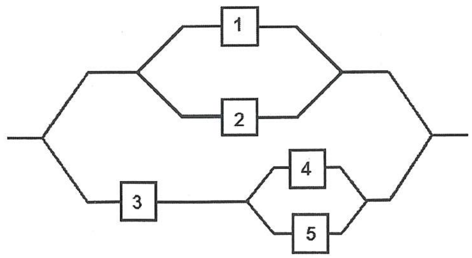
1. (5 points) Consider the system of 5 identical components connected as in the accompanying picture. If components work (mutually) independently of one another and P(component works) = 0.9, calculate P(system works).



$$P(\text{System Works}) = P(A_1 \cup A_2 \cup (A_3 \cap (A_4 \cup A_5)))$$
$$= 1 - P(A_1^c \cap A_2^c \cap (A_3^c \cup (A_4 \cup A_5)^c))$$
$$= 1 - P(A_1^c)P(A_2^c)P(A_3^c \cup (A_4^c \cap A_5^c)) \quad (\text{By indep.})$$
$$= 1 - (0.1)(0.1)(1 - P(A_3 \cap (A_4 \cup A_5)))$$
$$= 1 - 0.01 \times (1 - P(A_3).P(A_4 \cup A_5)) \quad (\text{By indep.})$$
$$= 1 - 0.01 \times (1 - P(A_3)(P(A_4) + P(A_5) - P(A_4).P(A_5)))$$
$$= 1 - 0.01 \times (1 - 0.9 \times (2 \times 0.9 - 0.9^2)) \quad (\text{By indep})$$
$$= 0.99891$$

2. (5 points) If a fair six-sided die is rolled 10 times, what is the probability that there are exactly 4 even numbers OR exactly 3 multiples of 5?

$\#$ even numbers $\sim Bin(10, \frac{1}{2})$

$\#$ multiples of 5 $\sim Bin(10, \frac{1}{6})$

$P(\text{exactly 4 even numbers}) = \binom{10}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^6 = .2051$

$P(\text{exactly 3 multiples of 5}) = \binom{10}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^7 = .1550$

also, $P(\text{exactly 4 even numbers } \underline{\text{and}} \text{ exactly 3 multiples of 5})$

$$= \binom{10}{4}\binom{6}{3}\left(\frac{1}{2}\right)^4\left(\frac{1}{6}\right)^3\left(1-\frac{1}{2}-\frac{1}{6}\right)^3 = 0.0450$$

Now, $P(\text{exactly 4 even numbers } \underline{\text{or}} \text{ exactly 3 multiples of 5})$.

$$= 0.2051 + 0.1550 - 0.0450 = 0.3151$$

3. (5 points) The Reviews editor for a certain scientific journal decides whether the review for any particular book should be short (1-2 pages), medium (3-4 pages) or long (5-6 pages). Data on recent reviews indicates that 60% of them are short, 30% are medium, and the other 10% are long. Reviews are submitted in either Word or LaTeX. For short reviews, 40% are in LaTex, whereas 30% of medium reviews are in LaTex and 70% of long reviews are in LaTex. Suppose a recent review is randomly selected.
a. (1 point) What is the probability that the selected review was submitted in Word format?
b. (1 point) If the selected review was submitted in LaTex format, what is the conditional probability of it being short?
c. (1 point) Check if the events that the selected review is short and the selected review is submitted in LaTex format are independent or not.
d. (1 point) Check if the events that the selected review is not short, the selected review is not medium and the selected review is not long are mutually independent or not.
e. (1 point) Check if the events that the selected review is not short, the selected review is not medium and the selected review is not long are conditionally (mutually) independent or not, given that it was submitted in LaTex format.

S: Short, M: Medium, L: Long, W: Word, La: LaTex

(a) $P(W) = P(W|S)P(S) + P(W|M)P(M) + P(W|L)P(L)$
$= (0.6)(0.6) + (0.7)(0.3) + (0.3)(0.1)$
$= 0.6$

(b) $P(S|La) = \dfrac{P(S \cap La)}{P(La)} = \dfrac{P(La|S).P(S)}{1-P(W)} = \dfrac{(0.4)(0.6)}{1-0.6} = 0.6$

(c) $P(S|La) = 0.6 = P(S)$
So, S and La are indep.

(d) $P(S^c \cap M^c) = P((M \cup L) \cap (S \cup L)) = P(L) = 0.1$
$P(S^c).P(M^c) = (1-0.6)(1-0.3) = 0.28$
$\Rightarrow$ S^c and M^c are not indep.
$\Rightarrow$ S^c, M^c and L^c are <u>not</u> mutually indep.

(e) $P(S^c \cap M^c | La) = P(L|La) = 0.175$
$P(S^c|La) = 1-0.6 = 0.4$, $P(M^c|La) = 1 - 0.225 = 0.775$

[From part (b), $P(S|La) = 0.6$.
Similarly, $P(M|La) = \dfrac{(0.3)(0.3)}{1-0.6} = \dfrac{9}{40} = 0.225$.
$P(L|La) = \dfrac{(0.7)(0.1)}{1-0.6} = \dfrac{7}{40} = 0.175$.]

$P(S^c|La).P(M^c|La) = 0.31$
$\Rightarrow$ S^c and M^c are not conditionally indep. given La.
$\Rightarrow$ S^c, M^c and L^c are <u>not</u> conditionally (mutually) indep. given La.

Alternatively, if you realize that $S^c \cap M^c \cap L^c = \phi$

Then, $P(S^c \cap M^c \cap L^c) = P(S^c \cap M^c \cap L^c | La) = 0$

But S^c, M^c, L^c all have positive probabilities unconditionally or conditionally given La.

So, $P(S^c \cap M^c \cap L^c) = 0 \neq P(S^c).P(M^c).P(L^c) > 0$

and $P(S^c \cap M^c \cap L^c | La) = 0 \neq P(S^c|La).P(M^c|La).P(L^c|La) > 0$

$\Rightarrow$ NOT mutually indep. unconditionally (part d) and conditionally given La (part e).

4. (15 points) A pair of coins with probabilities of heads 0.2 and 0.25 are flipped together independently of each other. Let X be the number of heads observed.
a. (2 points) Find the support and PMF of X.
b. (2 points) Find the expectation and variance of $(2-\sqrt{X})$.
c. (1 point) Find $F_{2X-1}(2)$, i.e. the CDF of 2X-1, evaluated at the point 2.
d. (2 points) You decide with your friend that you win a round if both coins show heads and your friend wins otherwise. Each round consists of one flip of each coin. You repeat this process independently with the same pair of coins. What is the probability that out of 25 total rounds of this game, you win exactly 4 rounds? What is the probability that your friend wins more than 20 rounds out of 25 such rounds?
e. (3 points) Find the (approximate) probability that out of 60 such rounds, the number of rounds your friend wins is within 1.5 standard deviations of its mean value?
f. (1 point) How many rounds are you expected to wait for your first win (including that round)?
g. (2 points) Suppose that your friend pays you $20 if you win a round and you pay her $1 if she wins a round. What are the expectation and variance of your net profit after 60 rounds of the game (assuming that both of you have enough money at the beginning of the game)?
h. (1 point) If you have lost $2 after 60 rounds of the game, should you continue playing or quit (assuming that you have enough money to keep playing forever)? Briefly explain the reason behind your decision.
i. (1 point) How much should you pay your friend if she wins a round for the game to be fair, i.e. not favoring either of you, especially in the long run (assuming she still pays you $20 when you win a round)?

(a) $S_X = \{0, 1, 2\}$

$$P_X(x) = \begin{cases} (0.2)(0.25) = 0.05 & \text{if } x=2 \\ (0.2)(0.75)+(0.8)(0.25) = 0.35 & \text{if } x=1 \\ (0.8)(0.75) = 0.6 & \text{if } x=0 \end{cases}$$

(b) PMF of $Y = 2-\sqrt{X}$:

y	0.5858	1	2
$P(y)$	0.05	0.35	0.6

$E(2-\sqrt{X}) = 2(0.6) + 1(0.35) + 0.5858(0.05) = 1.5793$

$V(2-\sqrt{X}) = 2^2(0.6) + 1^2(0.35) + (0.5858)^2(0.05) - (1.5793)^2 = 0.2730$

(c) $F_{2X-1}(2) = P(2X-1 \leq 2) = P(X \leq 1.5) = P_X(0) + P_X(1) = 0.95$

(d) Let Y_1 and Z_1 be the number of rounds you and your friend win after 25 rounds. $Y_1 \sim Bin(25, 0.05)$ and $Z_1 \sim Bin(25, 0.95)$

$P(Y_1 = 4) = b(4; 25, 0.05) = 0.993 - 0.966$ (from table) $= 0.027$

$P(Z_1 > 20) = 1 - B(20; 25, 0.95) = 1 - 0.007$ (from table) $= 0.993$

(e) Let Y_2 and Z_2 be the number of rounds you and your friend win after 60 rounds. $Y_2 \sim Bin(60, 0.05)$ and $Z_2 \sim Bin(60, 0.95)$

$\mu_{Z_2} = 60 \times 0.95 = 57$, $\sigma_{Z_2} = \sqrt{60(0.95)(0.05)} = 1.6882$

$P(\mu_{Z_2} - 1.5\sigma_{Z_2} \leq Z_2 \leq \mu_{Z_2} + 1.5\sigma_{Z_2}) = P(57 - 2.5323 \leq Z_2 \leq 57 + 2.5323)$

$= P(55 \leq Z_2 \leq 59)$

$= P(60-59 \leq 60 - Z_2 \leq 60-55)$

$= P(1 \leq Y_2 \leq 5)$

$Y_2 \sim Bin(60, 0.05)$
$60 \times 0.05 = 3 < 5$ and $60 > 50$.
$\therefore Y_2 \overset{approx}{\sim} Poi(3)$
$\Rightarrow P(1 \le Y_2 \le 5) \approx \sum_{x=1}^{5} \frac{e^{-3} 3^x}{x!} = F(5;3) - F(0;3)$
$= 0.916 - 0.050$ (from table) $= 0.866$

(f) Rounds you wait for your first win $= W$ (say)
$W \sim Geo(0.05)$
$\Rightarrow E(W) = \frac{1}{0.05} = 20$.

(g) $V =$ Your profit in one round (say)
PMF of V:

v	20	-1
$P(v)$	0.05	0.95

$E(V) = 20(0.05) - 1(0.95) = 0.05$
$E(V^2) = 20^2(0.05) + 1^2(0.95) = 20.95 \Rightarrow Var(V) = 20.95 - (0.05)^2 = 20.9475$
Expectation and Variance of your net profit after 60 rounds are $60 \times 0.05 = 3$ and $60 \times 20.9475 = 1256.85$ (in \$) respectively.
(By indep.)

Alternatively, Your net profit after 60 rounds $= 20Y_2 + (-1)Z_2$
$= 20Y_2 - (60 - Y_2)$
$= 21Y_2 - 60$

Now, $E(21Y_2 - 60) = 21 \cdot E(Y_2) - 60 = 21 \times 3 - 60 = 3$
$V(21Y_2 - 60) = (21)^2 \cdot V(Y_2) = (21)^2 \times 60 \times (0.05)(0.95) = 1256.85$.

(h) $E(V) > 0$, so the game favors you in the long run and eventually you'll start winning money.
You should continue playing despite the short term loss.

(i) Let that value be \$A.
We need : $E(V) = 0 \Rightarrow 20(0.05) - A(0.95) \Rightarrow A = \frac{1}{0.95} = 1.0526$
You should pay \$1.05 (approx).

5. (10 points) Suppose that a continuous random variable X has CDF

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{7x^2}{8} & if\ 0 \le x \le 1 \\ c & if\ 1 < x < 7 \\ dx & if\ 7 \le x \le 8 \\ 1 & if\ x > 8 \end{cases}$$

a. (1 point) Find the constants c and d.
b. (1 point) Find $P(X < 7.5 \mid X \ge 0.5)$.
c. (2 points) Find the PDF f(x) of X.
d. (1 point) What is the support of X?
e. (2 points) Find the 20th percentile of (5-X).
f. (3 points) Find E|X-5|, i.e. the mean of the absolute value of (X-5).

(a) By continuity of F, $c = F(1+) = F(1) = 7/8$ and from $F(7-) = F(7)$, we get $7/8 = d \cdot 7 \Rightarrow d = \frac{1}{8}$.

(b) $$P(X<7.5 \mid X \geqslant 0.5) = \frac{P(0.5 \le X < 7.5)}{P(X \geqslant 0.5)} = \frac{F(7.5) - F(0.5)}{1 - F(0.5)}$$
$$= \frac{7.5/8 - 7/8\,(0.5)^2}{1 - 7/8(0.5)^2} = 0.92.$$

(c) $$f(x) = F'(x) = \begin{cases} \frac{7x}{4} & \text{if } 0 \le x \le 1 \\ 1/8 & \text{if } 7 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$$

(d) $S_X = [0,1] \cup [7,8]$

(e) Let $\eta_{5-X}(0.2)$ be c, so that $P(5-X \le c) = 0.2$
$\Rightarrow P(X \geqslant 5-c) = 0.2 \Rightarrow P(X \le 5-c) = 0.8 \Rightarrow F(5-c) = 0.8$

Now, $F(1) = 7/8 > 0.875 \Rightarrow 0 < 5-c < 1$

So, $F(5-c) = 0.8 \Rightarrow \frac{7 \cdot (5-c)^2}{8} = 0.8 \Rightarrow 5 - c = .9562 \Rightarrow c = 4.0438$

$\therefore \eta_{5-X}(0.2) = 4.0438.$

(f) $$E|X-5| = \int_{-\infty}^{\infty} |x-5| f(x)dx = \int_0^1 (5-x)\frac{7x}{4}\,dx + \int_7^8 (x-5)\frac{1}{8}\,dx$$
$$= \frac{35}{4}\int_0^1 x\,dx - \frac{7}{4}\cdot\int_0^1 x^2 dx + \frac{1}{8}\int_7^8 x\,dx - \frac{5}{8}\int_7^8 dx$$
$$= \frac{35}{4}\left[\frac{x^2}{2}\right]_0^1 - \frac{7}{4}\left[\frac{x^3}{3}\right]_0^1 + \frac{1}{8}\left[\frac{x^2}{2}\right]_7^8 - \frac{5}{8}[x]_7^8$$
$$= \frac{35}{4}\cdot\frac{1}{2} - \frac{7}{4}\cdot\frac{1}{3} + \frac{1}{8}\cdot\frac{15}{2} - \frac{5}{8}\cdot 1$$
$$= 4.1042.$$