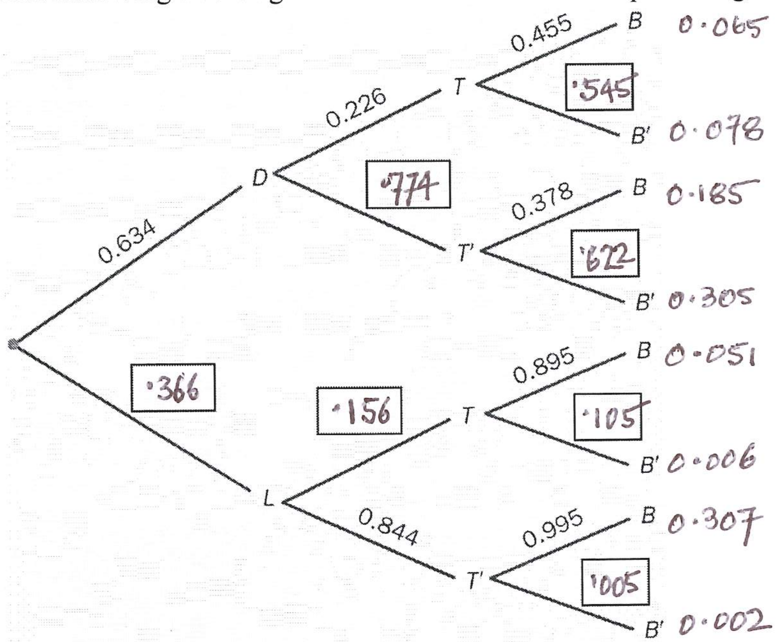
1. (15 points) After a minor collision, a driver must take his car to one of two body shops in the area. Consider the following events.
D = driver takes his car to shop D.
L = driver takes his car to shop L.
T = the work is completed on time.
B = the cost is less than or equal to the estimate (under budget).

The following tree diagram describes the relationships among these events.



(a) (2 points) Complete the tree diagram by filling in the missing path probabilities.
(b) (1 point) What is the probability that the car is repaired under budget, on time, and with company D?
(c) (2 points) What is the probability that the cost of the repair is over the estimate?
(d) (2 points) What is the probability that the car is repaired under budget, given that it is ready on time?
(e) (2 points) What is the probability that he went to shop L, given that the car is repaired under budget?
(f) (2 points) Are the events D and T independent? Justify your answer.
(g) (2 points) Are the three events D, T and B mutually independent? Justify your answer.
(h) (2 points) Are the events D and L independent? Justify your answer.

(b) $P(B\cap T\cap D) = P(D)P(T|D)P(B|T\cap D) = 0.634\times 0.226\times 0.455 = 0.065$

(c) $P(B') = P(B'\cap T\cap D) + P(B'\cap T'\cap D) + P(B'\cap T\cap L) + P(B'\cap T'\cap L)$
$= 0.078 + 0.305 + 0.006 + 0.002 = 0.391$

(d) $P(B|T) = \frac{P(B\cap T)}{P(T)} = \frac{0.065+0.051}{0.065+0.078+0.051+0.006} = \frac{0.116}{0.2} = 0.58$

(e) $P(L|B) = \frac{P(L\cap B)}{P(B)} = \frac{0.051+0.307}{1-0.391} = \frac{0.358}{0.609} = 0.588$

(f) $P(D\cap T) = 0.065+0.078 = 0.143 \neq (0.634)(0.2) = P(D)P(T)$
NOT indep.

(g) D, T and B are NOT mutually indep. as D and T are NOT pairwise indep.

(h) D and L are disjoint, so $P(D \cap L) = 0 \neq (0.634)(0.366)$
$= P(D)P(L)$

$\Rightarrow$ D and L are NOT indep.

2. (10 points) For the following cumulative distribution function for a discrete random variable X,

$$F(x) = \begin{cases} 0 & \text{if } x < -3 \\ 0.03 & \text{if } -3 \le x < 1 \\ 0.20 & \text{if } 1 \le x < 2.5 \\ 0.76 & \text{if } 2.5 \le x < 7 \\ 1 & \text{if } 7 \le x \end{cases}$$

(a) (1 point) Find the support of the distribution of X.
(b) (1 point) What is p(0), i.e. P(X=0)?
(c) (1 point) What is the value of the cumulative distribution function of X+1 at the point x=2?
(d) (3 points) Write down the probability mass function of X in the following table. You may add extra columns, if necessary. Why is it a valid probability mass function?

x	-3	1	2.5	7
P(X=x)	0.03	0.17	0.56	0.24

(e) (2 points) Given that X is positive, what is the probability that it will be at least 2?
(f) (2 points) Find out E(11X-17) and V(11X-17).

(a) $D = \{-3, 1, 2.5, 7\}$

(b) $p(0) = P(X=0) = 0$

(c) $F_{X+1}(2) = P(X+1 \le 2) = P(X \le 1) = F_X(1) = 0.2$

(d) PMF is valid as $P(X=x) \geq 0 \ \forall x$ and $\sum_{x \in D} P(X=x) = 1$

(e) $P(X \geq 2 \mid X > 0) = \dfrac{P(\{X \geq 2\} \cap \{X > 0\})}{P(X>0)} = \dfrac{P(X \geq 2)}{P(X > 0)}$

$= \dfrac{0.56 + 0.24}{0.17 + 0.56 + 0.24}$

$= \dfrac{80}{97} = 0.825$

(f) $E(X) = (-3)(0.03) + (1)(0.17) + (2.5)(0.56) + (7)(0.24)$

$= -0.09 + 0.17 + 1.4 + 1.68 = 3.16$

$E(X^2) = (9)(0.03) + (1)(0.17) + (6.25)(0.56) + (49)(0.24)$

$= 0.27 + 0.17 + 3.5 + 11.76 = 15.7$

$\Rightarrow V(X) = 15.7 - (3.16)^2 = 5.7144$

Now, $E(11X - 17) = 11 \times 3.16 - 17 = 17.76$

$V(11X - 17) = (11)^2 \cdot (5.7144) = 691.4424$

3. (4 points) A fair six sided die is rolled 6 times.
(a) (3 points) What is the probability that each number from 1 to 6 appears at least once?
(b) (1 point) Given that each number from 1 to 6 appears at least once, what is the probability that at least one number has appeared twice?

(a) Each number appears exactly once iff they appear at least once. (Event A)

$$P(A) = \frac{N(A)}{N} \quad \text{[all outcomes are equally likely]}$$

$$= \frac{\text{\# arrangements with exactly 1 of each value (1-6)}}{\text{Total \# arrangements}}$$

$$= \frac{6!}{6^6} = 0.015$$

(b) If A happens, no number can appear twice, i.e. B can't happen [Event B = At least one number appears twice]

$$\Rightarrow P(B|A) = 0$$

4. (3 points) Joe works on homework 25% of the time. He is on the computer 40% of the time. He likes to do homework on the computer, so 30% of the time that he is on the computer is spent doing homework. What percentage of the time that Joe is NOT on the computer is spent NOT doing homework?

$$\left.\begin{array}{l} P(H) = 0.25 \\ P(C) = 0.4 \\ P(H|C) = 0.3 \end{array}\right\} \Rightarrow P(H \cap C) = P(H|C)P(C) = (0.4)(0.3) = 0.12$$

$$P(H^c|C^c) = 1 - P(H|C^c) = 1 - \frac{P(H \cap C^c)}{P(C^c)} = 1 - \frac{P(H) - P(H \cap C)}{1 - P(C)}$$

$$= 1 - \frac{0.25 - 0.3 \times 0.4}{1 - 0.4}$$

$$= 1 - \frac{0.13}{0.6}$$

$$= \frac{47}{60} = 0.783$$

$$= 78.3\%$$

5. (3 points) If X has Bernoulli distribution with probability of success 0.6, calculate $E[(X+1)^3]$ and $V[(X+1)^3]$.

X has the PMF: $p_X(x) = \begin{cases} 0.6 & \text{if } x = 1 \\ 0.4 & \text{if } x = 0 \end{cases}$

$Y = X+1$ has the PMF: $p_Y(x) = \begin{cases} 0.6 & \text{if } x = 2 \\ 0.4 & \text{if } x = 1 \end{cases}$

$$E[(X+1)^3] = E(Y^3) = (2)^3(0.6) + (1)^3(0.4) = 5.2$$

$$V[(X+1)^3] = V(Y^3) = (8 - 5.2)^2(0.6) + (1 - 5.2)^2(0.4)$$

$$= (7.84)(0.6) + (17.64)(0.4) = 11.76$$

6. (10 points) An exam consists of 20 multiple choice questions with 5 possible answer choices per question. The students haven't studied, so they decide to make random guesses for each answer choice. Assume each question's answer is independent from the others and there is only one correct answer for each question.

(a) (2 points) What is the probability that a student who randomly guesses on each question gets exactly 5 questions correct, given that he got at least 1 question correct?
(b) (2 points) If a student has to pay \$2 to take the exam, but the instructor pays them 25 cents for every correct answer, what is a student's expected net profit (in dollars)? What is the standard deviation in the net profit?
(c) (1 point) How much should the instructor at least pay for every correct answer so that the exam looks profitable to the students in terms of money?
(d) (2 points) Assuming that there are infinitely many exams to grade, what is the probability that the instructor will have to grade 3 such exams to find one with exactly 5 correct answers?
(e) (3 points) If the exam had 100 questions with 25 possible answer choices per question and students followed the same strategy, what is the (exact) probability that a student gets at least 2 questions correct? Find an approximate answer and discuss its validity.

(a) X = # questions answered correctly by a student

$p = P(\text{Each question answered correctly}) = \frac{1}{5} = 0.2$

$\therefore X \sim Bin(20, 0.2)$

$$P(X=5 \mid X \geq 1) = \frac{P(\{X=5\} \cap \{X \geq 1\})}{P(X \geq 1)} = \frac{P(X=5)}{P(X \geq 1)}$$

$$= \frac{b(5; 20, 0.2)}{1 - b(0; 20, 0.2)}$$

$$= \frac{0.804 - 0.630}{1 - 0.012} \quad \text{(From Table)}$$

$$= 0.176$$

(b) Net Profit (in \$) $= \frac{X}{4} - 2$

$$E(\text{Net Profit}) = E\left(\frac{X}{4} - 2\right) = \frac{E(X)}{4} - 2 = \frac{20 \times (0.2)}{4} - 2 = -1$$

$$\sigma_{\text{Net Profit}} = \sqrt{V(\text{Net Profit})} = \sqrt{V\left(\frac{X}{4} - 2\right)} = \left(\frac{1}{4}\right)\sqrt{V(X)} = \frac{\sqrt{(20) \times (0.2) \times (0.8)}}{4} = 0.447$$

(c) $E(\text{Net Profit})$ should be > 0 (Let c cents be paid per question)

$$\Rightarrow E\left(\frac{cX}{100} - 2\right) = \frac{c}{100} \cdot E(X) - 2 = \frac{c}{100} \cdot 4 - 2 > 0$$

$$\Rightarrow \frac{c}{25} > 2 \Rightarrow c > 50$$

So, he should pay more than 50 cents.

(d) $Y = \#$ exams to grade to find one with exactly 5 correct answers

$p' = P(\text{exactly 5 correct answers})$
$= P(X=5) = b(5; 20, 0.2) = 0.174$ (From Table)

Hence, $Y \sim Geo(p')$

$P(Y=3) = (1-p')^2 p'$
$= (1-0.174)^2 (0.174)$
$= 0.119$

(e) Now, $X \sim Bin\left(100, \frac{1}{25}\right)$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - B(1; 100, 0.04)$$
$$= 1 - \binom{100}{0}(0.04)^0(0.96)^{100} - \binom{100}{1}(0.04)^1(0.96)^{99}$$
$$= 1 - 0.017 - 0.070$$
$$= 0.913$$

Here $n = 100 > 50$ and $np = 4 < 5$, so Poisson approx. is valid.

So, $X \overset{approx.}{\sim} Poi(\lambda = 4)$

$$P(X \geq 2) = 1 - P(X \leq 1)$$
$$\approx 1 - 0.092 \text{ (From Table)}$$
$$= 0.908$$

The approximation differs only by 0.005. ($\approx 0.55\%$ error)