

Now consider an event  $A$ , with  $N(A)$  denoting the number of outcomes contained in  $A$ . Then

$$P(A) = \sum_{E_i \text{ in } A} P(E_i) = \sum_{E_i \text{ in } A} \frac{1}{N} = \frac{N(A)}{N}$$

Thus when outcomes are equally likely, computing probabilities reduces to counting: determine both the number of outcomes  $N(A)$  in  $A$  and the number of outcomes  $N$  in  $\mathcal{S}$ , and form their ratio.

**EXAMPLE 2.16**

You have six unread mysteries and six unread science fiction books on your bookshelf. The first three of each type are hardcover, and the last three are paperback. Consider randomly selecting one of the six mysteries and then randomly selecting one of the six science fiction books to take on a post-finals vacation to Acapulco (after all, you need something to read on the beach). Number the mysteries 1, 2, ..., 6, and do the same for the science fiction books. Then each outcome is a pair of numbers such as (4, 1), and there are  $N = 36$  possible outcomes (For a visual of this situation, refer to the table in Example 2.3 and delete the first row and column). With random selection as described, the 36 outcomes are equally likely. Nine of these outcomes are such that both selected books are paperbacks (those in the lower right-hand corner of the referenced table): (4,4), (4,5), ..., (6,6). So the probability of the event  $A$  that both selected books are paperbacks is

$$P(A) = \frac{N(A)}{N} = \frac{9}{36} = .25$$

**EXERCISES Section 2.2 (11–28)**

11. A mutual fund company offers its customers a variety of funds: a money-market fund, three different bond funds (short, intermediate, and long-term), two stock funds (moderate and high-risk), and a balanced fund. Among customers who own shares in just one fund, the percentages of customers in the different funds are as follows:

Money-market	20%	High-risk stock	18%
Short bond	15%	Moderate-risk stock	25%
Intermediate bond	10%	Balanced	7%
Long bond	5%		

A customer who owns shares in just one fund is randomly selected.

- What is the probability that the selected individual owns shares in the balanced fund?
  - What is the probability that the individual owns shares in a bond fund?
  - What is the probability that the selected individual does not own shares in a stock fund?
12. Consider randomly selecting a student at a large university, and let  $A$  be the event that the selected student has a

Visa card and  $B$  be the analogous event for MasterCard. Suppose that  $P(A) = .6$  and  $P(B) = .4$ .

- Could it be the case that  $P(A \cap B) = .5$ ? Why or why not? [Hint: See Exercise 24.]
  - From now on, suppose that  $P(A \cap B) = .3$ . What is the probability that the selected student has at least one of these two types of cards?
  - What is the probability that the selected student has neither type of card?
  - Describe, in terms of  $A$  and  $B$ , the event that the selected student has a Visa card but not a MasterCard, and then calculate the probability of this event.
  - Calculate the probability that the selected student has exactly one of the two types of cards.
13. A computer consulting firm presently has bids out on three projects. Let  $A_i = \{\text{awarded project } i\}$ , for  $i = 1, 2, 3$ , and suppose that  $P(A_1) = .22$ ,  $P(A_2) = .25$ ,  $P(A_3) = .28$ ,  $P(A_1 \cap A_2) = .11$ ,  $P(A_1 \cap A_3) = .05$ ,  $P(A_2 \cap A_3) = .07$ ,  $P(A_1 \cap A_2 \cap A_3) = .01$ . Express in words each of the following events, and compute the probability of each event:
- $A_1 \cup A_2$
  - $A_1' \cap A_2'$  [Hint:  $(A_1 \cup A_2)' = A_1' \cap A_2'$ ]
  - $A_1 \cup A_2 \cup A_3$
  - $A_1' \cap A_2' \cap A_3'$
  - $A_1' \cap A_2' \cap A_3$
  - $(A_1' \cap A_2') \cup A_3$

## Short-sleeved

Size	Pattern		
	Pl	Pr	St
S	.04	.02	.05
M	.08	.07	.12
L	.03	.07	.08

## Long-sleeved

Size	Pattern		
	Pl	Pr	St
S	.03	.02	.03
M	.10	.05	.07
L	.04	.02	.08

- What is the probability that the next shirt sold is a medium, long-sleeved, print shirt?
  - What is the probability that the next shirt sold is a medium print shirt?
  - What is the probability that the next shirt sold is a short-sleeved shirt? A long-sleeved shirt?
  - What is the probability that the size of the next shirt sold is medium? That the pattern of the next shirt sold is a print?
  - Given that the shirt just sold was a short-sleeved plaid, what is the probability that its size was medium?
  - Given that the shirt just sold was a medium plaid, what is the probability that it was short-sleeved? Long-sleeved?
- One box contains six red balls and four green balls, and a second box contains seven red balls and three green balls. A ball is randomly chosen from the first box and placed in the second box. Then a ball is randomly selected from the second box and placed in the first box.
    - What is the probability that a red ball is selected from the first box and a red ball is selected from the second box?
    - At the conclusion of the selection process, what is the probability that the numbers of red and green balls in the first box are identical to the numbers at the beginning?
  - A system consists of two identical pumps, #1 and #2. If one pump fails, the system will still operate. However, because of the added strain, the remaining pump is now more likely to fail than was originally the case. That is,  $r = P(\#2 \text{ fails} \mid \#1 \text{ fails}) > P(\#2 \text{ fails}) = q$ . If at least one pump fails by the end of the pump design life in 7% of all systems and both pumps fail during that period in only 1%, what is the probability that pump #1 will fail during the pump design life?
  - A certain shop repairs both audio and video components. Let  $A$  denote the event that the next component brought in for repair is an audio component, and let  $B$  be the event that the next component is a compact disc player (so the event  $B$  is contained in  $A$ ). Suppose that  $P(A) = .6$  and  $P(B) = .05$ . What is  $P(B|A)$ ?
  - In Exercise 13,  $A_i = \{\text{awarded project } i\}$ , for  $i = 1, 2, 3$ . Use the probabilities given there to compute the following probabilities, and explain in words the meaning of each one.
    - $P(A_2|A_1)$
    - $P(A_2 \cap A_3|A_1)$
    - $P(A_2 \cup A_3|A_1)$
    - $P(A_1 \cap A_2 \cap A_3|A_1 \cup A_2 \cup A_3)$ .
  - Deer ticks can be carriers of either Lyme disease or human granulocytic ehrlichiosis (HGE). Based on a recent study, suppose that 16% of all ticks in a certain location carry Lyme disease, 10% carry HGE, and 10% of the ticks that carry at least one of these diseases in fact carry both of them. If a randomly selected tick is found to have carried HGE, what is the probability that the selected tick is also a carrier of Lyme disease?
  - For any events  $A$  and  $B$  with  $P(B) > 0$ , show that  $P(A|B) + P(A'|B) = 1$ .
  - If  $P(B|A) > P(B)$ , show that  $P(B'|A) < P(B')$ . [Hint: Add  $P(B'|A)$  to both sides of the given inequality and then use the result of Exercise 56.]
  - Show that for any three events  $A$ ,  $B$ , and  $C$  with  $P(C) > 0$ ,  $P(A \cup B|C) = P(A|C) + P(B|C) - P(A \cap B|C)$ .
  - At a certain gas station, 40% of the customers use regular gas ( $A_1$ ), 35% use plus gas ( $A_2$ ), and 25% use premium ( $A_3$ ). Of those customers using regular gas, only 30% fill their tanks (event  $B$ ). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.
    - What is the probability that the next customer will request plus gas and fill the tank ( $A_2 \cap B$ )?
    - What is the probability that the next customer fills the tank?
    - If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?
  - Seventy percent of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared.
    - If it has an emergency locator, what is the probability that it will not be discovered?
    - If it does not have an emergency locator, what is the probability that it will be discovered?
  - Components of a certain type are shipped to a supplier in batches of ten. Suppose that 50% of all such batches contain no defective components, 30% contain one defective component, and 20% contain two defective components. Two components from a batch are randomly selected and tested. What

66. Consider the following information about travelers on vacation (based partly on a recent Travelocity poll): 40% check work email, 30% use a cell phone to stay connected to work, 25% bring a laptop with them, 23% both check work email and use a cell phone to stay connected, and 51% neither check work email nor use a cell phone to stay connected nor bring a laptop. In addition, 88 out of every 100 who bring a laptop also check work email, and 70 out of every 100 who use a cell phone to stay connected also bring a laptop.
- What is the probability that a randomly selected traveler who checks work email also uses a cell phone to stay connected?
  - What is the probability that someone who brings a laptop on vacation also uses a cell phone to stay connected?
  - If the randomly selected traveler checked work email and brought a laptop, what is the probability that he/she uses a cell phone to stay connected?
67. There has been a great deal of controversy over the last several years regarding what types of surveillance are appropriate to prevent terrorism. Suppose a particular surveillance system has a 99% chance of correctly identifying a future terrorist and a 99.9% chance of correctly identifying someone who is not a future terrorist. If there are 1000 future terrorists in a population of 300 million, and one of these 300 million is randomly selected, scrutinized by the system, and identified as a future terrorist, what is the probability that he/she actually is a future terrorist? Does the value of this probability make you uneasy about using the surveillance system? Explain.
68. A friend who lives in Los Angeles makes frequent consulting trips to Washington, D.C.; 50% of the time she travels on airline #1, 30% of the time on airline #2, and the remaining 20% of the time on airline #3. For airline #1, flights are late into D.C. 30% of the time and late into L.A. 10% of the time. For airline #2, these percentages are 25% and 20%, whereas for airline #3 the percentages are 40% and 25%. If we learn that on a particular trip she arrived late at exactly one of the two destinations, what are the posterior probabilities of having flown on airlines #1, #2, and #3? Assume that the chance of a late arrival in L.A. is unaffected by what happens on the flight to D.C. [Hint: From the tip of each first-generation branch on a tree diagram, draw three second-generation branches labeled, respectively, 0 late, 1 late, and 2 late.]
69. In Exercise 59, consider the following additional information on credit card usage:
- 70% of all regular fill-up customers use a credit card.
  - 50% of all regular non-fill-up customers use a credit card.
  - 60% of all plus fill-up customers use a credit card.
  - 50% of all plus non-fill-up customers use a credit card.
  - 50% of all premium fill-up customers use a credit card.
  - 40% of all premium non-fill-up customers use a credit card.
- Compute the probability of each of the following events for the next customer to arrive (a tree diagram might help).
- {plus and fill-up and credit card}
  - {premium and non-fill-up and credit card}
  - {premium and credit card}
  - {fill-up and credit card}
  - {credit card}
  - If the next customer uses a credit card, what is the probability that premium was requested?

## 2.5 Independence

The definition of conditional probability enables us to revise the probability  $P(A)$  originally assigned to  $A$  when we are subsequently informed that another event  $B$  has occurred; the new probability of  $A$  is  $P(A|B)$ . In our examples, it frequently happened that  $P(A|B)$  differed from the unconditional probability  $P(A)$ . Then the information “ $B$  has occurred” resulted in a change in the likelihood of  $A$  occurring. Often the chance that  $A$  will occur or has occurred is not affected by knowledge that  $B$  has occurred, so that  $P(A|B) = P(A)$ . It is then natural to regard  $A$  and  $B$  as independent events, meaning that the occurrence or nonoccurrence of one event has no bearing on the chance that the other will occur.

### DEFINITION

Two events  $A$  and  $B$  are **independent** if  $P(A|B) = P(A)$  and are **dependent** otherwise.

system fails as soon as an entire column fails, and system lifetime exceeds  $t_0$  only if the life of every column does so. For this configuration,

$$\begin{aligned} P(\text{system lifetime is at least } t_0) &= [P(\text{column lifetime exceeds } t_0)]^3 \\ &= [1 - P(\text{column lifetime is } \leq t_0)]^3 \\ &= [1 - P(\text{both cells in a column have lifetime } \leq t_0)]^3 \\ &= [1 - (1 - .9)^2]^3 = .970 \end{aligned}$$

## EXERCISES Section 2.5 (70–89)

70. Reconsider the credit card scenario of Exercise 47 (Section 2.4), and show that  $A$  and  $B$  are dependent first by using the definition of independence and then by verifying that the multiplication property does not hold.
71. An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let  $A$  be the event that the Asian project is successful and  $B$  be the event that the European project is successful. Suppose that  $A$  and  $B$  are independent events with  $P(A) = .4$  and  $P(B) = .7$ .
- If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.
  - What is the probability that at least one of the two projects will be successful?
  - Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?
72. In Exercise 13, is any  $A_i$  independent of any other  $A_j$ ? Answer using the multiplication property for independent events.
73. If  $A$  and  $B$  are independent events, show that  $A'$  and  $B$  are also independent. [Hint: First establish a relationship between  $P(A' \cap B)$ ,  $P(B)$ , and  $P(A \cap B)$ .]
74. The proportions of blood phenotypes in the U.S. population are as follows:

A	B	AB	O
.40	.11	.04	.45

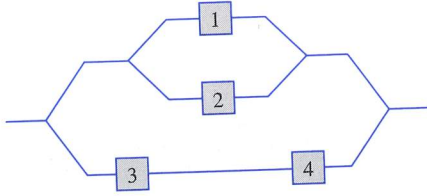
Assuming that the phenotypes of two randomly selected individuals are independent of one another, what is the probability that both phenotypes are O? What is the probability that the phenotypes of two randomly selected individuals match?

75. One of the assumptions underlying the theory of control charting (see Chapter 16) is that successive plotted points are independent of one another. Each plotted point can signal either that a manufacturing process is operating correctly or that there is some sort of malfunction.

Even when a process is running correctly, there is a small probability that a particular point will signal a problem with the process. Suppose that this probability is .05. What is the probability that at least one of 10 successive points indicates a problem when in fact the process is operating correctly? Answer this question for 25 successive points.

76. In October, 1994, a flaw in a certain Pentium chip installed in computers was discovered that could result in a wrong answer when performing a division. The manufacturer initially claimed that the chance of any particular division being incorrect was only 1 in 9 billion, so that it would take thousands of years before a typical user encountered a mistake. However, statisticians are not typical users; some modern statistical techniques are so computationally intensive that a billion divisions over a short time period is not outside the realm of possibility. Assuming that the 1 in 9 billion figure is correct and that results of different divisions are independent of one another, what is the probability that at least one error occurs in one billion divisions with this chip?
77. An aircraft seam requires 25 rivets. The seam will have to be reworked if any of these rivets is defective. Suppose rivets are defective independently of one another, each with the same probability.
- If 15% of all seams need reworking, what is the probability that a rivet is defective?
  - How small should the probability of a defective rivet be to ensure that only 10% of all seams need reworking?
78. A boiler has five identical relief valves. The probability that any particular valve will open on demand is .96. Assuming independent operation of the valves, calculate  $P(\text{at least one valve opens})$  and  $P(\text{at least one valve fails to open})$ .
79. Two pumps connected in parallel fail independently of one another on any given day. The probability that only the older pump will fail is .10, and the probability that only the newer pump will fail is .05. What is the probability that the pumping system will fail on any given day (which happens if both pumps fail)?

80. Consider the system of components connected as in the accompanying picture. Components 1 and 2 are connected in parallel, so that subsystem works iff either 1 or 2 works; since 3 and 4 are connected in series, that subsystem works iff both 3 and 4 work. If components work independently of one another and  $P(\text{component } i \text{ works}) = .9$  for  $i = 1, 2$  and  $= .8$  for  $i = 3, 4$ , calculate  $P(\text{system works})$ .



81. Refer back to the series-parallel system configuration introduced in Example 2.36, and suppose that there are only two cells rather than three in each parallel subsystem [in Figure 2.14(a), eliminate cells 3 and 6, and renumber cells 4 and 5 as 3 and 4]. Using  $P(A_i) = .9$ , the probability that system lifetime exceeds  $t_0$  is easily seen to be .9639. To what value would .9 have to be changed in order to increase the system lifetime reliability from .9639 to .99? [Hint: Let  $P(A_i) = p$ , express system reliability in terms of  $p$ , and then let  $x = p^2$ .]
82. Consider independently rolling two fair dice, one red and the other green. Let  $A$  be the event that the red die shows 3 dots,  $B$  be the event that the green die shows 4 dots, and  $C$  be the event that the total number of dots showing on the two dice is 7. Are these events pairwise independent (i.e., are  $A$  and  $B$  independent events, are  $A$  and  $C$  independent, and are  $B$  and  $C$  independent)? Are the three events mutually independent?
83. Components arriving at a distributor are checked for defects by two different inspectors (each component is checked by both inspectors). The first inspector detects 90% of all defectives that are present, and the second inspector does likewise. At least one inspector does not detect a defect on 20% of all defective components. What is the probability that the following occur?
- A defective component will be detected only by the first inspector? By exactly one of the two inspectors?
  - All three defective components in a batch escape detection by both inspectors (assuming inspections of different components are independent of one another)?
84. Consider purchasing a system of audio components consisting of a receiver, a pair of speakers, and a CD player. Let  $A_1$  be the event that the receiver functions properly throughout the warranty period,  $A_2$  be the event that the speakers function properly throughout the warranty period, and  $A_3$  be the event that the CD player functions properly throughout the warranty period. Suppose that these events are (mutually) independent with  $P(A_1) = .95$ ,  $P(A_2) = .98$ , and  $P(A_3) = .80$ .
- What is the probability that all three components function properly throughout the warranty period?
  - What is the probability that at least one component needs service during the warranty period?
  - What is the probability that all three components need service during the warranty period?
  - What is the probability that only the receiver needs service during the warranty period?
  - What is the probability that exactly one of the three components needs service during the warranty period?
  - What is the probability that all three components function properly throughout the warranty period but that at least one fails within a month after the warranty expires?
85. A quality control inspector is examining newly produced items for faults. The inspector searches an item for faults in a series of independent fixations, each of a fixed duration. Given that a flaw is actually present, let  $p$  denote the probability that the flaw is detected during any one fixation (this model is discussed in "Human Performance in Sampling Inspection," *Human Factors*, 1979: 99–105).
- Assuming that an item has a flaw, what is the probability that it is detected by the end of the second fixation (once a flaw has been detected, the sequence of fixations terminates)?
  - Give an expression for the probability that a flaw will be detected by the end of the  $n$ th fixation.
  - If when a flaw has not been detected in three fixations, the item is passed, what is the probability that a flawed item will pass inspection?
  - Suppose 10% of all items contain a flaw [ $P(\text{randomly chosen item is flawed}) = .1$ ]. With the assumption of part (c), what is the probability that a randomly chosen item will pass inspection (it will automatically pass if it is not flawed, but could also pass if it is flawed)?
  - Given that an item has passed inspection (no flaws in three fixations), what is the probability that it is actually flawed? Calculate for  $p = .5$ .
86. a. A lumber company has just taken delivery on a shipment of 10,000  $2 \times 4$  boards. Suppose that 20% of these boards (2000) are actually too green to be used in first-quality construction. Two boards are selected at random, one after the other. Let  $A = \{\text{the first board is green}\}$  and  $B = \{\text{the second board is green}\}$ . Compute  $P(A)$ ,  $P(B)$ , and  $P(A \cap B)$  (a tree diagram might help). Are  $A$  and  $B$  independent?
- With  $A$  and  $B$  independent and  $P(A) = P(B) = .2$ , what is  $P(A \cap B)$ ? How much difference is there between this answer and  $P(A \cap B)$  in part (a)? For purposes of calculating  $P(A \cap B)$ , can we assume that  $A$  and  $B$  of part (a) are independent to obtain essentially the correct probability?
  - Suppose the shipment consists of ten boards, of which two are green. Does the assumption of independence

Bonus Question:

Only 1 in 1000 adults is afflicted with a rare disease for which two diagnostic tests have been developed. The sensitivity and specificity of both the tests are 99% and 98% respectively. Moreover the results of these two tests are conditionally independent, given that the condition of a patient is known (i.e. independent given that a patient has the disease and also given that a patient does not have the disease).

- (a) If a randomly selected patient gets a positive result in the first test, what is the probability of getting a positive result in their second test?
- (b) If a randomly selected patient tests positive in both the tests, what is the probability that the individual has the disease?