7.

a. $p(1,1) = .030.$

b. $P(X \le 1 \text{ and } Y \le 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120.$

c. $P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100$; $P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300.$

d. $P(\text{overflow}) = P(X + 3Y > 5) = 1 - P(X + 3Y \le 5) = 1 - P((X,Y)=(0,0) \text{ or } \ldots\text{or } (5,0) \text{ or } (0,1) \text{ or } (1,1) \text{ or } (2,1)) = 1 - .620 = .380.$

e. The marginal probabilities for X (row sums from the joint probability table) are $p_X(0) = .05$, $p_X(1) = .10$, $p_X(2) = .25$, $p_X(3) = .30$, $p_X(4) = .20$, $p_X(5) = .10$; those for Y (column sums) are $p_Y(0) = .5$, $p_Y(1) = .3$, $p_Y(2) = .2$. It is now easily verified that for every (x,y), $p(x,y) = p_X(x) \cdot p_Y(y)$, so X and Y are independent.

26. Revenue = $3X + 10Y$, so $E(\text{revenue}) = E(3X + 10Y)$

$$= \sum_{x=0}^{5}\sum_{y=0}^{2}(3x+10y)\cdot p(x,y) = 0\cdot p(0,0) + \ldots + 35\cdot p(5,2) = 15.4 = \$15.40.$$

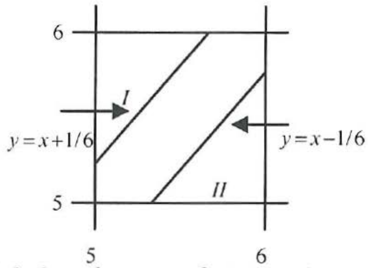
10.

a. Since $f_X(x) = \dfrac{1}{6-5} = 1$ for $5 \le x \le 6$, similarly $f_Y(y) = 1$ for $5 \le y \le 6$, and X and Y are independent,

$$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} 1 & 5 \le x \le 6, 5 \le y \le 6 \\ 0 & \text{otherwise} \end{cases}$$

b. $P(5.25 \le X \le 5.75, 5.25 \le Y \le 5.75) = P(5.25 \le X \le 5.75) \cdot P(5.25 \le Y \le 5.75)$ by independence $= (.5)(.5) = .25.$

c. The region A is the diagonal stripe below.



$$P((X,Y) \in A) = \iint_A 1\,dxdy = \text{the area of } A = 1 - (\text{area of } I + \text{area of } II)$$

$$= 1 - \left(\frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6} + \frac{1}{2}\cdot\frac{5}{6}\cdot\frac{5}{6}\right) = 1 - \frac{25}{36} = \frac{11}{36} = .306.$$

12.

a. $P(X > 3) = \int_3^\infty \int_0^\infty xe^{-x(1+y)}dydx = \int_3^\infty e^{-x}dx = .050.$

b. The marginal pdf of X is $f_X(x) = \int_0^\infty xe^{-x(1+y)}dy = e^{-x}$ for $x \ge 0$. The marginal pdf of Y is $f_Y(y) = \int_3^\infty xe^{-x(1+y)}dx = \dfrac{1}{(1+y)^2}$ for $y \ge 0$. It is now clear that $f(x,y)$ is not the product of the marginal pdfs, so the two rvs are not independent.

c. $P(\text{at least one exceeds } 3) = P(X > 3 \text{ or } Y > 3) = 1 - P(X \le 3 \text{ and } Y \le 3)$

$$= 1 - \int_0^3\int_0^3 xe^{-x(1+y)}dydx = 1 - \int_0^3\int_0^3 xe^{-x}e^{-xy}dy$$

$$= 1 - \int_0^3 e^{-x}(1 - e^{-3x})dx = e^{-3} + .25 - .25e^{-12} = .300.$$

32. $E(XY) = \int_0^\infty \int_0^\infty xy \cdot xe^{-x(1+y)} dydx = \cdots = 1$. Yet, since the marginal pdf of Y is $f_Y(y) = \frac{1}{(1-y)^2}$ for $y \geq 0$,

$E(Y) = \int_0^\infty \frac{y}{(1+y)^2} dy = \infty$. Therefore, Cov($X$, Y) and Corr(X, Y) do not exist, since they require this integral (among others) to be convergent.

27. The amount of time Annie waits for Alvie, if Annie arrives first, is $Y - X$; similarly, the time Alvie waits for Annie is $X - Y$. Either way, the amount of time the first person waits for the second person is $h(X, Y) = |X - Y|$. Since X and Y are independent, their joint pdf is given by $f_X(x) \cdot f_Y(y) = (3x^2)(2y) = 6x^2y$. From these, the expected waiting time is

$E[h(X,Y)] = \int_0^1 \int_0^1 |x-y| \cdot f(x,y) dxdy = \int_0^1 \int_0^1 |x-y| \cdot 6x^2 y dxdy$

$= \int_0^1 \int_0^x (x-y) \cdot 6x^2 y dydx + \int_0^1 \int_x^1 (x-y) \cdot 6x^2 y dydx = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$ hour, or 15 minutes.

37. The joint pmf of X_1 and X_2 is presented below. Each joint probability is calculated using the independence of X_1 and X_2; e.g., $p(25, 25) = P(X_1 = 25) \cdot P(X_2 = 25) = (.2)(.2) = .04$.

	$p(x_1, x_2)$	x_1: 25	40	65	
x_2	25	.04	.10	.06	.2
	40	.10	.25	.15	.5
	65	.06	.15	.09	.3
		.2	.5	.3	

a. For each coordinate in the table above, calculate $\bar{x}$. The six possible resulting $\bar{x}$ values and their corresponding probabilities appear in the accompanying pmf table.

$\bar{x}$	25	32.5	40	45	52.5	65
$p(\bar{x})$	.04	.20	.25	.12	.30	.09

From the table, $E(\bar{X}) = (25)(.04) + 32.5(.20) + ... + 65(.09) = 44.5$. From the original pmf, $\mu = 25(.2) + 40(.5) + 65(.3) = 44.5$. So, $E(\bar{X}) = \mu$.

b. For each coordinate in the joint pmf table above, calculate $s^2 = \frac{1}{2-1}\sum_{i=1}^{2}(x_i - \bar{x})^2$. The four possible resulting s^2 values and their corresponding probabilities appear in the accompanying pmf table.

s^2	0	112.5	312.5	800
$p(s^2)$	.38	.20	.30	.12

From the table, $E(S^2) = 0(.38) + \ldots + 800(.12) = 212.25$. From the original pmf, $\sigma^2 = (25 - 44.5)^2(.2) + (40 - 44.5)^2(.5) + (65 - 44.5)^2(.3) = 212.25$. So, $E(S^2) = \sigma^2$.

50.

a. $P(9{,}900 \leq \bar{X} \leq 10{,}200) \approx P\left(\frac{9{,}900 - 10{,}000}{500/\sqrt{40}} \leq Z \leq \frac{10{,}200 - 10{,}000}{500/\sqrt{40}}\right)$

$= P(-1.26 \leq Z \leq 2.53) = \Phi(2.53) - \Phi(-1.26) = .9943 - .1038 = .8905$.

b. According to the guideline given in Section 5.4, n should be greater than 30 in order to apply the CLT, thus using the same procedure for $n = 15$ as was used for $n = 40$ would not be appropriate.

56.

a. Let X = the number of erroneous bits out of 1000, so $X \sim$ Bin(1000, .10). If we approximate X by a normal rv with $\mu = np = 100$ and $\sigma^2 = npq = 90$, then with a continuity correction $P(X \leq 125) = P(X \leq 125.5) \approx P\left(Z \leq \frac{125.5 - 100}{\sqrt{90}}\right) = P(Z \leq 2.69) = \Phi(2.69) = .9964$.

b. Let Y = the number of errors in the second transmission, so $Y \sim$ Bin(1000, .10) and is independent of X. To find $P(|X - Y| \leq 50)$, use the facts that $E[X - Y] = 100 - 100 = 0$ and $V(X - Y) = V(X) + V(Y) = 90 + 90 = 180$. So, using a normal approximation to both binomial rvs, $P(|X - Y| \leq 50) = P(-50 \leq X - Y \leq 50) = P(-50.5 \leq X - Y \leq 50.5) \approx P\left(-\frac{50.5}{\sqrt{180}} \leq Z \leq \frac{50.5}{\sqrt{180}}\right)$
$= P(-3.76 \leq Z \leq 3.76) \approx 1 - 0 = 1$.

69.

a. $E(X_1 + X_2 + X_3) = 800 + 1000 + 600 = 2400$.

b. Assuming independence of X_1, X_2, X_3, $V(X_1 + X_2 + X_3) = (16)^2 + (25)^2 + (18)^2 = 1205$.
It is enough to assume that X_1, X_2, X_3 are (pairwise) uncorrelated here.

c. $E(X_1 + X_2 + X_3) = 2400$ as before, but now $V(X_1 + X_2 + X_3)$
$= V(X_1) + V(X_2) + V(X_3) + 2\text{Cov}(X_1, X_2) + 2\text{Cov}(X_1, X_3) + 2\text{Cov}(X_2, X_3) = 1745$, from which the standard deviation is 41.77.

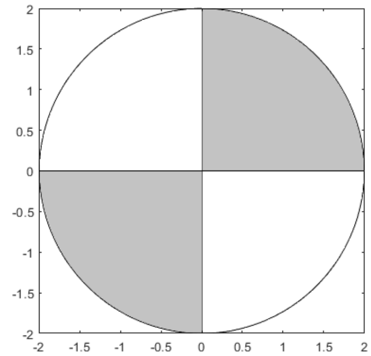
72. The total elapsed time between leaving and returning is $T_o = X_1 + X_2 + X_3 + X_4$, with $E(T_o) = 40$, $\sigma^2_{T_o} = 30$, $\sigma_{T_o} = 5.477$. T_o is normally distributed, and the desired value t is the 99th percentile of the lapsed time distribution added to 10 A.M.:
10:00 + [40 + 2.33(5.477)] = 10:52.76 A.M. = 10:52:45.68 A.M. = 10:52:46 A.M. (approximately)

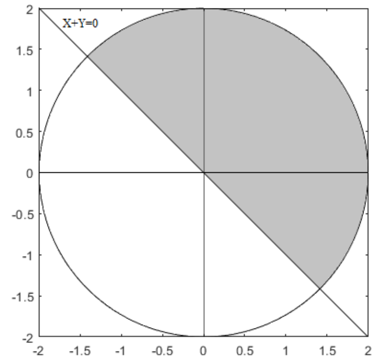
Bonus Problem:

(a) (X, Y) is jointly uniform on the circle D with center at $(0,0)$ and radius 2. Hence $c = \frac{1}{area(D)} = \frac{1}{\pi(2)^2} = \frac{1}{4\pi}$.

(b) Using the fact that $P\left((X, Y)\epsilon A\right) = \frac{area(A \cap D)}{area(D)}$, we shade the region $A \cap D$ for each of the probabilities and calculate the probabilities as ratios of areas.

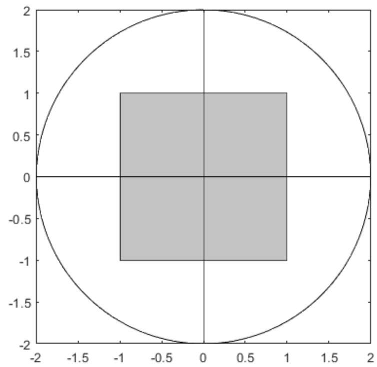
$$P(XY > 0) = \frac{1}{2} = 0.5$$



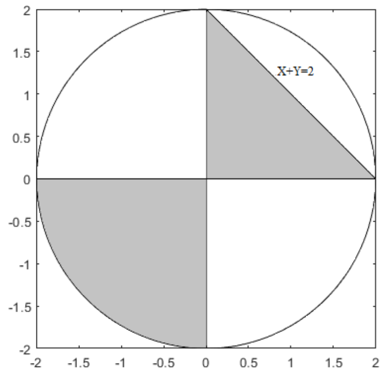


$$P(X + Y > 0) = \frac{1}{2} = 0.5$$

$$P(-1 \leq X \leq 1, -1 \leq Y \leq 1) = \frac{2^2}{4\pi} = \frac{1}{\pi} = 0.3183$$



$$P(XY > 0, X + Y < 2) = \frac{\left(\frac{4\pi}{4}\right) + \left(\frac{1}{2}.2.2\right)}{4\pi} = \frac{1}{4} + \frac{1}{2\pi} = 0.4092$$



(c) Marginal supports of both X and Y are $(-2,2)$, i.e. $S_X = S_Y = (-2,2)$.

Marginal PDF of X: $f_X(x) = \int_{-2}^{2} f(x,y)dy = \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{4\pi} dy = \frac{\sqrt{4-x^2}}{2\pi}$ for $x \in S_X$.

Similarly, by the symmetry of X and Y, $f_Y(y) = \frac{\sqrt{4-y^2}}{2\pi}$ for $y \in S_Y$.

(d) The joint support D is not rectangular, so X and Y are not independent.

(e) Let $x \in S_X = (-2,2)$ be fixed.

For $f_{Y|X}(y|x)$ to be positive, $(x,y) \in D$ i.e. $x^2 + y^2 \le 4$, i.e. $y^2 \le 4 - x^2$,

i.e. $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$.

$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/4\pi}{\sqrt{4-x^2}/2\pi} = \frac{1}{2\sqrt{4-x^2}}$ if $(x,y) \in D$

Hence $f_{Y|X}(y|x) = \begin{cases} \frac{1}{2\sqrt{4-x^2}}, if -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \\ 0, \ otherwise \end{cases}$

Clearly $f_{Y|X}(y|x)$ does not depend on y or is constant as a function of y. Hence this is a member of the univariate Uniform distribution family $U(a,b)$ with $a = -\sqrt{4-x^2}, b = \sqrt{4-x^2}$.

(f) From the conditional distribution Y|X=x derived in part e,

Conditional mean: $E(Y|X=x) = \frac{a+b}{2} = 0$

Conditional standard deviation: $\sigma_{Y|X=x} = \frac{b-a}{\sqrt{12}} = \frac{2\sqrt{4-x^2}}{\sqrt{12}} = \sqrt{\frac{4-x^2}{3}}$

Conditional third quartile: $\eta_{Y|X=x}(0.75) = a + \frac{3(b-a)}{4} = \frac{\sqrt{4-x^2}}{2}$

(g) $E(XY) = \iint_{x^2+y^2\le 4} xyf(x,y)\, dy\, dx =$

$\frac{1}{4\pi}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} xy\, dy\, dx = \frac{1}{4\pi}\int_{x=-2}^{2} x \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx = \frac{1}{4\pi}\int_{x=-2}^{2} x.0\, dx =$

$\frac{1}{4\pi}\int_{x=-2}^{2} 0\, dx = 0$

by using the hint with $c = \sqrt{4-x^2}$ for the inner integral as the function $g(y) = y$ is odd,

i.e. $g(-y) = -y = -g(y)$.

Also, $E(X) = \iint_{x^2+y^2\le 4} xf(x,y)\, dy\, dx =$

$\frac{1}{4\pi}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x\, dy\, dx = \frac{1}{4\pi}\int_{x=-2}^{2} x \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx = \frac{1}{2\pi}\int_{x=-2}^{2} x\sqrt{4-x^2}\, dx = 0$

by using the hint with $c = 2$ for the outer integral as the function $h(x) = x\sqrt{4-x^2}$ is odd,

i.e. $h(-x) = (-x)\sqrt{4-(-x)^2} = -x\sqrt{4-x^2} = -h(x)$.

By symmetry of X and Y, $E(Y) = E(X) = 0$.

Now $Cov(X,Y) = E(XY) - E(X)E(Y) = 0 - 0 = 0$ and $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{0}{\sigma_X\sigma_Y} = 0$. ($\sigma_X, \sigma_Y > 0$ as X and Y are clearly not degenerate).

Note: This is another example where X and Y are uncorrelated but not independent. The example shown in class had discrete X and Y whereas here X and Y have a joint density.