4.

a. $\int_{-\infty}^{\infty} f(x;\theta)dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{\infty} = 0-(-1) = 1$

b. $P(X \le 200) = \int_{-\infty}^{200} f(x;\theta)dx = \int_0^{200} \frac{x}{\theta^2} e^{-x^2/2\theta^2} dx = -e^{-x^2/2\theta^2}\Big]_0^{200} \approx -.1353+1 = .8647$.

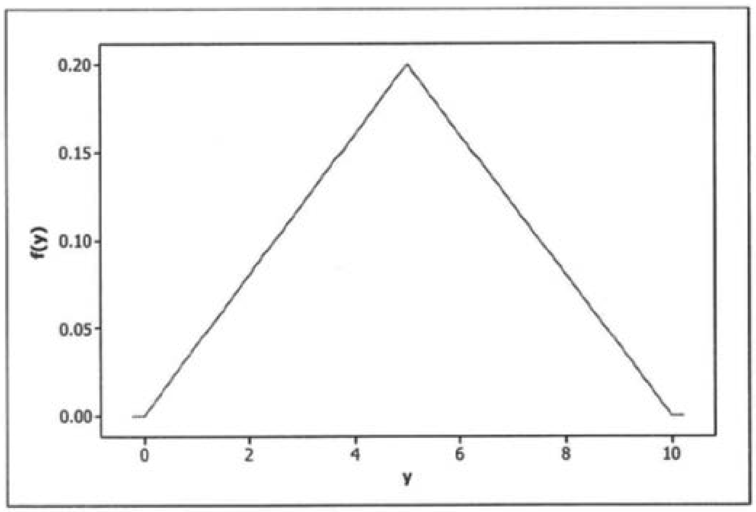
$P(X < 200) = P(X \le 200) \approx .8647$, since X is continuous.
$P(X \ge 200) = 1 - P(X < 200) \approx .1353$.

c. $P(100 \le X \le 200) = \int_{100}^{200} f(x;\theta)dx = -e^{-x^2/20{,}000}\Big]_{100}^{200} \approx .4712$.

d. For $x > 0$, $P(X \le x) = \int_{-\infty}^{x} f(y;\theta)dy = \int_0^{x} \frac{y}{\theta^2} e^{-y^2/2\theta^2} dx = -e^{-y^2/2\theta^2}\Big]_0^{x} = 1 - e^{-x^2/2\theta^2}$.

8.

a.



b. $\int_{-\infty}^{\infty} f(y)dy = \int_0^5 \frac{1}{25}y\,dy + \int_5^{10}(\frac{2}{5}-\frac{1}{25}y)dy = \frac{y^2}{50}\Big]_0^5 + \left(\frac{2}{5}y - \frac{1}{50}y^2\right)\Big]_5^{10} =$

$\frac{25}{50} + \left[(4-2) - \left(2-\frac{1}{2}\right)\right] = \frac{1}{2} + \frac{1}{2} = 1$

c. $P(Y \le 3) = \int_0^3 \frac{1}{25}y\,dy = \frac{y^2}{50}\Big]_0^3 = \frac{9}{50} = .18.$

d. $P(Y \le 8) = \int_0^5 \frac{1}{25}y\,dy + \int_5^8 (\frac{2}{5}-\frac{1}{25}y)dy = \frac{23}{25} = .92.$

e. Use parts **c** and **d:** $P(3 \le Y \le 8) = P(Y \le 8) - P(Y < 3) = .92 - .18 = .74.$

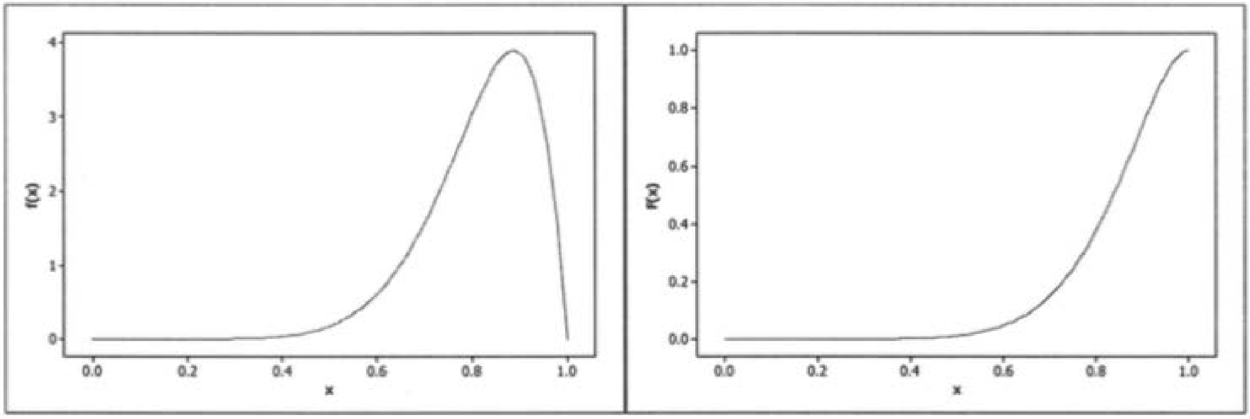
f. $P(Y < 2 \text{ or } Y > 6) = = \int_0^2 \frac{1}{25}y\,dy + \int_6^{10}(\frac{2}{5}-\frac{1}{25}y)dy = \cdots = .4$.

15.

a. Since X is limited to the interval (0, 1), $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.
For $0 < x < 1$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{0}^{x} 90y^8(1-y)dy = \int_{0}^{x}(90y^8 - 90y^9)dy = 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10}\ .$$

The graphs of the pdf and cdf of X appear below.



b. $F(.5) = 10(.5)^9 - 9(.5)^{10} = .0107$.

c. $P(.25 < X \le .5) = F(.5) - F(.25) = .0107 - [10(.25)^9 - 9(.25)^{10}] = .0107 - .0000 = .0107$.
Since X is continuous, $P(.25 \le X \le .5) = P(.25 < X \le .5) = .0107$.

d. The 75th percentile is the value of x for which $F(x) = .75$: $10x^9 - 9x^{10} = .75 \Rightarrow x = .9036$ using software.

e. $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^1 x \cdot 90x^8(1-x)dx = \int_0^1 (90x^9 - 90x^{10})dx = 9x^{10} - \frac{90}{11}x^{11}\Big]_0^1 = 9 - \frac{90}{11} = \frac{9}{11} = .8182$.

Similarly, $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)dx = \int_0^1 x^2 \cdot 90x^8(1-x)dx = \ldots = .6818$, from which $V(X) = .6818 - (.8182)^2 = .0124$ and $\sigma_X = .11134$.

f. $\mu \pm \sigma = (.7068, .9295)$. Thus, $P(\mu - \sigma \le X \le \mu + \sigma) = F(.9295) - F(.7068) = .8465 - .1602 = .6863$, and the probability X is <u>more</u> than 1 standard deviation from its mean value equals $1 - .6863 = 3137$.

19.

a. $P(X \le 1) = F(1) = .25[1 + \ln(4)] = .597$.

b. $P(1 \le X \le 3) = F(3) - F(1) = .966 - .597 = .369$.

c. For $x < 0$ or $x > 4$, the pdf is $f(x) = 0$ since X is restricted to (0, 4). For $0 < x < 4$, take the first derivative of the cdf:

$$F(x) = \frac{x}{4}\left[1 + \ln\left(\frac{4}{x}\right)\right] = \frac{1}{4}x + \frac{\ln(4)}{4}x - \frac{1}{4}x\ln(x) \Rightarrow$$

$$f(x) = F'(x) = \frac{1}{4} + \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) - \frac{1}{4}x\frac{1}{x} = \frac{\ln(4)}{4} - \frac{1}{4}\ln(x) = .3466 - .25\ln(x)$$

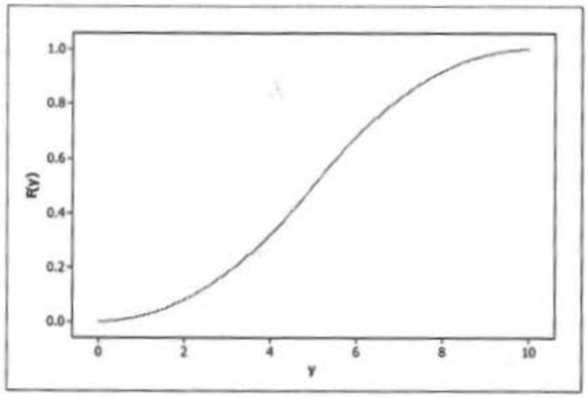
20.

a. For $0 \le y < 5$, $F(y) = \int_0^y \frac{u}{25}du = \frac{y^2}{50}$; for $5 \le y \le 10$,

$$F(y) = \int_0^y f(u)du = \int_0^5 f(u)du + \int_5^y f(u)du = \frac{5^2}{50} + \int_5^y \left(\frac{2}{5} - \frac{u}{25}\right)du = \cdots = -\frac{y^2}{50} + \frac{2}{5}y - 1$$

So, the complete cdf of Y is

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{50} & 0 \le y < 5 \\ -\frac{y^2}{50} + \frac{2}{5}y - 1 & 5 \le y < 10 \\ 1 & y \ge 10 \end{cases}$$



b. In general, set $F(y) = p$ and solve for y.

For $0 < p < .5$, $p = F(y) = \frac{y^2}{50} \Rightarrow \eta(p) = y = \sqrt{50p}$; for $.5 \le p < 1$,

$p = -\frac{y^2}{50} + \frac{2}{5}y - 1 \Rightarrow \eta(p) = y = 10 - 5\sqrt{2(1-p)}$.

c. $E(Y) = 5$ by straightforward integration, or by the symmetry of $f(y)$ about $y = 5$.

Similarly, by symmetry $V(Y) = \int_0^{10} (y-5)^2 f(y)dy = 2\int_0^5 (y-5)^2 f(y)dy = 2\int_0^5 (y-5)^2 \frac{y^2}{50}dy = \ldots = \frac{50}{12} =$ 4.1667. For the waiting time X for a single bus, $E(X) = 2.5$ and $V(X) = \frac{25}{12}$; not coincidentally, the mean and variance of Y are exactly twice that of X.

22.

a. For $1 \le x \le 2$, $F(x) = \int_1^x 2\left(1-\frac{1}{y^2}\right)dy = 2\left(y+\frac{1}{y}\right)\Big]_1^x = 2\left(x+\frac{1}{x}\right)-4$, so the cdf is

$$F(x)=\begin{cases}0 & x<1\\ 2\left(x+\frac{1}{x}\right)-4 & 1\le x\le 2\\ 1 & x>2\end{cases}$$

b. Set $F(x) = p$ and solve for x: $2\left(x+\frac{1}{x}\right)-4 = p \Rightarrow 2x^2-(p+4)x+2=0 \Rightarrow$

$\eta(p) = x = \frac{(p+4)+\sqrt{(p+4)^2-4(2)(2)}}{2(2)} = \frac{p+4+\sqrt{p^2+8p}}{4}$. (The other root of the quadratic gives solutions outside the interval [1, 2].) To find the median $\tilde{\mu}$, set $p = .5$: $\tilde{\mu} = \eta(.5) = \ldots = 1.640$.

c. $E(X) = \int_1^2 x\cdot 2\left(1-\frac{1}{x^2}\right)dx = 2\int_1^2\left(x-\frac{1}{x}\right)dx = 2\left(\frac{x^2}{2}-\ln(x)\right)\Big]_1^2 = 1.614$. Similarly,

$E(X^2) = 2\int_1^2\left(x^2-1\right)dx = 2\left(\frac{x^3}{3}-x\right)\Big]_1^2 = \frac{8}{3} \Rightarrow V(X) = .0626.$

d. The amount left is given by $h(x) = \max(1.5 - x, 0)$, so

$E(h(X)) = \int_1^2 \max(1.5-x,0)f(x)dx = 2\int_1^{1.5}(1.5-x)\left(1-\frac{1}{x^2}\right)dx = .061$.

Bonus problem:

Let the distribution of a random variable X be geometric with parameter p with $0 < p < 1$.

a. Show that $P(X > m + n \mid X > m) = P(X > n)$, for $m, n \in \{1,2,3,\ldots\}$. This is called the *memoryless property* of the geometric distribution.

b. Find $E(2^{-X})$.

c. Let $p = 1/3$ and $Y = |X - 5|$. Find the support and PMF of Y.

a. $P(X>n) = 1 - P(X \le n) = q^n$ (where $q = 1-p$) $\forall n \in \{1,2,3,\ldots\}$

Now, $P(X > m+n \mid X > m) = \dfrac{P(\{X > m+n\} \cap \{X > m\})}{P(X>m)}$

$$= \frac{P(X>m+n)}{P(X>m)} = \frac{q^{m+n}}{q^m} = q^n$$

$= P(X>n)$

($\forall\ m, n \in \{1,2,3,\ldots\}$, $m+n \in \{1,2,3,\ldots\}$)

b. $E(2^{-X}) = \sum_{x=1}^{\infty} 2^{-x} P(X=x) = \sum_{x=1}^{\infty} 2^{-x} p q^{x-1}$

$$= \frac{p}{q} \sum_{x=1}^{\infty} \left(\frac{q}{2}\right)^x$$

$$= \frac{p}{q} \cdot \frac{q/2}{1 - q/2}$$

[The sum represents an infinite geometric series with $a = q/2$ and $r = q/2$, which converges as $0 < r < 1/2$, since $0 < q < 1$]

$$= \frac{p}{q} \cdot \frac{q}{2-q}$$

$$= \frac{p}{2-(1-p)}$$

$$= \frac{p}{1+p}$$

c. $Y = |X-5|$ and $S_X = \{1, 2, 3, \dots\}$ (also, $p = \frac{1}{3}$ and $q = 1-p = \frac{2}{3}$)

Hence, $S_Y = \{0, 1, 2, 3, \dots\}$

$P(Y=0) = P(|X-5| = 0) = P(X=5)$

$P(Y=1) = P(|X-5| = 1) = P(X \in \{6, 4\}) = P(X=4) + P(X=5)$

In general, $P(Y=y) = P(X = 5-y) + P(X = 5+y) \quad \forall y \in \{1, 2, 3, \dots\}$

Moreover for $y \geq 5$, $P(X = 5-y) = 0$ as $X \geq 1$.

So the PMF of Y is:

$$p_Y(y) = \begin{cases} p_X(5) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4, & \text{if } y=0 \\ p_X(5-y) + p_X(5+y) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{4-y} + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{4+y}, & \text{if } 1 \leq y \leq 4 \\ p_X(5+y) = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{4+y}, & \text{if } y \geq 5 \end{cases}$$

Or more specifically,

$$p_Y(y) = \begin{cases} \frac{2^4}{3^5} = 0.0658, & \text{if } y=0 \\ \frac{2^3}{3^4} + \frac{2^5}{3^6} = 0.1427, & \text{if } y=1 \\ \frac{2^2}{3^3} + \frac{2^6}{3^7} = 0.1774, & \text{if } y=2 \\ \frac{2}{3^2} + \frac{2^7}{3^8} = 0.2417, & \text{if } y=3 \\ \frac{1}{3} + \frac{2^8}{3^9} = 0.3463, & \text{if } y=4 \\ \frac{2^{4+y}}{3^{5+y}}, & \text{if } y > 4 \end{cases}$$