36.

a. $P(X < 1500) = P(Z < 3) = \Phi(3) = .9987$; $P(X \geq 1000) = P(Z \geq -.33) = 1 - \Phi(-.33) = 1 - .3707 = .6293$.

b. $P(1000 < X < 1500) = P(-.33 < Z < 3) = \Phi(3) - \Phi(-.33) = .9987 - .3707 = .6280$

c. From the table, $\Phi(z) = .02 \Rightarrow z = -2.05 \Rightarrow x = 1050 - 2.05(150) = 742.5$ μm. The smallest 2% of droplets are those smaller than 742.5 μm in size.

d. P(at least one droplet in 5 that exceeds 1500 μm) = $1 - P$(all 5 are less than 1500 μm) = $1 - (.9987)^5 = 1 - .9935 = .0065$.

42. The probability X is within .1 of its mean is given by $P(\mu - .1 \leq X \leq \mu + .1) = P\left(\frac{(\mu-.1)-\mu}{\sigma} < Z < \frac{(\mu+.1)-\mu}{\sigma}\right) = \Phi\left(\frac{.1}{\sigma}\right) - \Phi\left(-\frac{.1}{\sigma}\right) = 2\Phi\left(\frac{.1}{\sigma}\right) - 1$. If we require this to equal 95%, we find $2\Phi\left(\frac{.1}{\sigma}\right) - 1 = .95 \Rightarrow \Phi\left(\frac{.1}{\sigma}\right) = .975 \Rightarrow \frac{.1}{\sigma} = 1.96$ from the standard normal table. Thus, $\sigma = \frac{.1}{1.96} = .0510$.

Alternatively, use the empirical rule: 95% of all values lie within 2 standard deviations of the mean, so we want $2\sigma = .1$, or $\sigma = .05$. (This is not quite as precise as the first answer.)

43. Since 1.28 is the 90th z-percentile ($z_{.1} = 1.28$) and -1.645 is the 5th z-percentile ($z_{.05} = 1.645$), the given information implies that $\mu + 1.28\sigma = 10.256$ and $\mu - 1.645\sigma = 9.671$.
Solve: By subtracting the equations, $2.925\sigma = .585$, so $\sigma = .2$, and then $\mu = 10$.

44.

a. $P(\mu - 1.5\sigma \leq X \leq \mu + 1.5\sigma) = P(-1.5 \leq Z \leq 1.5) = \Phi(1.50) - \Phi(-1.50) = .8664$.

b. $P(X < \mu - 2.5\sigma \text{ or } X > \mu + 2.5\sigma) = 1 - P(\mu - 2.5\sigma \leq X \leq \mu + 2.5\sigma)$
$= 1 - P(-2.5 \leq Z \leq 2.5) = 1 - .9876 = .0124$.

c. $P(\mu - 2\sigma \leq X \leq \mu - \sigma \text{ or } \mu + \sigma \leq X \leq \mu + 2\sigma) = P(\text{within 2 sd's}) - P(\text{within 1 sd}) =$
$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) - P(\mu - \sigma \leq X \leq \mu + \sigma) = .9544 - .6826 = .2718$.

46.

a. $P(67 < X < 75) = P\left(\frac{67-70}{3} < \frac{X-70}{3} < \frac{75-70}{3}\right) = P(-1 < Z < 1.67) = \Phi(1.67) - \Phi(-1) = .9525 - .1587 = .7938$.

b. By the Empirical Rule, c should equal 2 standard deviations. Since $\sigma = 3$, $c = 2(3) = 6$.
We can be a little more precise, as in Exercise 42, and use $c = 1.96(3) = 5.88$.

c. Let Y = the number of acceptable specimens out of 10, so $Y \sim \text{Bin}(10, p)$, where $p = .7938$ from part a. Then $E(Y) = np = 10(.7938) = 7.938$ specimens.

d. Now let Y = the number of specimens out of 10 that have a hardness of less than 73.84, so $Y \sim \text{Bin}(10, p)$, where

$$p = P(X < 73.84) = P\left(Z < \frac{73.84-70}{3}\right) = P(Z < 1.28) = \Phi(1.28) = .8997. \text{ Then}$$

$$P(Y \leq 8) = \sum_{y=0}^{8}\binom{10}{y}(.8997)^y(.1003)^{10-y} = .2651.$$

You can also compute $1 - P(Y = 9, 10)$ and use the binomial formula, or round slightly to $p = .9$ and use the binomial table: $P(Y \leq 8) = B(8; 10, .9) = .265$.

63.

a. If a customer's calls are typically short, the first calling plan makes more sense. If a customer's calls are somewhat longer, then the second plan makes more sense, viz. 99¢ is less than 20min(10¢/min) = \$2 for the first 20 minutes under the first (flat-rate) plan.

b. $h_1(X) = 10X$, while $h_2(X) = 99$ for $X \le 20$ and $99 + 10(X - 20)$ for $X > 20$. With $\mu = 1/\lambda$ for the exponential distribution, it's obvious that $E[h_1(X)] = 10E[X] = 10\mu$. On the other hand,

$$E[h_2(X)] = 99 + 10\int_{20}^{\infty}(x-20)\lambda e^{-\lambda x}\,dx = 99 + \frac{10}{\lambda}e^{-20\lambda} = 99 + 10\mu e^{-20/\mu}.$$

When $\mu = 10$, $E[h_1(X)] = 100¢ = \$1.00$ while $E[h_2(X)] = 99 + 100e^{-2} \approx \$1.13$.
When $\mu = 15$, $E[h_1(X)] = 150¢ = \$1.50$ while $E[h_2(X)] = 99 + 150e^{-4/3} \approx \$1.39$.
As predicted, the first plan is better when expected call length is lower, and the second plan is better when expected call length is somewhat higher.

69.

a. $\{X \ge t\}$ = {the lifetime of the system is at least t}. Since the components are connected in series, this equals {all 5 lifetimes are at least t} $= A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

b. Since the events A_i are assumed to be independent, $P(X \ge t) = P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot P(A_4) \cdot P(A_5)$. Using the exponential cdf, for any i we have
$P(A_i) = P(\text{component lifetime is} \ge t) = 1 - F(t) = 1 - [1 - e^{-.01t}] = e^{-.01t}$.
Therefore, $P(X \ge t) = (e^{-.01t}) \cdots (e^{-.01t}) = e^{-.05t}$, and $F_X(t) = P(X \le t) = 1 - e^{-.05t}$.
Taking the derivative, the pdf of X is $f_X(t) = .05e^{-.05t}$ for $t \ge 0$. Thus X also has an exponential distribution, but with parameter $\lambda = .05$.

c. By the same reasoning, $P(X \le t) = 1 - e^{-n\lambda t}$, so X has an exponential distribution with parameter $n\lambda$.

79. Notice that μ_X and σ_X are the mean and standard deviation of the lognormal variable X in this example; they are not the parameters μ and σ which usually refer to the mean and standard deviation of ln(X). We're given $\mu_X = 10{,}281$ and $\sigma_X/\mu_X = .40$, from which $\sigma_X = .40\mu_X = 4112.4$.

a. To find the mean and standard deviation of ln(X), set the lognormal mean and variance equal to the appropriate quantities: $10{,}281 = E(X) = e^{\mu+\sigma^2/2}$ and $(4112.4)^2 = V(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$. Square the first equation: $(10{,}281)^2 = e^{2\mu+\sigma^2}$. Now divide the variance by this amount:

$$\frac{(4112.4)^2}{(10{,}281)^2} = \frac{e^{2\mu+\sigma^2}(e^{\sigma^2}-1)}{e^{2\mu+\sigma^2}} \Rightarrow e^{\sigma^2} - 1 = (.40)^2 = .16 \Rightarrow \sigma = \sqrt{\ln(1.16)} = .38525$$

That's the standard deviation of ln(X). Use this in the formula for $E(X)$ to solve for μ:
$10{,}281 = e^{\mu + (.38525)^2/2} = e^{\mu + .07421} \Rightarrow \mu = 9.164$. That's $E(\ln(X))$.

b. $P(X \le 15{,}000) = P\left(Z \le \dfrac{\ln(15{,}000) - 9.164}{.38525}\right) = P(Z \le 1.17) = \Phi(1.17) = .8790$.

c. $P(X \ge \mu_X) = P(X \ge 10{,}281) = P\left(Z \ge \dfrac{\ln(10{,}281) - 9.164}{.38525}\right) = P(Z \ge .19) = 1 - \Phi(0.19) = .4247$. Even though the normal distribution is symmetric, the lognormal distribution is not a symmetric distribution. (See the lognormal graphs in the textbook.) So, the mean and the median of X aren't the same and, in particular, the probability X exceeds its own mean doesn't equal .5.

d. One way to check is to determine whether $P(X < 17{,}000) = .95$; this would mean 17,000 is indeed the 95th percentile. However, we find that $P(X < 17{,}000) = \Phi\left(\dfrac{\ln(17{,}000) - 9.164}{.38525}\right) = \Phi(1.50) = .9332$, so 17,000 is not the 95th percentile of this distribution (it's the 93.32%ile).

81.

a. $V(X) = (e^{\sigma^2} - 1) \cdot e^{2\mu+\sigma^2}$

$= (e^{0.06} - 1) \cdot e^{2\times 2.05 + 0.06}$

$= \underline{\underline{3.962}}.$

Hence. $\sqrt{V(X)} = \sqrt{3.962} = \underline{\underline{1.99}}$

b. $P(X > 12) = 1 - P(X \le 12) = 1 - P(Z \le \frac{\ln(12) - 2.05}{\sqrt{0.06}}) = 1 - P(Z \le 1.78) = \underline{\underline{0.0375}}$

c. $E(X) = e^{\mu + \sigma^2/2} = e^{2.05 + 0.06/2} = 8.004.$

$P(E(X) - \sqrt{V(X)} \le X \le E(X) + \sqrt{V(X)}) = P(8.004 - 1.99 \le X \le 8.004 + 1.99)$

$= P(6.014 \le X \le 9.994)$

$= P(\frac{\ln(6.014) - 2.05}{\sqrt{0.06}} \le Z \le \frac{\ln(9.994) - 2.05}{\sqrt{0.06}})$

$= P(-1.04 \le Z \le 1.03) = 0.8485 - 0.1492 = 0.6993$

d. In Exercise 80, it was shown that $\tilde{\mu} = e^{\mu}$. Here, $\tilde{\mu} = e^{2.05} = \underline{\underline{7.768}}.$

e. It was shown in Exercise 80 that the $100(1-\alpha)$th quantile is $e^{\mu+\sigma z_\alpha}$. We'd like to find the 99th percentile, so $\alpha = 0.01$ and $z_{0.01} = 2.33$. Therefore the 99-th percentile of the delay time distribution is $e^{2.05 + \sqrt{0.06}\cdot 2.33} = \underline{\underline{13.746}}.$

f. $P(X > 8) = 1 - P(X \le 8) = 1 - P(Z \le \frac{\ln(8) - 2.05}{\sqrt{0.06}}) = 1 - 0.5478 = 0.4522$

The number of products whose delay time exceeds 8 months follows a binomial distribution with $n = 10$ and $p = 0.4522$, so the expected number of items is $np = 10 \times 0.4522 = 4.522$

108.

a. Since X is exponential, $E(X) = \frac{1}{\lambda} = 1.075$ and $\sigma = \frac{1}{\lambda} = 1.075$.

b. $P(X > 3.0) = 1 - P(X \le 3.0) = 1 - F(3.0) = 1 - [1 - e^{-.93(3.0)}] = .0614$.
$P(1.0 \le X \le 3.0) = F(3.0) - F(1.0) = [1 - e^{-.93(3.0)}] - [1 - e^{-.93(1.0)}] = .333$.

c. The 90th percentile is requested; denoting it by c, we have $.9 = F(c) = 1 - e^{-(.93)c}$. Solving for c, we get $c = \frac{\ln(.1)}{(-.93)} = 2.476\mu\text{m}$.

Bonus Problem:

$X \sim Exp(\lambda)$

(a) $P(X > x) = \begin{cases} e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$

$$P(X \le x \mid X > c) = \begin{cases} \dfrac{P(\{X \le x\} \cap \{X > c\})}{P(X > c)} = \dfrac{P(c < X \le x)}{P(X > c)} = \dfrac{e^{-\lambda c} - e^{-\lambda x}}{e^{-\lambda c}} = 1 - e^{-\lambda(x-c)}, & \text{if } x \ge c \\ 0, & \text{if } x < c \end{cases}$$

(b) $F_{X|X>c}(x) = \begin{cases} 1 - e^{-\lambda(x-c)}, & \text{if } x \ge c \\ 0, & \text{if } x < c \end{cases}$

$f_{X|X>c}(x) = F'_{X|X>c}(x) = \begin{cases} \lambda e^{-\lambda(x-c)}, & \text{if } x \ge c \\ 0, & \text{if } x < c \end{cases}$

Clearly, $f_{X|X>c}(x) \ge 0 \ \forall x$

Now, $\int_{-\infty}^{\infty} f_{X|X>c}(x)\,dx = \int_c^{\infty} \lambda e^{-\lambda(x-c)}\,dx$

$= \int_0^{\infty} \lambda e^{-\lambda y}\,dy$ [Let $y = x - c$ so $dy = dx$]

$= [-e^{-\lambda y}]_0^{\infty} = (-0 - (-1))$

$= 1$

So $f_{X|X>c}(x)$ is a valid density

(c) $E(X|X>c) = \int_{-\infty}^{\infty} x f_{X|X>c}(x)\,dx$

$$= \int_{c}^{\infty} x \lambda e^{-\lambda(x-c)}\,dx$$

$$= \int_{0}^{\infty} (y+c)\lambda e^{-\lambda y}\,dy \quad \begin{pmatrix}\text{Let } y = x-c \\ \text{so } dy = dx\end{pmatrix}$$

$$= \int_{0}^{\infty} y\lambda e^{-\lambda y}\,dy + c\int_{0}^{\infty} \lambda e^{-\lambda y}\,dy$$

$$= \frac{1}{\lambda} + c \cdot 1$$

(The first integral is $E(X)$ when $X \sim Exp(\lambda)$. The second one is 1 from part b)

$$= c + \frac{1}{\lambda}$$

(d) Since X has the memoryless property, $(X-c)$, when $X>c$ is already known, has the same distribution as X, which is $Exp(\lambda)$.

Now, $E(X|X>c) = E(c + X - c \mid X>c)$

$$= c + E(X-c \mid X>c)$$

$$= c + E(X)$$

$$= c + \frac{1}{\lambda}.$$