So that

  I. Each unit has each treatment for one period

  II. In each period, each treatment is used on n units.

Source of Variation	DF
units	$np - 1$
periods	$p - 1$
treatment	$p - 1$
error	$np^2 - (n+2)p + 2$
total	$np^2 - 1$

- Residual effects and washout periods

- Balanced for residual effects:

  III. Each treatment follows every other treatment n times.

Graeco-Latin Square: An Example

An experiment is conducted to compare four gasoline additives by testing them on four cars with four drivers over four days. Only four runs can be conducted in each day. The response is the amount of automobile emission.

Treatment factor: gasoline additive, denoted by A, B, C and D.

Block factor 1: driver, denoted by 1, 2, 3, 4.

Block factor 2: day, denoted by 1, 2, 3, 4.

Block factor 3: car, denoted by α, β, γ, δ.

	days			
drivers	1	2	3	4
1	$A\alpha = 32$	$B\beta = 25$	$C\gamma = 31$	$D\delta = 27$
2	$B\delta = 24$	$A\gamma = 36$	$D\beta = 20$	$C\alpha = 25$
3	$C\beta = 28$	$D\alpha = 30$	$A\delta = 23$	$B\gamma = 31$
4	$D\gamma = 34$	$C\delta = 35$	$B\alpha = 29$	$A\beta = 33$

Graeco-Latin Square Design Matrix:

driver	day	additive	car
1	1	A	α
1	2	B	β
1	3	C	γ
1	4	D	δ
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4	1	D	γ
4	2	C	δ
4	3	B	α
4	4	A	β

- Two $p \times p$ Latin squares are said to be **orthogonal** if the two squares when superimposed have the property that each pair of letters appears once.
- the superimposed square is called Graeco-Latin square.
- tables available for $p \leq 8$ in Wu&Hamada.
- `PROC FACTEX` in SAS
- 6$\times$6 Graeco-Latin square does not exist

```
TITLE ’Graeco-Latin Square Design’;
/*  NLEV: must be a prime power number for fractional/blocked design */
PROC FACTEX;
   FACTORS drivers days additives cars / NLEV=4;
   SIZE DESIGN=16;
   MODEL RESOLUTION=MAX;
   OUTPUT OUT=gls4additives
                drivers NVALS=(1 2 3 4)
                days NVALS=(1 2 3 4)
                additives CVALS=(’A’ ’B’ ’C’ ’D’)
                cars CVALS=(’a’ ’b’ ’c’ ’d’);

PROC PRINT DATA=gls4additives NOOBS;
RUN; QUIT;
```

drivers	days	additives	cars
1	1	A	a
1	2	D	d
1	3	B	b
1	4	C	c
2	1	D	c
2	2	A	b
2	3	C	d
2	4	B	a
3	1	B	d
3	2	C	a
3	3	A	c
3	4	D	b
4	1	C	b
4	2	B	c
4	3	D	a
4	4	A	d

Model and Assumptions

$$y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + \zeta_l + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, p \end{cases}$$

μ - grand mean

α_i - ith block 1 effect (row effect) $\sum \alpha_i = 0$

τ_j - jth treatment effect $\sum \tau_j = 0$

β_k - kth block 2 effect (column effect) $\sum \beta_k = 0$

ζ_l - lth block 3 effect (Greek letter effect) $\sum \zeta_l = 0$

$\epsilon_{ijk} \sim \mathrm{N}(0, \sigma^2)$ (independent)

- Completely additive model (no interaction)

Estimation and ANOVA

- Rewrite observation as:

  $y_{ijkl} = \overline{y}_{....} + (\overline{y}_{i...} - \overline{y}_{....}) + (\overline{y}_{.j..} - \overline{y}_{....}) + (\overline{y}_{..k.} - \overline{y}_{....}) +$
  $(\overline{y}_{...l} - \overline{y}_{....}) + (y_{ijkl} - \overline{y}_{i...} - \overline{y}_{.j..} - \overline{y}_{..k.} - \overline{y}_{...l} + 3\overline{y}_{....})$
  $= \hat{\mu} + \hat{\alpha}_i + \hat{\tau}_j + \hat{\beta}_k + \hat{\zeta}_l + \hat{\epsilon}_{ijkl}$

- Partition $\mathrm{SS_T}$ into:

  $p\sum(\overline{y}_{i...} - \overline{y}_{....})^2 + p\sum(\overline{y}_{.j..} - \overline{y}_{....})^2 + p\sum(\overline{y}_{..k.} - \overline{y}_{....})^2 +$
  $p\sum(\overline{y}_{...l} - \overline{y}_{....})^2 + \sum\sum\hat{\epsilon}^2_{ijkl}$
  $= \mathrm{SS_{Row}} + \mathrm{SS_{Treatment}} + \mathrm{SS_{Col}} + \mathrm{SS_{Greek}} + \mathrm{SS_E}$

  with degree of freedom $p-1$, $p-1$, $p-1$, $p-1$ and $(p-3)(p-1)$, respectively.

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Rows	SS_{Row}	$p-1$	MS_{Row}	
Treatment	$SS_{\mathrm{Treatment}}$	$p-1$	$MS_{\mathrm{Treatment}}$	F_0
Column	SS_{Column}	$p-1$	MS_{Column}	
Greek	SS_{Greek}	$p-1$	MS_{Greek}	
Error	SS_{E}	$(p-3)(p-1)$	MS_{E}	
Total	SS_{T}	p^2-1		

$SS_{\mathrm{T}} = \sum\sum\sum y_{ijkl}^2 - y_{....}^2/p^2$; $SS_{\mathrm{Row}} = \frac{1}{p}\sum y_{i...}^2 - y_{....}^2/p^2$;

$SS_{\mathrm{Treatment}} = \frac{1}{p}\sum y_{.j..}^2 - y_{....}^2/p^2$ $SS_{\mathrm{Column}} = \frac{1}{p}\sum y_{..k.}^2 - y_{....}^2/p^2$;

$SS_{\mathrm{Greek}} = \frac{1}{p}\sum y_{...l}^2 - y_{....}^2/p^2$; $SS_{\mathrm{Error}} =$ Use subtraction;

Decision Rule: If $F_0 > F_{\alpha,p-1,(p-3)(p-1)}$ then reject H_0

Sas Code and Output

```
data new; input row col trt greek resp @@; datalines;
1 1 1 1 32 1 2 2 2 25
1 3 3 3 31 1 4 4 4 27
2 1 2 4 24 2 2 1 3 36
2 3 4 2 20 2 4 3 1 25
3 1 3 2 28 3 2 4 1 30
3 3 1 4 23 3 4 2 3 31
4 1 4 3 34 4 2 3 4 35
4 3 2 1 29 4 4 1 2 33
;
proc glm data=new;
class row col trt greek;
model resp=row col trt greek;
run;
```

Output

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	12	296.7500000	24.7291667	2.83	0.2122
Error	3	26.1875000	8.7291667		
CoTotal	15	322.9375000			

R-Square	Coeff Var	Root MSE	resp Mean
0.918908	10.20999	2.954516	28.93750

Source	DF	Type I SS	Mean Square	F Value	Pr >F
row	3	90.6875000	30.2291667	3.46	0.1674
col	3	68.1875000	22.729166	2.60	0.2263
trt	3	36.6875000	12.2291667	1.40	0.3942
greek	3	101.1875000	33.7291667	3.86	0.1481

Multiple comparison can be carried out using similar methods