36. Let X be the damage incurred (in \$) in a certain type of accident during a given year. Possible X values are 0, 1000, 5000, and 10000, with probabilities .8, .1, .08, and .02, respectively. A particular company offers a \$500 deductible policy. If the company wishes its expected profit to be \$100, what premium amount should it charge?

37. The n candidates for a job have been ranked $1, 2, 3, \ldots, n$. Let X = the rank of a randomly selected candidate, so that X has pmf

$$p(x) = \begin{cases} 1/n & x = 1, 2, 3, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

(this is called the *discrete uniform distribution*). Compute $E(X)$ and $V(X)$ using the shortcut formula. [*Hint:* The sum of the first n positive integers is $n(n + 1)/2$, whereas the sum of their squares is $n(n + 1)(2n + 1)/6$.]

38. Let X = the outcome when a fair die is rolled once. If before the die is rolled you are offered either (1/3.5) dollars or $h(X) = 1/X$ dollars, would you accept the guaranteed amount or would you gamble? [*Note:* It is not generally true that $1/E(X) = E(1/X)$.]

39. A chemical supply company currently has in stock 100 lb of a certain chemical, which it sells to customers in 5-lb batches. Let X = the number of batches ordered by a randomly chosen customer, and suppose that X has pmf

x	1	2	3	4
$p(x)$	.2	.4	.3	.1

Compute $E(X)$ and $V(X)$. Then compute the expected number of pounds left after the next customer's order is shipped and the variance of the number of pounds left. [*Hint:* The number of pounds left is a linear function of X.]

40. **a.** Draw a line graph of the pmf of X in Exercise 35. Then determine the pmf of $-X$ and draw its line graph. From these two pictures, what can you say about $V(X)$ and $V(-X)$?

b. Use the proposition involving $V(aX + b)$ to establish a general relationship between $V(X)$ and $V(-X)$.

41. Use the definition in Expression (3.13) to prove that $V(aX + b) = a^2 \cdot \sigma_X^2$ [*Hint:* With $h(X) = aX + b$, $E[h(X)] = a\mu + b$ where $\mu = E(X)$.]

42. Suppose $E(X) = 5$ and $E[X(X - 1)] = 27.5$. What is

a. $E(X^2)$? [*Hint:* $E[X(X - 1)] = E[X^2 - X] = E(X^2) - E(X)$]?

b. $V(X)$?

c. The general relationship among the quantities $E(X)$, $E[X(X - 1)]$, and $V(X)$?

43. Write a general rule for $E(X - c)$ where c is a constant. What happens when you let $c = \mu$, the expected value of X?

44. A result called **Chebyshev's inequality** states that for any probability distribution of an rv X and any number k that is at least 1, $P(|X - \mu| \geq k\sigma) \leq 1/k^2$. In words, the probability that the value of X lies at least k standard deviations from its mean is at most $1/k^2$.

a. What is the value of the upper bound for $k = 2$? $k = 3$? $k = 4$? $k = 5$? $k = 10$?

b. Compute μ and σ for the distribution of Exercise 13. Then evaluate $P(|X - \mu| \geq k\sigma)$ for the values of k given in part (a). What does this suggest about the upper bound relative to the corresponding probability?

c. Let X have possible values -1, 0, and 1, with probabilities $\frac{1}{18}$, $\frac{8}{9}$, and $\frac{1}{18}$, respectively. What is $P(|X - \mu| \geq 3\sigma)$, and how does it compare to the corresponding bound?

d. Give a distribution for which $P(|X - \mu| \geq 5\sigma) = .04$.

45. If $a \leq X \leq b$, show that $a \leq E(X) \leq b$.

3.4 The Binomial Probability Distribution

There are many experiments that conform either exactly or approximately to the following list of requirements:

1. The experiment consists of a sequence of n smaller experiments called *trials*, where n is fixed in advance of the experiment.
2. Each trial can result in one of the same two possible outcomes (dichotomous trials), which we generically denote by success (S) and failure (F).
3. The trials are independent, so that the outcome on any particular trial does not influence the outcome on any other trial.
4. The probability of success $P(S)$ is constant from trial to trial; we denote this probability by p.

DEFINITION

> An experiment for which Conditions 1–4 are satisfied is called a **binomial experiment.**

c. The bookstore has 15 new copies and 15 used copies in stock. If 25 people come in one by one to purchase this text, what is the probability that all 25 will get the type of book they want from current stock? [*Hint:* Let X = the number who want a new copy. For what values of X will all 25 get what they want?]

d. Suppose that new copies cost \$100 and used copies cost \$70. Assume the bookstore currently has 50 new copies and 50 used copies. What is the expected value of total revenue from the sale of the next 25 copies purchased? Be sure to indicate what rule of expected value you are using. [*Hint:* Let $h(X)$ = the revenue when X of the 25 purchasers want new copies. Express this as a linear function.]

53. Exercise 30 (Section 3.3) gave the pmf of Y, the number of traffic citations for a randomly selected individual insured by a particular company. What is the probability that among 15 randomly chosen such individuals

a. At least 10 have no citations?

b. Fewer than half have at least one citation?

c. The number that have at least one citation is between 5 and 10, inclusive?*

54. A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.

a. Among ten randomly selected customers who want this type of racket, what is the probability that at least six want the oversize version?

b. Among ten randomly selected customers, what is the probability that the number who want the oversize version is within 1 standard deviation of the mean value?

c. The store currently has seven rackets of each version. What is the probability that all of the next ten customers who want this racket can get the version they want from current stock?

55. Twenty percent of all telephones of a certain type are submitted for service while under warranty. Of these, 60% can be repaired, whereas the other 40% must be replaced with new units. If a company purchases ten of these telephones, what is the probability that exactly two will end up being replaced under warranty?

56. The College Board reports that 2% of the 2 million high school students who take the SAT each year receive special accommodations because of documented disabilities (*Los Angeles Times,* July 16, 2002). Consider a random sample of 25 students who have recently taken the test.

a. What is the probability that exactly 1 received a special accommodation?

b. What is the probability that at least 1 received a special accommodation?

c. What is the probability that at least 2 received a special accommodation?

d. What is the probability that the number among the 25 who received a special accommodation is within 2 standard deviations of the number you would expect to be accommodated?

e. Suppose that a student who does not receive a special accommodation is allowed 3 hours for the exam, whereas an accommodated student is allowed 4.5 hours. What would you expect the average time allowed the 25 selected students to be?

57. Suppose that 90% of all batteries from a certain supplier have acceptable voltages. A certain type of flashlight requires two type-D batteries, and the flashlight will work only if both its batteries have acceptable voltages. Among ten randomly selected flashlights, what is the probability that at least nine will work? What assumptions did you make in the course of answering the question posed?

58. A very large batch of components has arrived at a distributor. The batch can be characterized as acceptable only if the proportion of defective components is at most .10. The distributor decides to randomly select 10 components and to accept the batch only if the number of defective components in the sample is at most 2.

a. What is the probability that the batch will be accepted when the actual proportion of defectives is .01? .05? .10? .20? .25?

b. Let p denote the actual proportion of defectives in the batch. A graph of P(batch is accepted) as a function of p, with p on the horizontal axis and P(batch is accepted) on the vertical axis, is called the *operating characteristic curve* for the acceptance sampling plan. Use the results of part (a) to sketch this curve for $0 \le p \le 1$.

c. Repeat parts (a) and (b) with "1" replacing "2" in the acceptance sampling plan.

d. Repeat parts (a) and (b) with "15" replacing "10" in the acceptance sampling plan.

e. Which of the three sampling plans, that of part (a), (c), or (d), appears most satisfactory, and why?

59. An ordinance requiring that a smoke detector be installed in all previously constructed houses has been in effect in a particular city for 1 year. The fire department is concerned that many houses remain without detectors. Let p = the true proportion of such houses having detectors, and suppose that a random sample of 25 homes is inspected. If the sample strongly indicates that fewer than 80% of all houses have a detector, the fire department will campaign for a mandatory inspection program. Because of the costliness of the program, the department prefers not to call for such inspections unless sample evidence strongly argues for their necessity. Let X denote the number of homes with detectors among the 25 sampled. Consider rejecting the claim that $p \ge .8$ if $x \le 15$.

a. What is the probability that the claim is rejected when the actual value of p is .8?

b. What is the probability of not rejecting the claim when $p = .7$? When $p = .6$?

c. How do the "error probabilities" of parts (a) and (b) change if the value 15 in the decision rule is replaced by 14?

* "Between a and b, inclusive" is equivalent to $(a \le X \le b)$.

60. A toll bridge charges \$1.00 for passenger cars and \$2.50 for other vehicles. Suppose that during daytime hours, 60% of all vehicles are passenger cars. If 25 vehicles cross the bridge during a particular daytime period, what is the resulting expected toll revenue? [*Hint:* Let $X =$ the number of passenger cars; then the toll revenue $h(X)$ is a linear function of X.]

61. A student who is trying to write a paper for a course has a choice of two topics, A and B. If topic A is chosen, the student will order two books through interlibrary loan, whereas if topic B is chosen, the student will order four books. The student believes that a good paper necessitates receiving and using at least half the books ordered for either topic chosen. If the probability that a book ordered through interlibrary loan actually arrives in time is .9 and books arrive independently of one another, which topic should the student choose to maximize the probability of writing a good paper? What if the arrival probability is only .5 instead of .9?

62. **a.** For fixed n, are there values of p $(0 \le p \le 1)$ for which $V(X) = 0$? Explain why this is so.
b. For what value of p is $V(X)$ maximized? [*Hint:* Either graph $V(X)$ as a function of p or else take a derivative.]

63. **a.** Show that $b(x; n, 1-p) = b(n-x; n, p)$.
b. Show that $B(x; n, 1-p) = 1 - B(n-x-1; n, p)$. [*Hint:* At most x S's is equivalent to at least $(n-x)$ F's.]
c. What do parts (a) and (b) imply about the necessity of including values of p greater than .5 in Appendix Table A.1?

64. Show that $E(X) = np$ when X is a binomial random variable. [*Hint:* First express $E(X)$ as a sum with lower limit $x = 1$. Then factor out np, let $y = x - 1$ so that the sum is from $y = 0$ to $y = n - 1$, and show that the sum equals 1.]

65. Customers at a gas station pay with a credit card (A), debit card (B), or cash (C). Assume that successive customers make independent choices, with $P(A) = .5$, $P(B) = .2$, and $P(C) = .3$.
a. Among the next 100 customers, what are the mean and variance of the number who pay with a debit card? Explain your reasoning.
b. Answer part (a) for the number among the 100 who don't pay with cash.

66. An airport limousine can accommodate up to four passengers on any one trip. The company will accept a maximum of six reservations for a trip, and a passenger must have a reservation. From previous records, 20% of all those making reservations do not appear for the trip. Answer the following questions, assuming independence wherever appropriate.
a. If six reservations are made, what is the probability that at least one individual with a reservation cannot be accommodated on the trip?
b. If six reservations are made, what is the expected number of available places when the limousine departs?
c. Suppose the probability distribution of the number of reservations made is given in the accompanying table.

Number of reservations	3	4	5	6
Probability	.1	.2	.3	.4

Let X denote the number of passengers on a randomly selected trip. Obtain the probability mass function of X.

67. Refer to Chebyshev's inequality given in Exercise 44. Calculate $P(|X - \mu| \ge k\sigma)$ for $k = 2$ and $k = 3$ when $X \sim \text{Bin}(20, .5)$, and compare to the corresponding upper bound. Repeat for $X \sim \text{Bin}(20, .75)$.

3.5 Hypergeometric and Negative Binomial Distributions

The hypergeometric and negative binomial distributions are both related to the binomial distribution. The binomial distribution is the approximate probability model for sampling without replacement from a finite dichotomous (S–F) population provided the sample size n is small relative to the population size N; the hypergeometric distribution is the exact probability model for the number of S's in the sample. The binomial rv X is the number of S's when the number n of trials is fixed, whereas the negative binomial distribution arises from fixing the number of S's desired and letting the number of trials be random.

The Hypergeometric Distribution

The assumptions leading to the hypergeometric distribution are as follows:

1. The population or set to be sampled consists of N individuals, objects, or elements (a *finite* population).

a. How many loads can be expected to occur during a 2-year period?
b. What is the probability that more than five loads occur during a 2-year period?
c. How long must a time period be so that the probability of no loads occurring during that period is at most .1?

90. Let X have a Poisson distribution with parameter μ. Show that $E(X) = \mu$ directly from the definition of expected value. [*Hint:* The first term in the sum equals 0, and then x can be canceled. Now factor out μ and show that what is left sums to 1.]

91. Suppose that trees are distributed in a forest according to a two-dimensional Poisson process with parameter α, the expected number of trees per acre, equal to 80.
a. What is the probability that in a certain quarter-acre plot, there will be at most 16 trees?
b. If the forest covers 85,000 acres, what is the expected number of trees in the forest?
c. Suppose you select a point in the forest and construct a circle of radius .1 mile. Let X = the number of trees within that circular region. What is the pmf of X? [*Hint:* 1 sq mile = 640 acres.]

92. Automobiles arrive at a vehicle equipment inspection station according to a Poisson process with rate $\alpha = 10$ per hour. Suppose that with probability .5 an arriving vehicle will have no equipment violations.
a. What is the probability that exactly ten arrive during the hour and all ten have no violations?
b. For any fixed $y \geq 10$, what is the probability that y arrive during the hour, of which ten have no violations?
c. What is the probability that ten "no-violation" cars arrive during the next hour? [*Hint:* Sum the probabilities in part (b) from $y = 10$ to ∞.]

93. a. In a Poisson process, what has to happen in both the time interval $(0, t)$ and the interval $(t, t + \Delta t)$ so that no events occur in the entire interval $(0, t + \Delta t)$? Use this and Assumptions 1–3 to write a relationship between $P_0(t + \Delta t)$ and $P_0(t)$.
b. Use the result of part (a) to write an expression for the difference $P_0(t + \Delta t) - P_0(t)$. Then divide by Δt and let $\Delta t \to 0$ to obtain an equation involving $(d/dt)P_0(t)$, the derivative of $P_0(t)$ with respect to t.
c. Verify that $P_0(t) = e^{-\alpha t}$ satisfies the equation of part (b).
d. It can be shown in a manner similar to parts (a) and (b) that the $P_k(t)$s must satisfy the system of differential equations

$$\frac{d}{dt}P_k(t) = \alpha P_{k-1}(t) - \alpha P_k(t)$$
$$k = 1, 2, 3, \ldots$$

Verify that $P_k(t) = e^{-\alpha t}(\alpha t)^k/k!$ satisfies the system. (This is actually the only solution.)

SUPPLEMENTARY EXERCISES (94–122)

94. Consider a deck consisting of seven cards, marked 1, 2, . . . , 7. Three of these cards are selected at random. Define an rv W by W = the sum of the resulting numbers, and compute the pmf of W. Then compute μ and σ^2. [*Hint:* Consider outcomes as unordered, so that (1, 3, 7) and (3, 1, 7) are not different outcomes. Then there are 35 outcomes, and they can be listed. (This type of rv actually arises in connection with a statistical procedure called Wilcoxon's rank-sum test, in which there is an x sample and a y sample and W is the sum of the ranks of the x's in the combined sample; see Section 15.2.)

95. After shuffling a deck of 52 cards, a dealer deals out 5. Let X = the number of suits represented in the five-card hand.
a. Show that the pmf of X is

x	1	2	3	4
$p(x)$	.002	.146	.588	.264

[*Hint:* $p(1) = 4P$(all are spades), $p(2) = 6P$(only spades and hearts with at least one of each suit), and $p(4) = 4P$(2 spades $\cap$ one of each other suit).]
b. Compute μ, σ^2, and σ.

96. The negative binomial rv X was defined as the number of F's preceding the rth S. Let Y = the number of trials necessary to obtain the rth S. In the same manner in which the pmf of X was derived, derive the pmf of Y.

97. Of all customers purchasing automatic garage-door openers, 75% purchase a chain-driven model. Let X = the number among the next 15 purchasers who select the chain-driven model.
a. What is the pmf of X?
b. Compute $P(X > 10)$.
c. Compute $P(6 \leq X \leq 10)$.
d. Compute μ and σ^2.
e. If the store currently has in stock 10 chain-driven models and 8 shaft-driven models, what is the probability that the requests of these 15 customers can all be met from existing stock?

98. A friend recently planned a camping trip. He had two flashlights, one that required a single 6-V battery and another that used two size-D batteries. He had previously packed two 6-V and four size-D batteries in his camper. Suppose the probability that any particular battery works is p and that batteries work or fail independently of one another. Our friend wants to take just one flashlight. For what values of p should he take the 6-V flashlight?